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CONVECTION-DIFFUSION PROBLEMS INVOLVING MEASURE DATA AND $\overrightarrow{p}(.)$-ANISOTROPIC OPERATOR IN VARIABLE EXPONENT SPACE

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Abstract. In this paper, we investigate a nonlinear diffusion-convection problem with measure data, involving a general anisotropic operator with variable exponent and a maximal monotone graph. Utilizing Yosida's regularization, we apply an approximation technique to formulate a regularized problem. We then employ the theory of maximal monotone operators in Banach spaces to demonstrate the existence of at least one solution for the approximate problem. Subsequently, we show that the sequence of solutions to the approximate problem converges, in the limit, to a renormalized and/or entropic solution of the original problem. Finally, we establish the uniqueness of the solution under certain additional assumptions regarding the convection term.

1. Introduction

Partial Differential Equations (PDEs) are crucial from a mathematical perspective for modeling observed phenomena in the real world, particularly in fields such as physics and chemistry. Thus, a wide range of problems in the mechanics of continuous media and engineering sciences can be modeled using partial differential equations. Most often, the mathematical modeling of real-world phenomena is conducted in Sobolev spaces with a constant exponent. However, it should be noted that these spaces do not accurately capture materials or phenomena that change state over time. To account for the real behavior of such phenomena, the exponent needs to vary.

Recently, the study of problems in variable exponent spaces has been gaining momentum, driven by new mathematical developments. The study of problems in generalized Sobolev spaces is driven by their relevance in modeling the behavior of certain materials known as electrorheological and thermorheological fluids (see [15, 32] for more details). These fluids can rapidly change their physical state (within milliseconds) when exposed to a weak electric field. A common example is aluminum oxide, also called alumina (Al_2O_3). Another motivation for studying PDEs in generalized Sobolev spaces has recently emerged in the works of Chen *et al* (see [10]), where they highlight the significance of these spaces in image

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processing. For materials that exhibit different spreading behavior in multiple directions, it becomes essential to use anisotropic Sobolev spaces, where the exponent is represented by a vector field. These spaces are particularly useful for modeling the propagation of epidemic diseases (see [3]).
This paper aims to establish the existence and uniqueness of both renormalized and entropy solutions for a nonlinear anisotropic problem governed by Dirichlet boundary conditions, given in the following form

$$(P^{\phi}_{\beta,\mu})\begin{cases} -\displaystyle\sum_{j=1}^{N}\frac{\partial}{\partial x_j}a_j\left(x,\frac{\partial u}{\partial x_j}\right)+\beta(u)+\mathrm{div}\phi(u)\ \ni\ \mu & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega.\end{cases}$$

Let Ω be an open bounded domain of $\mathbb{R}^N$ $(N\geq 3)$ with a smooth boundary. Let β be a maximal monotone graph on $\mathbb{R}$ with $\overline{dom(\beta)}=[a,b]$, where $-\infty<a\leq 0\leq b<\infty$ and $0\in\beta(0)$. The measure μ is a bounded Radon diffuse measure, meaning that μ does not charge the sets of zero $p(.)-$capacity, and it has positive total variation. The convection-diffusion flux $\phi:\mathbb{R}\to\mathbb{R}^N$ is a continuous function satisfying a suitable growth condition.
The main feature of this contribution relies on the fact that we are solving the problem $(P^{\phi}_{\beta,\mu})$ with a measure data (which is strongly non-regular) on the right-hand side, along with a diffusion-convection term that is not necessarily zero.
There is a large literature on particular cases of this type of problems. Indeed, the case without diffusion-convection term (i.e. $\phi\equiv 0$) was analysed in most of the works and has concerned the following problem

$$(P^{0}_{\beta,\mu})\begin{cases} -\displaystyle\sum_{j=1}^{N}\frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j})+\beta(u)\ \ni\ \mu & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega.\end{cases} \tag{1.1}$$

When, $\beta\equiv 0$, the authors in [22, 26] employed minimization techniques to demonstrate the existence of weak solutions for the problem $(P^{0}_{\beta,\mu})$ (refer to [23, 29] for additional works).
In [20] (see also [16]), the first author and his collaborator applied approximation methods and the theory of monotonicity in Banach spaces to establish both the existence and uniqueness of the entropy solution for the problem $(P^{0}_{\beta,\mu})$ when β is a continuous non-decreasing function. They have also extended their analysis of the problem $(P^{0}_{\beta,\mu})$ to encompass the general nonlinear case, specially when β represents a maximal monotone graph defined over a bounded domain in $\mathbb{R}$ (see [21]).
Let us also emphasize that this problem has been widely investigated within the framework of isotropic $p(.)$-Leray-Lions type operators (see [17, 28, 30, 35, 36] and the references therein). Among them, one can mention the work [36], which has inspired our previous works [18, 19] on the topic of anisotropic problems involving diffusion-convection terms.
In detail, the present issue was recently addressed by the first and third authors (see [18]) when μ is an L^{∞}-function. They then advanced their analysis further

in [19] when μ is an L^1-function. In [18], by taking advantage of the regular characteristics of the right-hand side data, we were able to obtain L^∞-estimates on the sequence of approximations of the graph β, which allowed us to establish its weak convergence star in L^∞. To establish the existence of a renormalized solution when the right-hand side data μ belongs to L^1 (see [19]), we employed bi-monotone approximations of the data μ. This approach enabled us to derive strong compactness arguments in L^1, leading to the necessary convergence results. We emphasize that this technique cannot be applied when μ is treated as measure.

In this paper, we employ approximation methods and monotonicity in Banach spaces to establish the existence of a renormalised solution in three steps. First, we construct an approximating problem $(P^{\phi_r}_{\beta_r,\mu_r})$ using Yosida regularization and truncation techniques. Then, we utilize the approach of maximal monotone operators in Banach spaces to demonstrate the existence of a sequence of solutions $(u_r)_{r>0}$ for the problem $(P^{\phi_r}_{\beta_r,\mu_r})$. Finally, we pass to the limit as r approaches zero to show that the sequence $(u_r)_{r>0}$ converges to a solution of the initial problem. The main difficulties we encounter arise from the non-linearity of the graph β and the non-regularity of the measure μ. Since we only have an L^1-estimate for the sequence of approximations of the graph, we can only establish its weak convergence star in the set of bounded measures. Consequently, the definition of a renormalized or entropy solution must account for the measure arising from this convergence. For this reason, we adopt an adapted formulation of the renormalized or entropy solution, as in [21, 28, 30]. To overcome the obstacle posed by the non-regularity of the measure data, we apply the decomposition argument for diffuse measures from [28], namely, μ is an element of $L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.

Taking advantage of the fact that ϕ is null, the uniqueness of solutions was achieved in [21], through the comparison of two solutions. When the right-hand side is an L^1-function, the uniqueness of the renormalized solution was proven using the comparison principle. However, since the convection-diffusion term is merely continuous, its presence poses an obstacle to achieving uniqueness when the right-hand side is a measure. To address this difficulty, we investigate uniqueness under additional assumptions.

The structure of this paper is outlined as follows. In Section 2 we review some preliminary findings related to the anisotropic variable space. Section 3 introduces necessary assumptions on the data of the problem, and our main results. In Section 4, we demonstrate the existence of a solution via renormalized and/or entropy solutions. In Section 5, we establish the uniqueness of the solution under additional conditions on the convection term. Finally, in Section 6, we provide an example that illustrates our result.

2. **Preliminaries**

Here, we provide an overview of key definitions and essential properties of anisotropic Lebesgue and Sobolev spaces.

Define the set

$$C_+(\overline{\Omega}) = \left\{ z \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} z(x) > 1 \quad a.e.\ x \in \Omega \right\}.$$

For any $z \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined as

$$L^{z(.)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function such that } \int_\Omega |u|^{z(x)}dx < \infty \right\}$$

and it is equipped with the Luxembourg norm given by

$$|u|_{z(.)} := \inf \left\{ \beta > 0 : \int_\Omega \left| \frac{u(x)}{\beta} \right|^{z(x)} dx \leq 1 \right\}.$$

The $z(.)$-modular of the space $L^{z(.)}(\Omega)$ is defined by the mapping $\rho_{z(.)} : L^{z(.)}(\Omega) \longrightarrow \mathbb{R}$, where $\rho_{z(.)}(u) = \displaystyle\int_\Omega |u|^{z(x)}dx$.

$\forall u \in L^{z(.)}(\Omega)$, one has (see [13], [14])

$$\min \left\{ |u|^{z^-}_{z(.)};\ |u|^{z^+}_{z(.)} \right\} \leq \rho_{z(.)}(u) \leq \max \left\{ |u|^{z^-}_{z(.)};\ |u|^{z^+}_{z(.)} \right\}. \tag{2.1}$$

For any $u \in L^{z_1(.)}(\Omega)$ and $v \in L^{z_2(.)}(\Omega)$, with $\frac{1}{z_1(x)} + \frac{1}{z_2(x)} = 1$ in Ω, one has the Hölder type inequality

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{z_1^-} + \frac{1}{z_2^-} \right) |u|_{z_1(.)} |v|_{z_2(.)}. \tag{2.2}$$

If Ω is a bounded domain and $z_1,\ z_2 \in C_+(\overline{\Omega})$ with the condition that $z_1(x) \leq z_2(x)$ for all $x \in \Omega$, then the embedding $L^{z_1(.)}(\Omega) \hookrightarrow L^{z_2(.)}(\Omega)$ is continuous (see [24]).

The anisotropic Sobolev space is defined as

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p_j(.)}(\Omega),\ j = 1, ..., N \right\}.$$

Equipped with the norm $||u||_{\overrightarrow{p}(.)} = \displaystyle\sum_{j=1}^N \left| \frac{\partial u}{\partial x_j} \right|_{p_j(.)}$, this space is a separable and reflexive Banach space (see [26]).

We define the following numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1};\ q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q}.$$

Let $Z_-^*,\ Z_-^+$ and $Z_{-,\infty}$ be positive constants defined as follows

$$Z_-^* = \frac{N}{\displaystyle\sum_{j=1}^N \frac{1}{p_j^-} - 1},\ Z_-^+ = \max\left\{p_1^-, ..., p_N^-\right\} \text{ and } Z_{-,\infty} = \max\left\{Z_-^+,\ Z_-^*\right\}.$$

Theorem 2.1. [28] *Let $p(.) : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function, and let $\mu \in \mathcal{M}_b(\Omega)$. Then, μ belongs to the space $\mathcal{M}_b^{p(.)}(\Omega)$ if and only if it can be represented as an element of $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$. If $u \in W_0^{1,p(.)}(\Omega)$, then*

$$\int_\Omega div(u)dx = 0.$$

We represent the N-dimensional Lebesgue measure on $\mathbb{R}^N$ as $\mathcal{L}^N$ and denote the space of bounded Radon measures in Ω by $\mathcal{M}_b(\Omega)$, which is equipped with the standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$.
Notice that if μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ is its total variation on Ω.
Let μ be an element of $\mathcal{M}_b(\Omega)$. We say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ (referred to as the $p(.)$-capacity) if $\mu(B) = 0$ for every set B such that $Cap_{p(.)}(B, \Omega) = 0$.
For every $B \subset \Omega$, we denote

$$S_{p(.)}(B) := \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } B, u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset B with respect to Ω is defined by

$$Cap_{p(.)}(B, \Omega) := \inf_{u \in S_{q(.)}(B)} \left\{ \int_\Omega |\nabla u|^{p(x)} dx \right\}.$$

When $S_{q(.)}(B) = \emptyset$, we assign $Cap_{q(.)}(B, \Omega) = \infty$.
The set of bounded Radon diffuse measures that are diffuse with respect to the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$.
For any $l_0 > 0$, we consider a function h_0 such that:

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(t) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(t) = 1$ if $|r| \leq l_0$ and $h_0(t) = 0$ if $|t| \geq l_0 + 1$.

Lemma 2.3. [27] *Let j be a lower semi-continuous function on $\mathbb{R}$ with $\overline{dom(j)} = [a, b] \subset \mathbb{R}$, and let j_r be a sequence of lower semi-continuous functions satisfying the conditions*

$$j_r(t) \geq 0, \forall t \in [a, b], \text{ and } j_r \uparrow j \text{ as } r \downarrow 0.$$

Let $(v_r)_{r>0}$ and $(z_r)_{r>0}$ be two sequences of measurable functions on Ω satisfying

$$\begin{cases} v_r \longrightarrow v\mathcal{L}^N \ , \ v \in dom(j)\mathcal{L}^N a.e. \text{ in } \Omega \\ \\ \forall r > 0, \ z_r \in \partial j_r(v_r)\mathcal{L}^N \ a.e \text{ in } \Omega. \end{cases}$$

Let ζ be in $\mathcal{M}_b^{p(.)}(\Omega) \cap [(W^{1,p(.)}(\Omega))^* + L^1]$ such that for all $\varphi \in C_C^1(\Omega)$, $\varphi \geq 0$,

$$\liminf_{r \to 0} \int_\Omega (t - v_\epsilon)\varphi h_0(v_\epsilon) z_\epsilon dx \geq \int_\Omega (t - v)\varphi d\zeta, \ \forall t \in \mathbb{R}. \tag{2.3}$$

Then,

$$\begin{cases} \zeta = w\mathcal{L}^N + \zeta_s \text{ with } \nu \perp \mathcal{L}^N, \ w \in \partial j(v)\mathcal{L}^N \ a.e \text{ in } \Omega, \ w \in L^1(\Omega) \\ \\ \zeta_s^+ \text{ is concentrated on } \ [u = b], \ \zeta_s^- \text{ is concentrated on } \ [u = a]. \end{cases}$$

Let Γ be a maximal monotone operator on $\mathbb{R}$. We defined by Γ_0 the principal section of Γ. Namely

$$\Gamma_0(h) = \begin{cases} \text{minimal absolute value of } \ \Gamma(h) & \text{if } \Gamma(h) \ \neq \emptyset \\ +\infty & \text{if } [h, +\infty) \cap D(\Gamma) = \emptyset \\ -\infty & \text{if } (-\infty, h] \cap D(\Gamma) = \emptyset. \end{cases}$$

We also need the following useful convergence result (see [28]).

Lemma 2.4. *Let $(B_m)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $B_n \to B$ in the sense of graphs (for $(r,s) \in B$, there exists $(r_n, s_n) \in B_n$ such that $r_n \to r$ and $s_n \to s$). Let $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ be two sequences of $L^1(\Omega)$. We assume that $\forall n \geq 1, w_n \in B_n(\zeta)$, $(w_n)_{n\geq 1}$ are bounded in $L^1(\Omega)$ and $\zeta_n \to \zeta$ in $L^1(\Omega)$. Then,*

$$\zeta \in dom(B).$$

The following embedding results hold (refer to [26], Theorem 1).

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a smooth boundary. Furthermore, suppose that the relation (3.6) is satisfied. For every $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < Z_{-,\infty} \text{ for any } x \in \overline{\Omega},$$

the embedding $W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Theorem 2.6. [34] *Let $p_1, ..., p_N \in [1, +\infty)$, and $f \in W^{1,(p_1,...,p_N)}(\Omega)$. Define q as*

$$\begin{cases} q = (\overline{p})^* & \text{if } (\overline{p})^* < N, \\ q \in [1, +\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ that depends on $N, p_1, ..., p_N$ if $\overline{p} < N$, and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||f||_{L^q(\Omega)} \leq C_4 \prod_{j=1}^{N} \left\| \frac{\partial f}{\partial x_j} \right\|_{L^{p_j}(\Omega)}^{\frac{1}{N}}. \tag{2.4}$$

We now recall some useful properties on the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$, which is defined as the set of measurable functions $f : \Omega \longrightarrow \mathbb{R}$.
By denoted

$$\alpha_f(t) = meas(\{x \in \Omega : |f(x)| > t\}), \ t \geq 0, \tag{2.5}$$

one has

$$\alpha_f(t) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{2.6}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$.

$$||f||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \forall k > 0\}. \tag{2.7}$$

Given $h > 0$, we denote with $T_h(s)$ the truncation function at level h, defined by

$$T_h(s) = \max\{-h, \min\{h; s\}\}. \tag{2.8}$$

It is obvious that $\lim\limits_{h\to\infty} T_h(s) = s$ and $|T_h(s)| = \min\{|s|; h\}$.

We define $\mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that,

for every $k \in \mathbb{N}$, the truncation function $T_k(u)$ belongs to $W_0^{1,\overrightarrow{p}(.)}(\Omega)$.
Let u be a measurable function on Ω and $l \in \mathbb{R}$. The sets $[u = l]$, $[u \leq l]$ and $[u \geq l]$ denote $\{x \in \Omega : u(x) = l\}$, $\{x \in \Omega : u(x) \leq l\}$, $\{x \in \Omega : u(x) \geq l\}$ respectively.

3. Assumptions and main results

3.1. Assumptions.

The data and exponents are supposed to satisfy the following assumptions.
Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$. Let us consider the vector-valued function $\overrightarrow{p}(.) = (p_1(.), ..., p_N(.))$ where for each $j = 1, ..., N$, the component $p_j(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function. Additionally, these functions are required to satisfy the following properties

$$(H_1) \qquad 1 < p_j^- := \inf_{x\in\Omega} p_j(x) \leq p_j^+ := \sup_{x\in\Omega} p_j(x) < \infty. \tag{3.1}$$

We represent by

$$q(x) := \max\{p_1(x), ..., p_N(x)\} \quad \text{and} \quad p(x) := \min\{p_1(x), ..., p_N(x)\}.$$

For each $j = 1, ..., N$, the function $a_j : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function that satisfies the following conditions:
(H_2) a positive constant C_1 exists such that

$$|a_j(x, \varphi)| \leq C_1 \bigg(k_j(x) + |\varphi|^{p_j(x)-1} \bigg), \tag{3.2}$$

for a.e. $x \in \Omega$ and for every $\varphi \in \mathbb{R}$, where k_j is a non-negative function belonging to $L^{p'_j(.)}(\Omega)$, satisfying the relation $\frac{1}{p_j(x)} + \frac{1}{p'_j(x)} = 1$;
(H_3) $\forall\varphi,\ \varphi' \ \in \mathbb{R}$, $\varphi \neq \varphi'$ and for a.e. $x \in \Omega$,

$$(a_j(x, \varphi) - a_j(x, \varphi'))(\varphi - \varphi') \geq \begin{cases} C_2|\varphi - \varphi'|^{p_j(x)} & \text{if } \ |\varphi - \varphi'| \geq 1, \\ C_2|\varphi - \eta|^{p_j^-} & \text{if } \ |\varphi - \varphi'| < 1 \end{cases} \tag{3.3}$$

where C_2 is a positive constant,
(H_4) there exists a positive constant C_3 such that

$$a_j(x, \varphi).\varphi \geq C_3|\varphi|^{p_j(x)}, \tag{3.4}$$

$\forall\varphi \in \mathbb{R}$ and a.e. $x \in \Omega$.
We assume throughout the paper that

$$(H_5) \qquad \frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_j^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_j^+ - p_j^- - 1}{p_j^-} < \frac{\overline{p}-N}{\overline{p}(N-1)} \tag{3.5}$$

and

$$(H_6) \qquad \sum_{j=1}^{N} \frac{1}{p_j^-} > 1, \tag{3.6}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{j=1}^{N} \frac{1}{p_j^-}$.

(H_7) $$\operatorname{int}(\operatorname{dom}(\beta)) = (a, b) \text{ with } -\infty < a \leq 0 \leq b < +\infty.$$

(H_8) For any $s \in \mathbb{R}$, $|\phi(s)| \leq C|s|^{p(x)-1}$.

3.2. **Concepts of solution and main results.**

Let us first clarify what we mean by a renormalized or entropy solution.

Definition 3.1. A pair of functions $(u, w) \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \times L^1(\Omega)$ is said to be a renormalized solution of problem $(P^{\phi}_{\beta,\mu})$ if the following conditions are satisfied:

• there exists $\nu \in \mathcal{M}_b^{p(.)}(\Omega)$ verifying $\nu \perp \mathcal{L}^N$ and

$$\begin{cases} u \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \ w \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \\ \nu^+ \text{ is concentrated on } \ [u = b], \\ \nu^- \text{ is concentrated on } \ [u = a]. \end{cases} \tag{3.7}$$

• For all $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{j=1}^{N} \int_\Omega a_j\Big(x, \frac{\partial u}{\partial x_j}\Big) \frac{\partial}{\partial x_j}[h(u)\varphi]dx + \int_\Omega w h(u)\varphi dx + \int_\Omega h(u)\varphi d\nu$$
$$- \int_\Omega \phi(u).\nabla[h(u)\varphi]dx = \int_\Omega h(u)\varphi d\mu. \tag{3.8}$$

Moreover,

$$\lim_{n\to+\infty} \sum_{j=1}^{N} \int_{\{n<|u|<n+1\}} \left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)} dx = 0. \tag{3.9}$$

Definition 3.2. An entropy solution of the problem $(P^{\phi}_{\beta,\mu})$ is a pair of function $(u, w) \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \times L^1(\Omega)$ that satisfies the following conditions

$$\sum_{j=1}^{N} \int_\Omega a_j\Big(x, \frac{\partial u}{\partial x_j}\Big) \frac{\partial}{\partial x_j} T_t(u - \varphi)dx + \int_\Omega w T_t(u - \varphi)dx - \int_\Omega \phi(u).\nabla T_t(u - \varphi)dx$$
$$\leq \int_\Omega T_t(u - \varphi)d\mu, \tag{3.10}$$

where $t > 0$ and $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, with $\varphi \in dom\beta$.

Theorem 3.3. *Let us assume that $(H_1) - (H_7)$ hold true. Then, the problem $(P^{\phi}_{\beta,\mu})$ admits at least one renormalized solution.*

Theorem 3.4. *Let $(H_1) - (H_7)$ hold and ϕ be a Lipschitz function. Then a solution u of problem $(P^{\phi}_{\beta,\mu})$ verifies*

$$\nu^+ \leq \mu_s \lfloor \ [u = b], \tag{3.11}$$

$$\nu^- \leq -\mu_s \lfloor\ [u = a]. \tag{3.12}$$

The relationship between our concept of a renormalized solution and the entropic formulation of the solution (refer to [4]) is described in the following theorem.

Theorem 3.5. *A renormalized solution of problem* $(P^{\phi}_{\beta,\mu})$ *is likewise an entropy solution.*

Proof. Let (u, w) be a renormalized solution of $(P^{\phi}_{\beta,\mu})$, and $\varphi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \in dom(\beta)$.
For all $t > 0$, using $T_t(u - \varphi)$ as test function in (3.8) one obtains

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_j} T_t(u-\varphi)dx - \int_\Omega \phi(u).\nabla T_t(u-\varphi)dx + \int_\Omega w T_t(u-\varphi)dx$$
$$+ \int_\Omega T_t(u-\varphi)d\nu = \int_\Omega T_t(u-\varphi)d\mu. \tag{3.13}$$

Neglecting the positive term $\int_\Omega T_t(u-\varphi)d\nu$ (see [21]), we obtain (3.10). □

4. Proof of Theorem 3.3

This section focuses on providing the proofs of our main results. We emphasize that this work is an extension of our previous works [18, 19, 20, 21]. As a result, some proofs are presented without full details to avoid unnecessary repetition of arguments.

4.1. **The approximate problem.**

For each $r > 0$, we define the Yosida regularization β_r of β (see [9]), which is expressed as follows.

$$\beta_r = \frac{1}{r}(I - (I + r\beta)^{-1}).$$

It should be noted that β_r is a non-decreasing and Lipschitz-continuous function. Since μ belongs to $\mathcal{M}_b^{p(.)}(\Omega)$, it can be expressed as $\mu = f - \operatorname{div}(F)$, where $f \in L^1(\Omega)$ and $F \in (L^{p'(.)}(\Omega))^N$ (see Theorem 2.1).
To carry out the regularization of μ, we introduce the function as follows.

$$\forall r > 0,\ f_r(x) = T_{\frac{1}{r}}(f(x)) \ \ a.e.\ x \in \Omega.$$

Then, setting

$$\mu_r = f_r - \nabla.\ F \text{ for any } r > 0,$$

we have $\mu_r \in \mathcal{M}_b^{p(.)}(\Omega)$, $\mu_r \rightharpoonup \mu$ and $\mu_r \in L^\infty(\Omega)$.
We consider the approximating problems

$$(P^{\phi_r}_{\beta_r,\mu_r}) \begin{cases} \beta_r(T_{\frac{1}{r}}(u_r)) - \displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) + \operatorname{div}\phi(T_{\frac{1}{r}}(u_r)) = \mu_r \text{ in } \ \Omega, \\ u_r = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \ \partial\Omega. \end{cases} \tag{4.1}$$

Since $\mu_r \in L^\infty(\Omega)$, we can apply the Theorem 1.1 from [18] to obtain the following result.

Theorem 4.1. *The problem* $(P^{\phi_r}_{\beta_r,\mu_r})$ *has at least one weak solution* u_r *satisfying the conditions* $u_r \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$, $\beta_r(u_r) \in L^1(\Omega)$ *and*

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial v}{\partial x_j}dx + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))vdx$$

$$-\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla v dx = \int_\Omega v d\mu_r, \tag{4.2}$$

where $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

4.2. **A priori estimates.**

We will derive a priori estimates for the sequence of solutions $(u_r)_{r>0}$, which will be useful to prove the convergence results as r approaches zero.

Proposition 4.2. *Let* $k > 0$, *and suppose that* u_r *is a weak solution to problem* $(P^{\phi_r}_{\beta_r,\mu_r})$. *Then*

(i)

$$\sum_{j=1}^{N} \int_{\{|u_r|\le k\}} \left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)} dx \le \frac{k|\mu|(\Omega)}{C_3}. \tag{4.3}$$

(ii) *Both sequences,* $(\beta_r(T_{\frac{1}{r}}(u_r)))_{r>0}$ *and* $(\beta_r(T_{\frac{1}{r}}(T_k(u_r))))_{r>0}$, *are uniformly bounded in* $L^1(\Omega)$.

Proof. By using $v = T_k(u_r)$ as a test function in (4.2), one obtains

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial T_k(u_r)}{\partial x_j}dx + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))T_k(u_r)dx - \int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla T_k(u_r)dx$$

$$= \int_\Omega f_r T_k(u_r)dx + \int_\Omega F\cdot\nabla T_k(u_r)dx. \tag{4.4}$$

Using the divergence theorem of Gauss (see Lemma 2.2), one obtains

$$\int_\Omega \phi(T_{\frac{1}{r}}(u_r))\nabla T_k(u_r)dx = \int_\Omega \phi(T_{\frac{1}{r}}(T_k(u_r)).\nabla T_k(u_r)dx$$

$$= \int_\Omega \nabla\left(\int_0^{T_k(u_r)} \phi\circ T_{\frac{1}{r}}(s)ds\right)dx = 0. \tag{4.5}$$

The rest of the proof follows from [21]. □

Remark 4.3. [18](Lemma 3.1 and Remark 3.2)

$$\int_{\{|u_r|\le k\}} |\nabla T_k(u_r)|^{p^-}dx \le C_5. \tag{4.6}$$

$$\lim_{l\to+\infty} |\{|u_r| \ge l\}| = 0. \tag{4.7}$$

Proposition 4.4. [20, 21, 28, 30] *Given any* $k > 0$ *that is sufficiently large, and assuming that* u_r *is a weak solution of* $(P^{\phi_r}_{\beta_r,\mu_r})$, *we obtain*

$$meas\{|u_r| > k\} \le \frac{C(\mu,\Omega)}{\min\{\beta_r(k), |\beta_r(-k)|\}}, \tag{4.8}$$

$$meas\left\{\left|\frac{\partial u_r}{\partial x_j}\right| > k\right\} \leq \frac{C(\mu,\Omega)}{k^{\frac{1}{(p_M)'}}} \tag{4.9}$$

and

$$meas\{|\nabla u_r| > k\} \leq \frac{C_6(k+1)}{k^{p^-}} + \frac{C(\mu,\Omega)}{\min\{\beta_r(k), |\beta_r(-k)|},} \tag{4.10}$$

where C_6 is a positive constant.

Lemma 4.5. [8, 16, 22] *Let u_r be a solution of the problem $(P^{\phi_r}_{\beta_r,\mu_r})$, then one obtains*

(i) $||u_r||_{\mathcal{M}^{q^*}(\Omega)} \leq C_7$,

(ii) $\|\frac{\partial u_r}{\partial x_j}\|_{\mathcal{M}^{p_j^- \frac{q}{p}}(\Omega)} \leq C_8, \quad \forall j = 1, ..., N.$

where $C_7 > 0$ and $C_8 >$ are constants.

4.3. **Convergence results.**

Proposition 4.6. *Let u_r be a solution of the problem $(P^{\phi_r}_{\beta_r,\mu_r})$. Then, there exists a function $u \in \mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega)$ that satisfies the following propertie.*

$$u_r \longrightarrow u \textit{ in measure and a.e. in } \Omega \textit{ as } r \longrightarrow 0. \tag{4.11}$$

Remark 4.7. By applying Lemma 2.4 exactly as in [21], we obtain $u \in dom(\beta)$ a.e. in Ω.

Lemma 4.8. [8, 16, 22] *Let u_r be a solution of the problem $(P^{\phi_r}_{\beta_r,\mu_r})$. Then, for $j = 1, ..., N$, as $r \to 0$, one has*

$$a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \longrightarrow a_j\left(x, \frac{\partial u}{\partial x_j}\right) \textit{ in } L^1(\Omega) \textit{ and a.e. } x \in \Omega. \tag{4.12}$$

Proposition 4.9. [8, 16, 22] *Let us assume that $(H_1) - (H_7)$ hold, and suppose that u_r is a weak solution of $(P^{\phi_r}_{\beta_r,\mu_r})$. Then, for $j = 1, ..., N$, and $k > 0$, one has*

(i) $\frac{\partial u_r}{\partial x_j}$ *converges in measure to the weak partial gradient of u,*

(ii) $a_j(x, \frac{\partial}{\partial x_j}T_k(u_r))$ *converges to $a_j(x, \frac{\partial}{\partial x_j}T_k(u))$ in $L^1(\Omega)$ strongly and in $L^{p_j'(.)}(\Omega)$ weakly,*

(iii) $a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial u_r}{\partial x_j} \longrightarrow a_j(x, \dfrac{\partial u}{\partial x_j})\dfrac{\partial u}{\partial x_j}$ *in $L^1(\Omega)$ and a.e. $x \in \Omega$.*

Lemma 4.10. [21] *Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, and suppose u_r a weak solution to problem $(P^{\phi_r}_{\beta_r,\mu_r})$. Then, for any $j = 1, ..., N$, one has:*

$$\frac{\partial}{\partial x_j}(h(u_r)\varphi) \longrightarrow \frac{\partial}{\partial x_j}(h(u)\varphi) \textit{ strongly in } L^1(\Omega) \textit{ as } r \to 0.$$

Lemma 4.11. [30] *Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. If u_r is a weak solution to problem $(P^{\phi_r}_{\beta_r,\mu_r})$, then*

$$\nabla[h(u_r)\varphi] \longrightarrow \nabla[h(u)\varphi] \textit{ strongly in } L^{p(.)}(\Omega) \textit{ as } r \to 0.$$

Lemma 4.12. [21] *For any* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, *one has*

$$\lim_{r\to 0}\int_\Omega h(u_r)\varphi d\mu_\epsilon = \int_\Omega h(u)\varphi d\mu. \tag{4.13}$$

$$\lim_{r\to 0}\int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial}{\partial x_j}[h(u_r)\varphi]dx = \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u}{\partial x_j}\right)\frac{\partial}{\partial x_j}[h(u)\varphi]dx \tag{4.14}$$

and

$$\lim_{r\to 0}\int_\Omega f_r h(u_r)\varphi dx = \int_\Omega f h(u)\varphi dx. \tag{4.15}$$

Lemma 4.13. *For every* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, *one has*

$$\lim_{r\to 0}\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla[h(u_r)\varphi]dx = \int_\Omega \phi(u).\nabla[h(u)\varphi]dx. \tag{4.16}$$

Proof. Let us suppose that A is a measurable subset of Ω. We aim to prove the equi-integrability of the function $\phi(T_{\frac{1}{r}}(u_r)).[h'(u_r)\varphi\nabla u_r + h(u_r)\nabla\varphi]$.
As h and h' are compactly supported in $\mathbb{R}$, the following holds for sufficiently small r.

$$\forall s \in \mathbb{R},\ \phi(T_{\frac{1}{r}}(s))h(s) = \phi(s)h(s) \text{ and } \phi(T_{\frac{1}{r}}(s))h(s) = \phi(s)h'(s).$$

$$\begin{aligned}
&\int_A \phi(T_{\frac{1}{r}}(u_r)).[h'(u_r)\varphi\nabla u_r + h(u_r)\nabla\varphi]dx\\
&= \int_A \phi(T_{\frac{1}{r}}(u_r))h'(u_r)\varphi\nabla u_r)dx + \int_A \phi(T_{\frac{1}{r}}(u_r))h(u_r)\nabla\varphi dx\\
&\le \|\phi h'\|_\infty\|\varphi\|_\infty \int_A |\nabla u_r|dx + \|\phi h\|_\infty \int_A |\nabla\varphi|dx\\
&\le \|\phi h'\|_\infty\|\varphi\|_\infty\left(\int_A |1|^{(p^-)'}dx\right)^{\frac{1}{(p^-)'}}\left(\int_A |\nabla u_r|^{(p^-)}dx\right)^{\frac{1}{(p^-)}}\\
&+ \|\phi h\|_\infty\left(\int_A |1|^{(p^-)'}dx\right)^{\frac{1}{(p^-)'}}\left(\int_A |\nabla\varphi|^{(p^-)}dx\right)^{\frac{1}{(p^-)}}\\
&\le (\text{meas}(A))^{\frac{1}{(p^-)'}}\left[\|\phi h'\|_\infty\|\varphi\|_\infty\left(\int_A |\nabla u_r|^{(p^-)}dx\right)^{\frac{1}{(p^-)}}\right.\\
&\left.+ \|\phi h\|_\infty\left(\int_A |\nabla\varphi|^{(p^-)}\right)^{\frac{1}{(p^-)}}dx\right]\\
&\le (\text{meas}(A))^{\frac{1}{(p^-)'}}\Big(\|\phi h'\|_\infty\|\varphi\|_\infty\|u_r\|_{1,p^-} + \|\phi h\|_\infty\|\varphi\|_{1,p^-}\Big).
\end{aligned}$$

In view of the following

$$\phi(T_{\frac{1}{r}}(u_r)).[h'(u_r)\varphi\nabla u_r) + h(u_r)\nabla\varphi \to \phi(u).[h'(u)\varphi\nabla u) + h(u)\nabla\varphi] \text{ a.e. in } \Omega \text{ as } r \to 0,$$

and adding the fact that u_r and φ belong to $W^{1,p^-}(\Omega)$, we can conclude that $\phi(T_{\frac{1}{r}}(u_r)).[h'(u_r)\varphi\nabla u_r) + h(u_r)\nabla\varphi]$ is equi-integrable.

Thus, we have

$$\begin{aligned}\lim_{r\to 0}\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla[h(u_r)\varphi]dx &= \lim_{r\to 0}\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).[h'(u_r)\varphi\nabla u_r)+h(u_r)\nabla\varphi]dx\\ &= \int_\Omega \phi(u).\nabla[h(u)\varphi]dx. \qquad (4.17)\end{aligned}$$

□

Let us focus on the sequence $(\beta_r(T_{\frac{1}{r}}(u_r)))_{r>0}$ in order to determine the conditions required to be able to pass to the limit as r tends to 0.

Lemma 4.14. *Assume $h_0 \in C_c^1(\mathbb{R})$. Let $k_0 > 0$ be chosen such that $\overline{dom(\beta(u))} = [a,b] \subset [-k_0,+k_0]$. Then, there exists $z \in \mathcal{M}_b^{p(.)}(\Omega)\cap[(W^{1,p(.)}(\Omega))^*+L^1]$ satisfying*

$$h_0(u_r)\beta_r(T_{\frac{1}{r}}(u_r)) \overset{*}{\rightharpoonup} z \ \text{as}\ r\to 0.$$

Moreover,

$$z = -\sum_{j=1}^{N}\frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j}) + div\ \phi(u)\ in\ \mathcal{D}'(\Omega)$$

and

$$z = w\mathcal{L}^N + \nu,$$

where w and ν satisfy the following properties

$$\begin{cases}\nu\in\mathcal{M}_b^{p(.)}(\Omega)\ with\ \nu\perp\mathcal{L}^N,\ w\in\beta(u)\mathcal{L}^N - a.e.\ in\ \Omega,\ w\in L^1(\Omega),\\ \nu^+\ is\ concentrated\ on\ \ [u=b],\\ \nu^-\ is\ concentrated\ on\ \ [u=a].\end{cases} \qquad (4.18)$$

Proof. Let $h_0\in C_c^1(\mathbb{R})$ and $k_0>0$ be such that $\overline{dom(\beta(u))} = [a,b]\subset[-k_0,+k_0]$. Notice that the sequence $(h_0(u_r)\beta_r(T_{\frac{1}{r}}(u_r)))_{r>0}$ is bounded in $L^1(\Omega)$.
Thus, there exists $z\in\mathcal{M}_b(\Omega)$ such that $h_0(u_r)\beta_r(T_{\frac{1}{r}}(u_r)) \overset{*}{\rightharpoonup} z$ in $\mathcal{M}_b(\Omega)$ as $r\to 0$.
Keeping in mind the results of convergence mentioned above, for all $\varphi\in W_0^{1,\vec{p}(.)}(\Omega)\cap$

$L^\infty(\Omega)$, one has

$$\begin{aligned}\int_\Omega \varphi dz &= \int_\Omega h_0(u)\varphi dz = \lim_{r\to 0}\int_\Omega h_0(u_r)\beta_r(u_r)\varphi dx \\ &= -\lim_{r\to 0}\int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial u_r}{\partial x_j})\frac{\partial}{\partial x_j}[h_0(u_r)\varphi]dx + \lim_{r\to 0}\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla[h_0(u_r)\varphi]dx \\ &\qquad + \lim_{r\to 0}\int_\Omega h(u_r)\varphi d\mu_r \\ &= -\int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial u}{\partial x_j})\frac{\partial}{\partial x_j}[h_0(u)\varphi]dx + \int_\Omega \phi(u).\nabla[h_0(u)\varphi]dx + \int_\Omega h_0(u)\varphi d\mu. \\ &= -\int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial u}{\partial x_j})\frac{\partial}{\partial x_j}\varphi dx + \int_\Omega \phi(u).\nabla\varphi dx + \int_\Omega \varphi d\mu,\end{aligned}$$

which implies that $z = -\sum_{j=1}^N \frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j}) + \mathrm{div}\ \phi(u) + \mu\ in\ \mathcal{D}'(\Omega)$ and $z \in \mathcal{M}_b^{p(.)}(\Omega)\cap[(W^{1,p(.)}(\Omega))^* + L^1]$.
For the rest of the proof, it suffices to establish the relation (2.3) of Lemma 2.3 by setting $v_r = u_r$ and $z_r = \beta_r(T_{\frac{1}{r}}(u_r))$.
Therefore, for any $\varphi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, one has

$$\begin{aligned}&\lim_{r\to 0}\int_\Omega (t-u_r)h_0(u_r)\beta_r(u_r)\varphi dx \\ &= \lim_{r\to 0}\int_\Omega (t-u_r)h_0(u_r)\varphi d\mu_r - \lim_{r\to 0}\int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial T_{k_0}(u_r)}{\partial x_j})\frac{\partial}{\partial x_j}[(t-u_r)h_0(u_r)\varphi]dx \\ &+ \lim_{r\to 0}\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla[(t-u_r)h_0(u_r)\varphi]dx \\ &= \int_\Omega (t-u)h_0(u)\varphi d\mu - \int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial u}{\partial x_j})\frac{\partial}{\partial x_j}[(t-u)h_0(u)\varphi]dx \\ &+ \int_\Omega \phi(u).\nabla[(t-u)h_0(u)\varphi]dx \\ &= \int_\Omega (t-u)\varphi d\mu - \int_\Omega \sum_{j=1}^N a_j(x,\frac{\partial u}{\partial x_j})\frac{\partial}{\partial x_j}[(t-u)\varphi]dx + \int_\Omega \phi(u).\nabla[(t-u)\varphi]dx \\ &= \int_\Omega (t-u)\varphi dz.\end{aligned}$$

By combining Lemma 2.3 with the equality above, one can complete the proof.
□

Let us now prove equality (3.8) from Theorem 3.3. To this end, we substitute $h(u_r)\varphi$ as a test function in (4.2) to obtain

$$\sum_{j=1}^{N}\int_{\Omega}a_j\Big(x,\frac{\partial u_r}{\partial x_j}\Big)\frac{\partial}{\partial x_j}[h(u_r)\varphi]dx-\int_{\Omega}\phi(T_{\frac{1}{r}}(u_r)).\nabla[h(u_r)\varphi]dx+\int_{\Omega}\beta_r(u_r)h(u_r)\varphi dx$$
$$=\int_{\Omega}f_rh(u_r)\varphi dx+\int_{\Omega}F_r\cdot\nabla[h(u_r)\varphi]dx, \tag{4.19}$$

for all $\varphi\in W_0^{1,\overrightarrow{p}(.)}(\Omega)\cap L^{\infty}(\Omega)$ and $h\in C_c^1(\mathbb{R})$.
Passing to the limit as $r\to 0$, one obtains

$$\sum_{j=1}^{N}\int_{\Omega}a_j\Big(x,\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_j}[h(u)\varphi]dx-\int_{\Omega}\phi(u).\nabla[h(u)\varphi]dx+\lim_{r\to 0}\int_{\Omega}\beta_r(u_r)h(u_r)\varphi dx$$
$$=\int_{\Omega}h(u)\varphi d\mu. \tag{4.20}$$

According to Lemma 4.14, one has

$$\begin{aligned}\lim_{r\to 0}\int_{\Omega}\beta_r(u_r)h(u_r)\varphi dx&=\int_{\Omega}h(u)\varphi d\mu-\sum_{j=1}^{N}\int_{\Omega}a_j\Big(x,\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_j}[h(u)\varphi]dx\\&\quad+\int_{\Omega}\phi(u).\nabla[h(u)\varphi]dx\\&=\int_{\Omega}h(u)\varphi dz\\&=\int_{\Omega}h(u)\varphi wdx+\int_{\Omega}h(u)\varphi d\nu.\end{aligned}$$

Hence, we conclude that

$$\sum_{j=1}^{N}\int_{\Omega}a_j\Big(x,\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_j}[h(u)\varphi]dx-\int_{\Omega}\phi(u).\nabla[h(u)\varphi]dx$$
$$+\int_{\Omega}wh(u)\varphi dx+\int_{\Omega}h(u)\varphi d\nu=\int_{\Omega}h(u)\varphi d\mu. \tag{4.21}$$

Remark 4.15. Given that the domain of β is bounded, there is no need for renormalization with the function h in Theorem 3.3.
Indeed, given that (4.21) is true for each $h\in C_c^1(\mathbb{R})$, we can set $h=h_{l_0}$ where $[a,b]\subset[-l_0,+l_0]$ such that

$$\sum_{j=1}^{N}\int_{\Omega}a_j\Big(x,\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_j}\varphi dx-\int_{\Omega}\phi(u).\nabla\varphi dx$$
$$+\int_{\Omega}wh\varphi dx+\int_{\Omega}h(u)\varphi d\nu=\int_{\Omega}\varphi d\mu. \tag{4.22}$$

Therefore, we can replace (4.21) with (4.22) in Theorem 3.3.

We conclude the proof of Theorem 3.3 by verifying (3.9).
Indeed, choosing $T_1(u_r - T_n(u_r))$ as test function in (4.2), yields

$$\sum_{j=1}^{N}\int_\Omega a_j\Big(x,\frac{\partial u_r}{\partial x_j}\Big)\frac{\partial T_1(u_r-T_n(u_r))}{\partial x_j}dx+\int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))T_1(u_r-T_n(u_r))dx$$

$$-\int_\Omega \phi(T_{\frac{1}{r}}(u_r)).\nabla T_1(u_r-T_n(u_r))dx=\int_\Omega T_1(u_r-T_n(u_r))d\mu_r. \tag{4.23}$$

By setting $D_n=\{n<|u_r|<n+1\}$ and acknowledging that

$$\int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))T_1(u_r-T_n(u_r))dx\geq 0$$

and

$$\frac{\partial}{\partial x_j}T_1(u_r-T_k(u_r))=\frac{\partial u_r}{\partial x_j}\chi_{D_n},\;\; \nabla T_1(u_r-T_k(u_r))=\nabla u_r\chi_{D_n},$$

the equality (4.23) can be expressed as

$$\sum_{j=1}^{N}\int_{D_n} a_j\Big(x,\frac{\partial u_r}{\partial x_j}\Big)\frac{\partial u_r}{\partial x_j}dx+\int_{D_n}\phi(T_{\frac{1}{r}}(u_r)).\nabla u_r dx$$

$$\leq \int_\Omega f_r T_1(u_r-T_n(u_r))dx+\int_{D_n}F_r\cdot\nabla u_r dx.$$

Utilizing (3.4), we can infer that

$$C_3\sum_{j=1}^{N}\int_{D_n}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)}dx+\int_{D_n}\phi(T_{\frac{1}{r}}(u_r)).\nabla u_r dx$$

$$\leq \int_\Omega f_r T_1(u_r-T_n(u_r))dx+\int_{D_n}F_r\cdot\nabla u_r dx. \tag{4.24}$$

Let us establish that

$$\lim_{n\to+\infty}\lim_{r\to 0}\int_{D_n}F_r\cdot\nabla u_r dx=0. \tag{4.25}$$

Applying Hölder inequality, it follows that

$$\int_{D_n}F_r\cdot\nabla u_r dx\leq\Big(\int_{D_n}|F_r|^{(p^-)'}dx\Big)^{\frac{1}{(p^-)'}}\times\Big(\int_{D_n}|\nabla u_r|^{p^-}dx\Big)^{\frac{1}{p^-}}.$$

Furthermore, we have

$$\int_{D_n}|F_r|^{(p^-)'}dx\leq\int_{\{|u_r|>n\}}|F_r|^{(p^-)'}dx$$

and

$$\lim_{r\to 0}\int_{\{|u_r|>n\}}|F_r|^{(p^-)'}dx\leq\int_{\{|u|\geq n\}}|F|^{(p^-)'}dx.$$

Since

$$\text{meas}(\{|u|\geq n\})\to 0,\;\text{ as } n\to+\infty(\text{ see (4.8) in Proposition 4.4}),$$

one has

$$\lim_{n\to+\infty}\lim_{r\to 0}\int_{D_n}|F_r|^{(p^-)'}dx=0.$$

Next, we need to establish that $\displaystyle\int_{\{n<|u_r|<n+1\}}|\nabla u_r|^{p^-}dx$ is bounded with respect to n.

On other hand, one has the following estimate (see [18] Remark 3.2),

$$||\nabla u_r||^{p^-}_{L^{p^-}(\Omega)}\leq\frac{1}{C}\bigg(\sum_{j=1}^{N}\int_\Omega\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)}dx+Nmeas(\Omega)\bigg),\tag{4.26}$$

where $C>0$ is a constant.

By applying (4.3), we derive

$$\begin{aligned}\int_{D_n}|\nabla u_r|^{p^-}dx&\leq\int_\Omega|\nabla u_r|^{p^-}dx\\&\leq\frac{1}{C}\bigg(\sum_{j=1}^{N}\int_\Omega\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)}dx+Nmeas(\Omega)\bigg)\\&\leq\frac{1}{C}\bigg(\frac{k|\mu|(\Omega)}{C_3}+Nmeas(\Omega)\bigg).\end{aligned}$$

Therefore, $\displaystyle\int_{D_n\}}|\nabla u_r|^{p^-}dx$ is bounded with respect to n.

By using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{r\to 0}\int_\Omega f_rT_1(u_r-T_n(u_r))dx=\int_\Omega fT_1(u-T_n(u))dx.\tag{4.27}$$

Since $T_1(u-T_n(u))\longrightarrow 0$ a.e. in Ω as $n\to+\infty$, we use the Lebesgue dominated convergence Theorem to obtain

$$\lim_{n\to+\infty}\lim_{r\to 0}\int_\Omega f_rT_1(u_r-T_n(u_r))dx=0.\tag{4.28}$$

Let us set $\Psi_r(t)=\displaystyle\int_0^t\phi(T_{\frac{1}{r}}(\delta))d\delta$. Then $\Psi_r(T_n(u_r))\in(W_0^{1,p(x)}(\Omega))^N$.

By using Lemma 2.2, we have

$$\begin{aligned}&\int_{D_n}\phi(T_{\frac{1}{r}}(u_r)).\nabla u_rdx\\&=\int_\Omega\phi(T_{\frac{1}{r}}(T_{n+1}(u_r))).\nabla T_{n+1}(u_r)dx-\int_\Omega\phi(T_{\frac{1}{r}}(T_n(u_r))).\nabla T_n(u_r)dx\\&=\int_\Omega div\Psi_r((T_{n+1}(u_r)))dx-\int_\Omega \mathrm{div}\Psi_r(T_n(u_r))dx=0.\end{aligned}\tag{4.29}$$

Thanks to (4.25)-(4.29), we can pass to the limit in (4.24) to obtain the renormalized condition

$$\lim_{n\to+\infty}\sum_{j=1}^{N}\int_{\{n<|u|<n+1\}}\left|\frac{\partial u}{\partial x_j}\right|^{p_j(x)}dx=0.$$

The following lemma will be useful in the proof of Theorem 3.4.

Lemma 4.16. *Let us assume that ϕ is a Lipschitz function, $\psi \in W_0^{1,p(.)}(\Omega)$, Z in $\mathcal{M}_b(\Omega)$ and $\delta \in \mathbb{R}$ be such that*

$$\begin{cases} \psi \leq \delta \text{ a.e. in } \Omega \text{ (resp. } \psi \geq \delta) \\ Z = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j(x, \frac{\partial u}{\partial x_j}) + div\ \phi(\psi) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{4.30}$$

Then

$$\int_{[\psi=\delta]} \varphi dZ \geq 0, \tag{4.31}$$

(resp.)

$$\int_{[\psi=\delta]} \varphi dZ \leq 0, \tag{4.32}$$

for any $\varphi \in C_c^1(\Omega)$, $\varphi \geq 0$.

Proof. Let us recall that $Z = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j(x, \frac{\partial u}{\partial x_j}) + \text{div}\phi(\psi)$ in $\mathcal{D}'(\Omega)$ and $z \in \mathcal{M}_b^{p(.)}(\Omega)$.
For any $n \geq 1$, we define the function φ_n by

$$\varphi_n(r) = \inf(1, (nr + 1 - n\delta)^+), \ \forall r \in \mathbb{R}.$$

For every $r \in \mathbb{R}$, we have, $\varphi_n(r) \longrightarrow \chi_{[\delta,\infty)}(r)$ as $n \to +\infty$. This implies that $\varphi_n(\psi(x)) \longrightarrow \chi_{[\delta,\infty)}(\psi(x))$ at each x where $\psi(x)$ is defined. Since ψ is defined almost everywhere and $\chi_{[\delta,\infty)} \circ \psi = \chi_{\{x\in\Omega:\psi(x)=\delta\}}$, the convergence of $\varphi_n(\psi)$ to $\chi_{[\delta,\infty)}(\psi)$ holds almost everywhere. Given that Z is diffuse, it follows that $\varphi_n(\psi)$ converges to $\chi_{\{x\in\Omega:\psi(x)=\delta\}}$ is Z-a.e. in Ω.
Next, we use the Lebesgue dominated convergence theorem and (3.4) to get

$$\begin{aligned} \int_{[\psi=\delta]} \varphi dZ &= \lim_{n\to+\infty} \int_\Omega \varphi\varphi_n(\psi) dZ \\ &= \lim_{n\to+\infty} \int_\Omega \sum_{j=1}^{N} a_j\left(x, \frac{\partial \psi}{\partial x_j}\right) \frac{\partial}{\partial x_j}(\varphi_n(\psi)) dx \\ &+ \lim_{n\to+\infty} \int_\Omega \text{div } (\phi(\psi))(\varphi_n(\psi)) dx \\ &\geq \lim_{n\to+\infty} \int_\Omega \sum_{j=1}^{N} a_j\left(x, \frac{\partial \psi}{\partial x_j}\right) \varphi_n(\psi) \frac{\partial \varphi}{\partial x_j} dx \\ &+ \lim_{n\to+\infty} \int_\Omega \text{div } (\phi(\psi))(\varphi_n(\psi)) dx. \end{aligned}$$

Since the function ϕ is Lipschitz function, we have

$$\int_\Omega \text{div } \phi(\psi)(\varphi_n(\psi)) dx = \int_\Omega (\varphi_n(\psi))\phi'(\psi) \cdot \nabla\psi dx.$$

It follows that

$$\begin{aligned}\left|\int_\Omega \operatorname{div}\, \phi(\psi)(\varphi_n(\psi))dx\right| &= \left|\int_\Omega (\varphi_n(\psi))\phi'(\psi)\cdot\nabla\psi dx\right| \\ &\leq \|\varphi\|_\infty \int_{\{\delta-\frac{1}{n}\leq\psi\leq\delta\}} |\phi'(\psi)|\ |\nabla\psi| dx \\ &\longrightarrow 0 \text{ as } n\to+\infty.\end{aligned}$$

One has (see [21])

$$a_j(x,0)=0 \text{ for any } j=1,...,N \text{ and for a.e. } x\in\Omega.$$

On other hand, one has

$$\begin{aligned}&\left|\int_\Omega \sum_{j=1}^N a_j\left(x,\frac{\partial\psi}{\partial x_j}\right)\varphi_n(\psi)\frac{\partial}{\partial x_j}(\varphi)dx\right| \\ &\leq \max_j ||\frac{\partial\varphi}{\partial x_j}||_\infty \sum_{j=1}^N \int_{\{x\in\Omega:\delta-\frac{1}{n}\leq\psi(x)\leq\delta\}} |a_j(x,\frac{\partial\psi}{\partial x_j})|dx \longrightarrow 0 \text{ as } n\to+\infty.\end{aligned}$$

Since $\lim_{n\to+\infty}\sum_{j=1}^N \int_{\{x\in\Omega:\delta-\frac{1}{n}\leq\psi(x)\leq\delta\}} |a_j(x,\frac{\partial\psi}{\partial x_j})|dx = \sum_{j=1}^N\int_\Omega |a_j(x,0)|dx = 0$, one deduces (4.31), namely

$$\int_{[\psi=\delta]}\varphi dZ\geq 0.$$

In the case $\psi\geq\delta$, one can reasons similarly after setting $\tilde{\psi}=-\psi$, $\tilde{\delta}=-\delta$ and $\tilde{a}_i(x,\psi)=-a_j(x,-\psi)$ to obtain (4.32). □

4.4. **Proof of Theorem 3.4.**

Proof. Since

$$\nu=\sum_{j=1}^N\frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j})-w\mathcal{L}^N+\mu,$$

we have

$$\mu-\nu-w\mathcal{L}^N=-\sum_{j=1}^N\frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j}).$$

By Lemma 4.16, for any $\varphi\in C_c^1(\Omega)$ such that $\varphi\geq 0$, we have

$$\int_{[u=b]}\varphi d\nu^+\leq\int_{[u=b]}\varphi d\mu-\int_{[u=b]}\varphi w dx$$

and

$$\int_{[u=a]}\varphi d\nu^-\leq-\int_{[u=a]}\varphi d\mu+\int_{[u=a]}\varphi w dx.$$

The first inequality implies that

$$\int_\Omega\varphi d\nu^+\leq\int_\Omega\varphi d\mu\lfloor[u=b]-\int_\Omega\varphi w\chi_{[u=b]}dx.$$

Consequently (3.11) holds. In the say way, we establish (3.12). □

5. Uniqueness of solution

The study of uniqueness of solution depends on additional conditions on the convection term ϕ.
If there is no convection term (i.e $\phi \equiv 0$), was established in [21] as follows.

Theorem 5.1. [21] *Let (u, w) and $(\tilde{u}, \tilde{w})$ be two solutions of $(P^{\phi}_{\beta,\mu})$ with $\phi \equiv 0$. Then, we have*

$$u = \tilde{u} \text{ a.e. in } \Omega \text{ and } w = \tilde{w} \text{ a.e. in } \Omega.$$

Theorem 5.2. *Assuming that ϕ is Lipschitz function. If (u, w) and $(\tilde{u}, \tilde{w})$ are two solutions of $(P^{\phi}_{\beta,\mu})$, then*

$$\int_{\Omega} (w - \tilde{w}) sign_0(u - \tilde{u}) dx = 0. \tag{5.1}$$

Proof. By choosing $\varphi = \tilde{u}$ and $\varphi = u$ as test functions in (3.10) for (u, w) and $(\tilde{u}, \tilde{w})$ respectively, we obtain

$$\sum_{j=1}^{N} \int_{\Omega} a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_j} T_k(u - \tilde{u}) dx + \int_{\Omega} w T_k(u - \tilde{u}) dx - \int_{\Omega} \phi(u).\nabla T_k(u - \tilde{u}) dx$$
$$\leq \int_{\Omega} T_k(u - \tilde{u}) d\mu \tag{5.2}$$

and

$$\sum_{j=1}^{N} \int_{\Omega} a_j\left(x, \frac{\partial \tilde{u}}{\partial x_j}\right) \frac{\partial}{\partial x_j} T_k(\tilde{u} - u) dx + \int_{\Omega} \tilde{w} T_k(\tilde{u} - u) dx - \int_{\Omega} \phi(u).\nabla T_k(\tilde{u} - u) dx$$
$$\leq \int_{\Omega} T_k(\tilde{u} - u) d\mu. \tag{5.3}$$

Adding (5.2) and (5.3) yields

$$\sum_{j=1}^{N} \int_{\Omega} \left(a_j(x, \frac{\partial u}{\partial x_j} - a_j(x, \frac{\partial \tilde{u}}{\partial x_j} \right) \frac{\partial}{\partial x_j} T_k(u - \tilde{u}) dx + \int_{\Omega} (w - \tilde{w}) T_k(u - \tilde{u}) dx$$
$$- \int_{\Omega} \Big(\phi(u) - \phi(\tilde{u}) \Big).\nabla T_k(\tilde{u} - u) dx \leq 0. \tag{5.4}$$

Since $a_j(x, .)$ is monotone, the first term of (5.4) is non negative and we deduce from (5.4) that

$$\int_{\Omega} (w - \tilde{w}) T_k(u - \tilde{u}) dx - \int_{\Omega} \Big(\phi(u) - \phi(\tilde{u}) \Big).\nabla T_k(u - \tilde{u}) dx \leq 0.$$

Dividing the above inequality by $k > 0$, we get

$$\frac{1}{k} \int_{\Omega} (w - \tilde{w}) T_k(u - \tilde{u}) dx - \frac{1}{k} \int_{\Omega} \Big(\phi(u) - \phi(\tilde{u}) \Big).\nabla T_k(u - \tilde{u}) dx \leq 0. \tag{5.5}$$

For the second term of (5.5), we have

$$\begin{aligned}
&\left| -\frac{1}{k}\int_\Omega \Big(\phi(u)-\phi(\tilde u)\Big).\nabla T_k(u-\tilde u)dx\right| \\
&= \left|\frac{1}{k}\int_\Omega \Big(\phi(u)-\phi(\tilde u)\Big).\nabla (u-\tilde u)\chi_{\{0\le |u-\tilde u|\le k\}}dx\right| \\
&\le \frac{1}{k}\int_\Omega \left|\Big(\phi(u)-\phi(\tilde u)\Big).\nabla (u-\tilde u)\chi_{\{0\le |u-\tilde u|\le k\}}\right|dx \\
&\le \frac{C}{k}\int_\Omega |u-\tilde u|\ |\nabla (u-\tilde u)|\chi_{\{0\le |u-\tilde u|\le k\}}dx \\
&\le C\int_\Omega |\nabla (u-\tilde u)|\chi_{\{0\le |u-\tilde u|\le k\}}dx.
\end{aligned}$$

Since $|\nabla(\tilde u - u)|\chi_{\{0\le |u-\tilde u|\le k\}} \longrightarrow 0$ a.e. in Ω as $k\to 0$ and

$$\Big||\nabla(\tilde u - u)|\chi_{\{0\le |u-\tilde u|\le k\}}\Big| \le |\nabla(\tilde u - u)| \in L^1(\Omega),$$

the Lebesgue dominated convergence theorem, implies that

$$\lim_{k\to 0}\int_\Omega |\nabla(\tilde u - u)|\chi_{\{0\le |u-\tilde u|\le k\}}dx = 0,$$

therefore

$$\lim_{k\to 0} -\frac{1}{k}\int_\Omega \Big(\phi(u)-\phi(\tilde u)\Big).\nabla T_k(\tilde u-u)dx = 0.$$

For the first term of (5.5), we have

$$\frac{1}{k}(w-\tilde w)T_k(u-\tilde u) \to (w-\tilde w)\mathrm{sign}_0(u-\tilde u)\ a.e.\text{ in } \Omega \text{ as } k\to 0$$

and

$$\left|\frac{1}{k}(w-\tilde w)T_k(u-\tilde u)\right| \le (w-\tilde w)\in L^1(\Omega).$$

Therefore

$$\lim_{k\to 0}\frac{1}{k}\int_\Omega (w-\tilde w)T_k(u-\tilde u)dx = 0.$$

Passing to limit as $k\to 0$ in (5.5), we obtain (5.1) □

Corollary 5.3. *If ϕ is a Lipschitz function and β is a continuous and non-decreasing function on $\mathbb{R}$, we have $w=\tilde w$ a.e. in Ω.*

Proof. Indeed, if β is a continuous and non decreasing function on $\mathbb{R}$, we have $(w-\tilde w)\mathrm{sign}_0(u-\tilde u) = |w-\tilde w|$. Then, using Theorem 5.2, we get

$$\|w-\tilde w\|_{L^1(\Omega)} = 0. \tag{5.6}$$

Hence $w=\tilde w$ a.e. in Ω. □

6. Example

An example that illustrates and falls within the scope of our assumptions is the following anisotropic $\overrightarrow{p}(.)$-harmonic problem: set

$$a_j(x, \frac{\partial u}{\partial x_j}) = \left| \frac{\partial u}{\partial x_j} \right|^{p_j(x)-2} \frac{\partial u}{\partial x_j}, \text{ where } \ p_j(.) = p \text{ for } \ \text{any } j = 1, ..., N,$$

and

$$\beta(u) = (u-1)^+ - (u-1)^-, \ \phi = (\phi_j)_{j=1,\dots,N} : \mathbb{R} \to \mathbb{R}^N.$$

Then, we obtain the problem

$$\begin{cases} (u-1)^+ - (u-1)^- - \displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\left| \frac{\partial u}{\partial x_j} \right|^{p_j(x)-2} \frac{\partial u}{\partial x_j} \right) + \text{div } \phi(u) = \ \mu & \text{in } \Omega, \\ u = \ 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

The functions $a_j(x, \varphi)$ are Carathéodory function satisfying the growth condition (3.2), the coercivity(3.4) and the monotonicity condition (3.3). Observing that all the assumptions of Theorem 3.3 are satisfied, we deduce that for any bounded Radon diffuse measure, the problem (6.1) has at least one solution.

7. Conclusion

In this research, we proved the existence and uniqueness of both renormalized and entropic solutions for a large class of anisotropic nonlinear elliptic problems characterized by inclusion equations and Dirichlet boundary conditions. The solutions were derived through a combination of approximation methods and the monotonicity technique in the framework of Banach spaces. The novelty of this work lies in the fact that it tackles problems with a diffusion-convection term and measure data on the right-hand side of $(P^{\phi}_{\beta,\mu})$, which introduces additional mathematical challenges.

The key contributions of this work are summarized as follows :

1) For any bounded Radon diffuse measure, the problem $(P^{\phi}_{\beta,\mu})$ possesses at least one solution, either in the renormalized or entropic sense.

2) If the convection term ϕ is a Lipschitz continuous function, then any two solutions (u, w) and $(\tilde{u}, \tilde{w})$ of problem $(P^{\phi}_{\beta,\mu})$ satisfy the following equality

$$\int_{\Omega} (w - \tilde{w}) \text{sign}_0(u - \tilde{u}) dx = 0.$$

3) In the particular case where the convection term ϕ is a Lipschitz function and the graph datum is a continuous and non-decreasing function on $\mathbb{R}$, the problem $(P^{\phi}_{\beta,\mu})$ admits a unique solution.

4) The key results of this study significantly generalize numerous previous works found in the literature. Notably, this research extends the most recent contributions presented in [16, 20, 18, 19].

References

1. Antontsev, S., & Rodrigues, J. F. (2006). On stationary thermorheological viscous flows. Ann. Univ. Ferrara Sez. VII Sci. Mat., 52, 19-36. DOI:10.1007/s11565-006-0002-9
2. Benboubker, M. B., Chrayteh, H., Moumni, M. E., & Hjaji, H. (2015). Entropy and renormalized Solutions for Nonlinear Elliptic Problem involving variable exponent and measure Data. Acta Mathematica Sinica, English Series., 31, No. 1, 151–169. DOI:10.1007/S10114-015-3555-7
3. Bendahmane, M., Langlais, M., & Saad M. (2003). On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease. Nonlinear Anal. TMA., 54(4), 617-636. https://doi.org/10.1016/S0362-546X(03)00090-7
4. Bénilan, P., Boccardo, L., Gallouet, T., Gariepy, R., Pierre, M., & V*á*zquez, J. L. (1995). An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Sc. Norm. Sup. Pisa, Cl. Sc 22(2), 241-272.
5. Boccardo, L., & Gallouet, T. (1989). Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal., 87, 149-169. DOI:10.1016/0022-1236(89)90005-0
6. Boccardo, L., & Gallouet, T. (1992). Nonlinear elliptic equations with right hand side measures. Comm Partial Differential Equations. 17, 641-655. DOI:10.1080/03605309208820857
7. Boccardo, L., Gallouet, T., & Orsina, L. (1996). Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. Ann. Inst. Henri Poincaré, Anal. Non Linéaire., 13 (5), 539-551. DOI10.1016/S0294-1449(16)30113-5
8. Bonzi, B. K., Ouaro S., & Zongo, F. D. Y. (2013). Entropy solution for nonlinear elliptic anisotropic homogeneous Neumann Problem. Int. J. Differ. Equ, Article ID 476781, 14p. DOI:1.1155/2013/476781
9. Brezis, H. (1973). Opérateurs maximaux monotones et semigroupes de contraction dans les espaces de Hilbert. North Holland, Amsterdam. DOI:10.1016/s0304-0208(08)x7125-7
10. Chen, Y., Levine, S., & Rao M. (2006). Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math., 66, 1383-1406.
11. Diening, L. (2002). Theoretical and Numerical Results for Electrorheological Fluids. PhD. thesis, University of Frieburg, Germany.
12. Dolzmann, G., Hungerbehler, N. & Muller S. v. (1997). Non-linear elliptic systems with measure-valued right-hand side. Math Z. 4, 545-574. https://doi.org/10.1007/PL00004354
13. Fan, X. (2011). Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(.)$-Laplacien equations. Complex variable and Elliptic equations. An International Journal, 56, 7-9, 623-642. DOI:10.1080/17476931003728412
14. Fan, X., & Zhao, D. (2001). On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. J. Math. Anal. Appl. 263, 424-446. https://doi.org/10.1006/jmaa.2000.7617
15. Gwiazda, P., Wierczewska-Gwiazda, A. & Wròblewska, A. (2010). Monotonicity methods in generalized Orlicz spaces for a class of non-newtonian fluids. Math. Methods Appl. Sci. 33 , 125-137. https://doi.org/10.1002/mma.1155
16. Ibrango, I., & Ouaro, S. (2015). Entropy solutions for anisotropic nonlinear Dirichlet problems. Ann. Univ. Craiova. Mat. Inform. 42 (2), 347-364. DOI:https://doi.org/10.52846/ami.v42i2.672
17. Igbida, N., Ouaro, S., & Soma S. (2012). Elliptic problem involving diffuse measures dat. J. Differ. Equations, 253(12), 3159-3183. https://doi.org/10.1016/j.jde.2012.08.025
18. Konaté, I., & Ouédraogo A. (2023). Existence of renormalized solutions to nonlinear elliptic equations with variable exponents and L^∞-data. Journal of Pure and Applied Functional Analysis. 8(1), 261-277. http://yokohamapublishers.jp/online2/oppafa/vol8/p261.html.

19. Konaté, I., & Ouédraogo A. (2023). Existence and uniqueness of renormalized Solution to nonlinear anisotropic elliptic problems with variable exponent and L^1-data. Int. J. Differ. Equ., Article ID 9454714, 11 p. https://doi.org/10.1155/2023/9454714
20. Konaté, I., & Ouaro, S. (2016). Good Radon measure for anisotropic problems with variable exponent. Electron. J. Diff. Equ. 221, 1-19. https://ejde.math.txstate.edu/Volumes/2016/221/konate.pdf
21. Konaté, I., & Ouaro, S. (2018). Nonlinear multivalued problems with variable exponent and diffuse measure data in anisotropic space. Gulf Journal of Mathematics. 6(2), 13-30. DOI:https://doi.org/10.56947/gjom.v6i2.130
22. Koné, B., Ouaro, S., & Soma, S. (2011). Weak solutions for anisotropic nonlinear elliptic problems with variable exponent and measure data. Int. J. Evol. Equ., 5(3), 1-23.
23. Koné, B., Ouaro, S., & Traoré, S. (2009). Weak solutions for anisotropic nonlinear elliptic equations with variable exponents. Electron. J. Diff. Equ. 144, 1-11. URL:http://ejde.math.txstate.eduorhttp://ejde.math.unt.eduftpejde.math.txstate.edu
24. Kovacik, O., & Rakosnik, J. (1991). On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czech. Math. J., 41(1), 592-618. DOI:10.21136/CMJ.1991.102493
25. Maso, G. D., Murat, F., Orsina, L. & Prignet A. (1999). Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa. Cl. Sci. 28(4), 741-808.
26. Mihailescu, M., Pucci, P. & Radulescu, V. (2008). Eigenvalue problems for anisotropic quasi-linear elliptic equations with variable exponent. J. Math. Appl. 340(1), 687-698. doi:10.1016/j.jmaa.2007.09.015
27. Nassouri, E., Ouaro, S., & Urbain, T. (2019). Nonlinear Neumann problems involving $p(x)$-Laplace operator and measure data. Discussiones Mathematicae Differential Inclusions, Control and Optimization. 39, 181-212.
28. Nyanquini, I., Ouaro, S., & Soma, S. (2013). Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data. Ann. Univ. Craiova ser. Mat. Inform. 40(2), 174-198. DOI:https://doi.org/10.52846/ami.v40i2.536
29. Ouaro, S. (2010). Well-posedness results for anisotropic nonlinear elliptic equations with variable exponent and L^1-data, Cubo J. 12, 133-148. http://dx.doi.org/10.4067/S0719-06462010000100012
30. Ouaro, S., Ouédraogo, A. & Soma, S. (2016). Multivalued homogeneous Neumann problem involving diffuse measure data and variable exponent. Nonlinear Dyn. Syst. Theory. 16(1), 102-114. https://www.e-ndst.kiev.ua/v16n1/9(54).pdf
31. Pfeiffer, C., Mavroidis, C., Bar-Cohen, Y., & Dolgin, B. (1999). Electrorheological fluid based force feedback device. In Proc. 1999 SPIE Telemanipulator and Telepresence Technologies VI Conf. (Boston, MA), 3840, pp. 88-99. https://doi.org/10.1117/12.369269
32. Ruzicka, M. (2000). Electrorheological fluids: modelling and mathematical theory. Springer, Berlin. DOI:10.1007/BFb0104029
33. Showalter, R. E. (1991). Monotone operators in Banach space and nonlinear partial differential equations. Mathematical surveys and monographs, 49, American Mathematical Society. DOI:https://doi.org/10.1090/surv/049
34. Troisi, M. (1969). Teoremi di inclusione per spazi di Sobolev non isotropi. Recherche. Mat, 18, 3-24.
35. Valea, T., & Ouédraogo, A. (2022). Existence and uniqueness of renormalized solution to multivalued homogeneous Neumann problem with L^1data. Gulf Journal of Mathematics. 13(2), 46-66. DOI:https://doi.org/10.56947/gjom.v13i2.590
36. Wittbold, P., Zimmermann, A. (2010). Existence and uniqueness solutions to nonlinear elliptic equations with variable exponent and L^1-data. Nonlinear Analysis: Theory, Methods & Applications, 72, 2990-3008.

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