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# ON THE GENERALIZED FRACTIONAL HANKEL TRANSFORMS OF ARBITRARY ORDER IN THE ZEMANIAN SPACES

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Abstract. In this paper, we introduce the fractional Hankel transform in Zemanian spaces $\mathcal{H}_\mu$, the space of test functions and in its dual $\mathcal{H}'_\mu$, the space of distributions of slow growth, for any real value $\mu$. Moreover, we prove some of its properties. As application, we solve some differential equations to illustrate the theoretical results and the importance of using this transform.

## 1. Introduction

In [10], Namias has introduced the fractional powers of Hankel type transform as generalization of the conventional Hankel-type transform in the fractional order which is developed in [6, 7, 8, 9, 12, 13, 14, 15]. Kerr [3] has defined the fractional Hankel transform that depends on a parameter $\alpha$ in $L^2(\mathbb{R}^+)$ by

$$h_{\mu,\alpha}f(y) = \int_0^\infty A_{\mu,\alpha} f(x) e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} \left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right) dx, \tag{1.1}$$

where $J_\mu$ is the Bessel function of the first kind and order $\mu$ and $\widehat{\alpha} = sgn\alpha$, $f \in L^2(\mathbb{R}^+)$, $\alpha \in \mathbb{R} \setminus \{2k\pi\}$, $k \in \mathbb{Z}$, $A_{\mu,\alpha} = |\sin\frac{\alpha}{2}|^{-\frac{1}{2}} e^{i(\frac{\pi}{2}\widehat{\alpha} - \frac{\alpha}{2})(\mu+1)}$ and $\mu > 1$. To get the classical Hankel transform, we replace $\alpha$ by $\pi$ we find

$$h_\mu f(y) = \int_0^\infty \sqrt{xy} f(x) J_\mu(xy) dx.$$

We were introduced in [1] the fractional Hankel transform in the space of test functions $\mathcal{H}_\mu$ and in its dual $\mathcal{H}'_\mu$, the slowly growing distributions space, when $\mu \geq -\frac{1}{2}$. It is interesting to consider what happens if $\mu < -\frac{1}{2}$, this is the aim of our paper. We define a generalized fractional Hankel transformation for any real value of the order $\mu$ (including values of $\mu$ less than $-\frac{1}{2}$) in such a way that an inverse fractional Hankel transform also exists.

In this paper we extend the fractional Hankel transform and its inverse in $\mathcal{H}_\mu$ for any real value of the order $\mu$ by means of recurrence relation similar to idea given by Zemainan[19], in such a way this transformation coincide with the

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fractional Hankel transform discussed in [1] when $\mu \geq -\frac{1}{2}$. Moroever, we extend the fractional Hankel transform in the distributional generalized sence (i,e in the dual space $\mathcal{H}'_\mu$) to negative real values of the order $\mu$ (i.e., $\mu < -\frac{1}{2}$). Our extended fractional Hankel transformation generates an operational calculus by means of which certain differential equations involving generalized functions can be resolved. The extension of the fractional Hankel transformation that we shall present in this paper possesses the following properties:

- **i)** The fractional Hankel transform possesses an inverse for every real value of the order $\mu$.
- **ii)** The fractional Hankel transform and its inverse transform of order $\mu$ are defined on $\mathcal{H}'_\mu$.
- **iii)** If $\mu \geq -\frac{1}{2}$ the extended direct and inverse fractional Hankel transform coincide with the fractional Hankel transform discussed in [1].

The rest of this paper is organized as follows. In section 2, we recall the definition of Zemanian spaces $\mathcal{H}_\mu$ and its dual $\mathcal{H}'_\mu$ and we present some of the results obtained in [1]. In section 3, we introduce the fractional Hankel transform in $\mathcal{H}_\mu$ for every real value of $\mu$ and discuss a few fundamental properties. In section 4, we define the fractional Hankel transform in the context of $\mathcal{H}'_\mu$ whatever be the real number $\mu$ and some properties of this transform are proved. In the last section we give some examples in the conext of using this transform to solve some differential equations.

Throughout this paper we will denote by $I$ the open interval $(0, \infty)$ and $x$ and $y$ are variables in $I$.

## 2. Preliminaries

The results of our recent paper [1] are developed for the generalized fractional Hankel transform of order $\mu \geq -\frac{1}{2}$, which is based on the testing function space $\mathcal{H}_\mu$ and its dual $\mathcal{H}'_\mu$. We summarize some of the results obtained in [1].

Let $\mu$ be a fixed real member. $\mathcal{H}_\mu$ is the topological linear space consisting of all smooth complex-valued functions $\phi(x)$ defined on $I$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in I}\left\{(1+x^2)^m|(\frac{1}{x}D)^k[x^{-\mu-1/2}\phi(x)]|\right\} < \infty.$$

The space $\mathcal{H}_\mu$ is linear. Moreover, each $\gamma^\mu_{m,k}$ is a seminorm on $\mathcal{H}_\mu$ and since the $\gamma^\mu_{m,0}$ are norms, so the set $\{\gamma^\mu_{m,k}\}^\infty_{m,k=0}$ is a multinorm. The topology on $\mathcal{H}_\mu$ is produced by the set $\{\gamma^\mu_{m,k}\}^\infty_{m,k=0}$. $\mathcal{H}_\mu$ is complete and a Frechet space. $\mathcal{H}'_\mu$ denotes the dual of $\mathcal{H}_\mu$, According to [17, Theorem 1.8-3], $\mathcal{H}'_\mu$ is also complete. The members of $\mathcal{H}'_\mu$ are distributions of slow growth as $x \to \infty$.

The fractional Hankel transform maps $\mathcal{H}_\mu$ into itself. In fact, $h_{\mu,\alpha}$ is an homeomorphism on $\mathcal{H}_\mu$ [1] so long as $\mu \geq -\frac{1}{2}$, where $h^{-1}_{\mu,\alpha} = h_{\mu,-\alpha}$. This allow us to define the generalized fractional Hankel transform $h_{\mu,\alpha}$ on $\mathcal{H}'_\mu$ by extending the Parseval equation. In particular, if $\mu \geq -\frac{1}{2}$, $\phi \in \mathcal{H}_\mu$, and $f \in \mathcal{H}'_\mu$, then

$$\langle h^*_{\mu,\alpha} f, h_{\mu,\alpha}\phi\rangle = \langle f, \phi\rangle.$$

Thus, $h^*_{\mu,\alpha}$ as the adjoint of $h^{-1}_{\mu,\alpha}$. The fact that $h_{\mu,\alpha}$ is an homeomorphism on $\mathcal{H}_\mu$ implies that $h^*_{\mu,\alpha}$ is an homeomorphism on $\mathcal{H}'_\mu$ with inverse $(h^*_{\mu,\alpha})^{-1}$ define as the adjoint of $h_{\mu,\alpha}$

$$\langle (h^*_{\mu,\alpha})^{-1} f, \phi\rangle = \langle f, h_{\mu,\alpha}\phi\rangle.$$

Our objective in this work is to extend some of these results to the case where $\mu < -\frac{1}{2}$. In the sequel, we need the following result which is established in[18].

**Lemma 2.1.** *For any positive or negative integer $k$ and for any real value of $\mu$, the mapping $\phi(x) \to x^k\phi(x)$ is an isomorphism from $\mathcal{H}_\mu$ onto $\mathcal{H}_{\mu+k}$.*

We will now establish similar results for certain differentiation operators. We recall the definition of the differentiation operator $M_{\mu,\alpha}$, $N_{\mu,\alpha}$, $P_{\mu,\alpha}$, and $Q_{\mu,\alpha}$.

$$\begin{aligned}
M_{\mu,\alpha}\phi(x) &= x^{\mu+\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_x x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x).\\
N_{\mu,\alpha}\phi(x) &= \overline{M}_{\mu,\alpha}\phi(x) = x^{\mu+\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_x x^{-\mu-\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x),\\
P_{\mu,\alpha}\phi(x) &= x^{-\mu-\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_x x^{\mu+\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x),\\
Q_{\mu,\alpha}\phi(x) &= \overline{P}_{\mu,\alpha}\phi(x) = x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_x x^{\mu+\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x).
\end{aligned}$$

We define a linear integral operator $M^{-1}_\mu$ by

$$M^{-1}_{\mu,\alpha}(x) = x^{\mu+\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}} \int_\infty^x t^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}t^2\cot\frac{\alpha}{2}}\phi(t)dt,$$

$M^{-1}_{\mu,\alpha}$ is certainly defined on every locally integrable function of rapid descent and therefore on every $\phi \in \mathcal{H}_\mu$.

**Lemma 2.2.** *For any real value $\mu$, the mapping $\phi \to M_\mu\phi$ is an isomorphism from $\mathcal{H}_\mu$ to $\mathcal{H}_{\mu+1}$, the reverse mapping being $\phi \to M^{-1}_\mu\phi$.*

*Proof.* We know that $\phi \to M_{\mu,\alpha}\phi$ is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_{\mu+1}$, see [1]. Clearly $M^{-1}_{\mu,\alpha}$ is linear and the inverse of $M_{\mu,\alpha}$, we prove that $M^{-1}_{\mu,\alpha}$ maps $\mathcal{H}_{\mu+1}$ continuously into $\mathcal{H}_\mu$. Let $\phi(x) \in \mathcal{H}_{\mu+1}$ and $k$ is a fixed positive integer.

Then,

$$
\begin{aligned}
&|x^m(x^{-1}D)^k x^{-\mu-\frac{1}{2}} M^{-1}_{\mu,\alpha}\phi(x)| \\
&\quad = \Big| x^m(\frac{1}{x}D)^k e^{\frac{1}{2}x^2\cot(\frac{\alpha}{2})}\int_\infty^x t^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}t^2\cot(\frac{\alpha}{2})}\phi(t)dt \\
&\quad + x^m\sum_{p=1}^{k}\binom{k}{p}(\frac{1}{x}D)^{k-p}(e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}})(\frac{1}{x}D)^p(\int_\infty^x t^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}t^2\cot\frac{\alpha}{2}}\phi(t)dt)\Big| \\
&\quad = \Big| x^m(\frac{1}{x}D)^k e^{\frac{1}{2}x^2\cot(\frac{\alpha}{2})}\int_\infty^x t^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}t^2\cot(\frac{\alpha}{2})}\phi(t)dt \\
&\quad + x^m\sum_{p=1}^{k}\Big(\binom{k}{p}(i\cot\frac{\alpha}{2})^{k-p}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}} \\
&\quad \times\Big[\sum_{n=0}^{p-1}\binom{p-1}{n}(-i\cot\frac{\alpha}{2})^n(\frac{1}{x}D)^{p-1-n}(x^{-\mu-\frac{1}{2}}\phi(x))\Big]\Big)\Big| \\
&\quad \le \Big| x^m(\frac{1}{x}D)^k e^{\frac{1}{2}x^2\cot(\frac{\alpha}{2})}\int_\infty^x t^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}t^2\cot(\frac{\alpha}{2})}\phi(t)dt\Big| \\
&\quad + \Big| x^m\sum_{p=1}^{k}\Big(\binom{k}{p}(i\cot\frac{\alpha}{2})^{k-p}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}} \\
&\quad \times\Big[\sum_{n=0}^{p-1}\binom{p-1}{n}(-i\cot\frac{\alpha}{2})^n(\frac{1}{x}D)^{p-1-n}(x^{-\mu-\frac{1}{2}}\phi(x))\Big]\Big)\Big| \\
&\quad \le (\cot(\frac{\alpha}{2}))^k\frac{\pi}{2}\Big(\gamma^{\mu+1}_{m+1,0}(\phi)+\gamma^{\mu+1}_{m+3,0}(\phi)\Big) \\
&\quad + \sum_{p=1}^{k}\binom{k}{p}|\cot\frac{\alpha}{2}|^{k-p}\Big(\sum_{n=0}^{p-1}\binom{p-1}{n}|\cot\frac{\alpha}{2}|^n\gamma^{\mu+1}_{m,p-1-n}(\phi)\Big).
\end{aligned}
$$

For the case $k=0$, we have

$$
\begin{aligned}
|x^m x^{-\mu-\frac{1}{2}}M^{-1}_{\mu,\alpha}\phi(x)| &= |x^m e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\int_\infty^x t^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}t^2\cot\frac{\alpha}{2}}\phi(t)dt| \\
&\le x^m\int_x^\infty |t^{-\mu-\frac{1}{2}}\phi(t)|dt \\
&\le \int_x^\infty |t^m t^{-\mu-\frac{1}{2}}\phi(t)|dt \\
&\le \int_0^\infty |\frac{1}{1+t^2}(t^{m+1}+t^{m+3})t^{-\mu-\frac{3}{2}}\phi(t)|dt \\
&\le \int_0^\infty \frac{1}{1+t^2}dt\sup_{0<t<\infty}|(t^{m+1}+t^{m+3})t^{-\mu-\frac{3}{2}}\phi(t)|.
\end{aligned}
$$

Then,

$$
\gamma^{\mu}_{m,0}(M^{-1}_{\mu,\alpha}\phi)\le\frac{\pi}{2}\Big(\gamma^{\mu+1}_{m+1,0}(\phi)+\gamma^{\mu+1}_{m+3,0}(\phi)\Big).
$$

Therefore, the mapping $\phi\to M^{-1}_{\mu,\alpha}\phi$ is a continuous linear mapping of $\mathcal{H}_{\mu+1}$ into $\mathcal{H}_\mu$. □

**Lemma 2.3.** [1] *$N_{\mu,\alpha}$ is a continuous linear mappings of $\mathcal{H}_\mu$ into $\mathcal{H}_{\mu+1}$.*

**Lemma 2.4.** [1] *$P_{\mu,\alpha}$ and $Q_{\mu,\alpha}$ are a continuous linear mappings of $\mathcal{H}_{\mu+1}$ into $\mathcal{H}_\mu$.*

*Remark* 2.5. From Lemma 2.3 and lemma 2.4. Then, the operator

$$\begin{aligned} Q_{\mu,\alpha}N_{\mu,\alpha}\phi &= x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{2\mu+1}D_x x^{-\mu-\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)\\ &= D_x^2\phi+(2ix\cot\frac{\alpha}{2})D_x\phi+i\cot\frac{\alpha}{2}\phi+(x\cot\frac{\alpha}{2})^2\phi+\frac{\frac{1}{4}-\mu^2}{x^2}\phi \end{aligned}$$

is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$.

## 3. The generalized fractional Hankel transform of arbitrary order

Let $\mu$ be any fixed real number and $k$ be any positive integer such that $\mu+k\geq -\frac{1}{2}$. We define a certain transform on $\mathcal{H}_\mu$, which coincides with (1) whenever $\mu\geq -\frac{1}{2}$. For any $\Phi\in\mathcal{H}_\mu$ we set

$$h_{\mu,k,\alpha}[\Phi(y)] = (-1)^k\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^{-k}h_{\mu+k,\alpha}M_{\mu+k-1,\alpha}...M_{\mu+1,\alpha}M_{\mu,\alpha}\Phi(y), \tag{3.1}$$

$$(h_{\mu,k,\alpha})^{-1}\Phi(x) = (-1)^k M^{-1}_{\mu,\alpha}M^{-1}_{\mu+1,\alpha}...M^{-1}_{\mu+k-1,\alpha}h_{\mu+k,-\alpha}\Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^{k}\Phi(x), \tag{3.2}$$

where $h_{\mu+k,-\alpha}$ the inverse of $h_{\mu+k,\alpha}$, see [1].

**Lemma 3.1.** *The transform $h_{\mu,k,\alpha}$ is an homeomorphism on $\mathcal{H}_\mu$ whatever be the real number $\mu$. It inverse is $h^{-1}_{\mu,k,\alpha}$ as defined before. Finally, on $\mathcal{H}_\mu$, $h_{\mu,k,\alpha}$ coincides with $h_{\mu,\alpha}$ as defined by (1) whenever $\mu\geq-\frac{1}{2}$.*

*Proof.* The first assertion follows from the fact that

$$\Phi\to M_{\mu+k-1,\alpha}...M_{\mu+1,\alpha}M_{\mu,\alpha}\Phi$$

is an isomorphism from $\mathcal{H}_\mu$ onto $\mathcal{H}_{\mu+k}$, $\Phi\to h_{\mu+k,\alpha}\Phi$ is an homeomorphism on $\mathcal{H}_{\mu+k,\alpha}$, and $\Phi\to x^{-k}\Phi$ is an isomorphism from $\mathcal{H}_{\mu+k}$ onto $\mathcal{H}_\mu$.
By assumption, $\mu+k\geq-\frac{1}{2}$. It is classical fact that $h^{-1}_{\mu+k,\alpha}$ is the inverse of $h_{\mu+k,\alpha}$ when it acts on smooth functions in $L^2(I)$. Since $\mathcal{H}_{\mu+k}(I)\subset L^2(I)$, the second assertion follows from this fact and Lemmas 2.1 and 2.2 again.
For the third assertion, assume that $\Phi(y)\in\mathcal{H}_\mu$ and $\mu\geq-\frac{1}{2}$. Consider the case $k=1$,

$$\begin{aligned} h_{\mu,1,\alpha}\Phi = -(\frac{x}{|\sin\frac{\alpha}{2}|})^{-1}\int_0^\infty\Bigg[&\big[y^{\mu+\frac{1}{2}}e^{\frac{i}{2}y^2\cot\frac{\alpha}{2}}D_y y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\Phi(y)\big]A_{\mu,\alpha}\\ &\times e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Bigg]dy \end{aligned}$$

An integration by parts and the formula [2, p154], $D_y y^{\mu+1}J_{\mu+1}(xy) = xy^{\mu+1}J_\mu(xy)$. Then, we can get

$$\begin{aligned} h_{\mu,1,\alpha}\Phi &= \Big[ - A_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|})^{-1}\Phi(y)e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big]_0^\infty \\ &\quad + \int_0^\infty A_{\mu,\alpha}\Phi(y)e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})dy. \end{aligned}$$

The limit terms are zero because $\Phi(y)$ is of rapid descent and

$$(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})$$

remains bounded as $y\to\infty$ whereas,for $y\to 0^+$, where $\mu \geq -\frac{1}{2}$. Thus,

$$\begin{aligned} h_{\mu,1,\alpha}\Phi &= -(\frac{x}{|\sin\frac{\alpha}{2}|})^{-1}h_{\mu+1,\alpha}M_{\mu,\alpha}\Phi \\ &= \int_0^\infty A_{\mu,\alpha}\Phi(y)e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})dy \end{aligned}$$

The general statement for larger integral values of $k$ follows by induction from this result. This completes the proof.

Incidentally, a similar argument shows that, if $k$ is a positive integer $\geq \mu+\frac{1}{2}$,

$$h_{\mu,k,\alpha}\Phi = (-1)^k\Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^{-k}h_{\mu-k,\alpha}M^{-1}_{\mu-k,\alpha}...M^{-1}_{\mu-2,\alpha}M^{-1}_{\mu-1,\alpha}\Phi(x)$$

is an homeomorphism on $\mathcal{H}_\mu$, and its inverse mapping is

$$(h_{\mu,k,\alpha})^{-1}\Phi = (-1)^k M_{\mu-1,\alpha}M_{\mu-2,\alpha}...M_{\mu-k,\alpha}h_{\mu-k,-\alpha}\Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^k\Phi(x)$$

However, we shall make no use of this fact.

A consequently of Lemma 3.1 is that $h_{\mu,k,\alpha} = h_{\mu,p,\alpha}$ so long as the positive integers $k$ and $p$ are both larger than $-\mu-\frac{1}{2}$. Indeed, assuming that $k > p$, we have that $h_{\mu+p,k-p,\alpha} = h_{\mu+p,\alpha}$ according to the last statement of Lemma 3.1. Thus, for $\Phi \in \mathcal{H}_\mu$, we obtain

$$h_{\mu,k,\alpha}\Phi = (-1)^p\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^{-p}h_{\mu+p,k-p,\alpha}M_{\mu+p-1,\alpha}...M_{\mu+1,\alpha}M_{\mu,\alpha}\Phi = h_{\mu,p,\alpha}\Phi$$

Since $h_{\mu,-\alpha}\Phi = h^{-1}_{\mu,\alpha}\Phi$, whenever $\Phi\in\mathcal{H}_\mu$ and $\mu\geq-\frac{1}{2}$, Lemma 3.1 also implies that $h^{-1}_{\mu,k,\alpha}$ coincides with $h_{\mu,\alpha}$ when $\mu\geq-\frac{1}{2}$. Moreover, by virtue of the preceding paragraph, $h^{-1}_{\mu,k,\alpha}$ is independant of the choice of $k$ so long as $\mu+k\geq-\frac{1}{2}$.

In view of these results, it is resonable to define the Hankel transform $h_{\mu,\alpha}$ for $\mu < -\frac{1}{2}$ on $\mathcal{H}_\mu$ by $h_{\mu,\alpha}\Phi = h_{\mu,k,\alpha}\Phi$ where $k$ is any positive integer no less than $-\mu-\frac{1}{2}$. The inverse Hankel transform $h^{-1}_{\mu,\alpha}$ is defined by $h^{-1}_{\mu,\alpha}\Phi = h^{-1}_{\mu,k,\alpha}\Phi$, $\Phi\in\mathcal{H}_\mu$. when $\mu\geq-\frac{1}{2}$, we have $h_{\mu,-\alpha} = h^{-1}_{\mu,\alpha}$, but this is not the case when $\mu < -\frac{1}{2}$. □

**Proposition 3.2.** *Let* $\phi\in\mathcal{H}_\mu$ *such that* $\mu\geq-\frac{1}{2}$. *Then,*

- **i)** $N_{\mu,\alpha}h_{\mu,k,\alpha}(\phi) = h_{\mu+1,k,\alpha}(-\frac{x}{|\sin\frac{\alpha}{2}|}\phi(x))$;
- **ii)** $h_{\mu+1,k,\alpha}(M_{\mu,\alpha}\phi) = -\frac{y}{|\sin\frac{\alpha}{2}|}h_{\mu,k,\alpha}(\phi)$.

*Proof.* **i)** From [2, p:154], we can get

$$D_x x^{-\mu-k} J_{\mu+k}(xy) = -y x^{-\mu-k} J_{\mu+k+1}(xy)$$

and differentiating under an integral sign as follows

$$\begin{aligned} & N_{\mu,\alpha} h_{\mu,k,\alpha}(\phi)(x) \\ &= (-1)^k \Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^{-k} \\ &\times \int_0^\infty \Big[ A_{\mu,\alpha} \big[ y^{\mu+k+\frac{1}{2}} e^{\frac{i}{2} y^2 \cot\frac{\alpha}{2}} (y^{-1} D_y)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2} y^2 \cot\frac{\alpha}{2}} \phi(y) \big] \\ &\times (\frac{-y}{|\sin\frac{\alpha}{2}|}) e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} (\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}} J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|}) \Big] dy \\ &= h_{\mu+1,k,\alpha}(-\frac{x}{|\sin\frac{\alpha}{2}|}\phi(x)). \end{aligned}$$

**ii)** According to [2, p:154], we obtain

$$D_x x^{\mu+k+1} J_{\mu+k+1}(xy) = y x^{\mu+k+1} J_{\mu+k}(xy) \tag{3.3}$$

and an integration by parts, we can get

$$\begin{aligned} & h_{\mu+1,\alpha}(M_{\mu,\alpha}\phi)(y) \\ &= (-1)^k \Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^{-k} \\ &\times \Big[ A_{\mu,\alpha} x^{\mu+k+\frac{1}{2}} (x^{-1} D_x)^k [x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2} x^2 \cot\frac{\alpha}{2}} \phi(x)] \\ &\times e^{-\frac{i}{2} y^2 \cot\frac{\alpha}{2}} \phi(x) (\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|}) \Big]_0^\infty \\ &- (-1)^k \Big(\frac{x}{|\sin\frac{\alpha}{2}|}\Big)^{-k} \frac{y}{|\sin\frac{\alpha}{2}|} \\ &\times \int_0^\infty \Big[ A_{\mu,\alpha} x^{\mu+k+\frac{1}{2}} e^{\frac{i}{2} x^2 \cot\frac{\alpha}{2}} (x^{-1} D_x)^k [x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2} x^2 \cot\frac{\alpha}{2}} \phi(x)] \\ &\times e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} (\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+k}(\frac{xy}{|\sin\frac{\alpha}{2}|}) \Big] dx \\ &= \frac{-y}{|\sin\frac{\alpha}{2}|} h_{\mu,k,\alpha}(\phi(y)). \end{aligned}$$

The limit terms are equal to zero since $\phi(x)$ is of rapid descent as $x \to \infty$ and as $x \to 0^+$( From [17, Lemma 5.2-1, p:130-131]),

$$(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|}) = \mathcal{O}(x)$$

and $\phi(x) = \mathcal{O}(1)$ when $\mu \geq -\frac{1}{2}$. Then,

$$h_{\mu+1,k,\alpha}(M_{\mu,\alpha}\phi)(y) = \frac{-y}{|\sin\frac{\alpha}{2}|} h_{\mu,k,\alpha}(\phi(y)).$$

□

**Proposition 3.3.** *Let $\mu$ be any fixed real number and $k$ a positive integer, such that $k \geq -\mu - \frac{1}{2}$. Then, for every $\phi \in \mathcal{H}_\mu$, we get*

$$Q_{\mu,\alpha}N_{\mu,\alpha}(h_{\mu,k,\alpha}\phi) = h_{\mu,k,\alpha}\Big( - \Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2 \phi\Big). \tag{3.4}$$

*Proof.* The right-hand side of (3.4) has meaning because, by Lemma 2.1, $y^2\phi \in \mathcal{H}_{\mu+2}$ and [18, Lemma 1], $\mathcal{H}_{\mu+2} \subset \mathcal{H}_\mu$.

On the other hand, we have the following

$$\begin{aligned}
&Q_{\mu,\alpha}N_{\mu,\alpha}(h_{\mu,k,\alpha}\phi)\\
&\quad= \frac{(-1)^k A_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}} x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_x x^{2\mu+1} D_x^{-\mu-k}\\
&\quad\times \int_0^\infty \Big[ y^{\mu+k+\frac{1}{2}}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)] J_{\mu+k}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big] dy.
\end{aligned}$$

Some simplifications and differentiations under the integral sign and using the formula [2, p154], $D_x x^{-\mu-k} J_{\mu+k}(xy) = -yx^{-\mu-k} J_{\mu+k+1}(xy)dy$. Then, we can obtain the following

$$\begin{aligned}
Q_{\mu,\alpha}N_{\mu,\alpha}(h_{\mu,k,\alpha}\phi) &= \frac{(-1)^k A_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}} x^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\times \int_0^\infty \Big[ y^{\mu+k+2}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)]\\
&\times D_x[x^{-2k} x^{\mu+k+1} J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})]\Big] dy
\end{aligned}$$

The differentiation under the integral sign is valid. Indeed, by using (3.3), we may write $J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})$ is bounded on $0 < \frac{xy}{|\sin\frac{\alpha}{2}|} < \infty$ since $\mu + k + 1 \geq \frac{1}{2}$. Also,

$$y^{\mu+k+2}(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)$$

is of order $\mathcal{O}(y^{\mu+k+2})$ as $y \to 0^+$ and of rapid descent as $y \to \infty$ in view of the fact that $\gamma^\mu_{m,k}(e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi) < \infty$ for every $m$. Moreover, $\mu + k + 2 \geq \frac{3}{2}$. Thus, the integral converges uniformly on every compact subset of $0 < x < \infty$, which validates our differentiation under the integral sign.

$$\begin{aligned}
D_x[x^{-2k} x^{\mu+k+1} J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})] &= -2kx^{-k-\frac{1}{2}} J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\\
&\quad + yx^{-k+\frac{1}{2}} J_{\mu+k}(\frac{xy}{|\sin\frac{\alpha}{2}|}).
\end{aligned}$$

This indicates that a differentiation under the integral sign yields

$$
\begin{aligned}
&Q_{\mu,\alpha}N_{\mu,\alpha}(h_{\mu,k,\alpha}\phi)\\
&=\frac{(-1)^k 2kA_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\times\int_0^\infty\Big[y^{\mu+k+2}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)]x^{-k-\frac{1}{2}}J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big]dy\\
&+\frac{(-1)^{k+1}2kA_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\times\int_0^\infty\Big[y^{\mu+k+3}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)]x^{-k+\frac{1}{2}}J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big]dy.
\end{aligned}
$$

By using the fact that $\mu+k\geq-\frac{1}{2}$, we again see as before that this differentiation under the integral sign is valid. An integration by parts of the first term and using (3.3), we can get

$$
\begin{aligned}
&Q_{\mu,\alpha}N_{\mu,\alpha}h_{\mu,k,\alpha}\phi\\
&\quad=\frac{(-1)^{k+1}2kA_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\quad\times\int_0^\infty y^{\mu+k+1}[(y^{-1}D_y)^{k-1}y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(y)]x^{-k+\frac{1}{2}}J_{\mu+k}(\frac{xy}{|\sin\frac{\alpha}{2}|})dy\\
&\quad+\frac{(-1)^{k+1}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\quad\times\int_0^\infty y^{\mu+k+3}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)]x^{-k+\frac{1}{2}}J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})dy.
\end{aligned}
$$

On the other han, the limit terms

$$
\begin{aligned}
&\frac{(-1)^k 2kA_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}[(y^{-1}D_y)^{k-1}y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(y)]y^{\mu+k+1}x^{-k-\frac{1}{2}}\\
&\qquad\qquad\times J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big|_{y=0}^{y=\infty}
\end{aligned}
$$

equal to zero because of the facts that $\gamma^{\mu}_{0,k-1}(e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi)<\infty$ and

$$
J_{\mu+k+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})=\mathcal{O}(y^{\mu+k+1})\ \text{as}\ y\to\infty,\ \text{and}\ \mu+k\geq-\frac{1}{2}.
$$

We next consider the right-hand side of (3.4), we obtain

$$
\begin{aligned}
&h_{\mu,k,\alpha}\Big(-\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2\phi\Big)\\
&\quad=\frac{(-1)^{k+1}A_{\mu,\alpha}}{|\sin\frac{\alpha}{2}|^{-k+\frac{1}{2}}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\\
&\quad\times\int_0^\infty y^{\mu+k+1}[(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}y^2\phi(y)]x^{-k+\frac{1}{2}}J_{\mu+k}(\frac{xy}{|\sin\frac{\alpha}{2}|})dy.
\end{aligned}
$$

But,

$$\begin{aligned}(y^{-1}D_y)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} y^2\phi(y) =& 2(y^{-1}D_y)^{k-1} y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)\\ &+(y^{-1}D_y)^{k-1} y^2 (y^{-1}D) y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)\\ &\vdots\\ =& 2k(y^{-1}D_y)^{k-1} y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y)\\ &+y^2(y^{-1}D)^k y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}\phi(y).\end{aligned}$$

Hence, we obtain the desired equality in Proposition 3.3. □

## 4. A distributional fractional Hankel transform

In this section, we define the fractional Hankel transform for some slowly growing distributions. Let $\mu$ be a real number, the fractional Hankel transform $h^*_{\mu,k,\alpha}$ on $\mathcal{H}'_\mu$ as the adjoint of $h^{-1}_{\mu,k,\alpha}$ on $\mathcal{H}_\mu$ is defined by

$$\langle h^*_{\mu,k,\alpha}(f), h_{\mu,k,\alpha}(\phi)\rangle = \langle f,\phi\rangle, \quad \text{for all } \phi\in\mathcal{H}_\mu,\ f\in\mathcal{H}'_\mu. \tag{4.1}$$

Or using the different symbols, we write

$$\langle h^*_{\mu,k,\alpha}(f), \Phi\rangle = \langle f, h^{-1}_{\mu,k,\alpha}(\Phi)\rangle, \quad \text{for all } \Phi\in\mathcal{H}_\mu,\ f\in\mathcal{H}'_\mu.$$

From this equality we immediately obtain the uniquness of $h^*_{\mu,k,\alpha}$.

**Lemma 4.1.** *Let $f_1, f_2 \in \mathcal{H}'_\mu$, such that $F_1 = h^*_{\mu,k,\alpha}(f_1)$ and $F_2 = h^*_{\mu,k,\alpha}(f_2)$. If $F_1 = F_2$. Then, $f_1 = f_2$ in the sense of distributions.*

**Theorem 4.2.** *The fractional Hankel transform $h^*_{\mu,k,\alpha} : \mathcal{H}'_\mu \to \mathcal{H}'_\mu$ is an homeomorphism.*

The expression (4.1) also defines the inverse of $(h^*_{\mu,k,\alpha})^{-1}$ as the adjoint of $h_{\mu,k,\alpha}$ by

$$\langle (h^*_{\mu,k,\alpha})^{-1}F, \Phi\rangle = \langle F, h_{\mu,k,\alpha}\Phi\rangle, \quad \text{for all } \Phi\in\mathcal{H}_\mu,\ F\in\mathcal{H}'_\mu.$$

When $\mu \geq -\frac{1}{2}$, the definition of $h^*_{\mu,k,\alpha}$ coincides with that in [1], and as this has been shown in this reference, so it agrees with (1) whenever $f \in L_2(I)$. In this later case, the inverse transform $(h^*_{\mu,k,\alpha})^{-1}$ agree with $h_{\mu,-\alpha}$ if, in addition, $f$ is bounded variation on every compact subinterval of $I$, see [2, §14.4].

4.1. **Some operations on transform formulas:** In $\mathcal{H}'_\mu$ we introduce the differential operators descried in section 2, when they are acting on certain generalized functions. In order to establish certain properties cited in [1, Theoreme 4.3] for the generalized fractional Hankel transform in $\mathcal{H}'_\mu$.

Let $\phi \in \mathcal{H}_{\mu+1}$, then $P_{\mu,\alpha}\phi$, $Q_{\mu,\alpha}\phi \in \mathcal{H}_\mu$. The operators $N^*_{\mu,\alpha}$ and $M^*_{\mu,\alpha}$ are defined on $\mathcal{H}'_\mu$ as the adjoints of $-P_{\mu,\alpha}$ and $-Q_{\mu,\alpha}$ respectively by,

$$\langle N^*_{\mu,\alpha}f,\phi\rangle = \langle f, -P_{\mu,\alpha}\phi\rangle, \quad \text{for all } f\in\mathcal{H}'_\mu.$$

$$\langle M^*_{\mu,\alpha}f,\phi\rangle = \langle f, -Q_{\mu,\alpha}\phi\rangle, \quad \text{for all } f\in\mathcal{H}'_\mu. \tag{4.2}$$

By Lemma 2.3, $f \to N^*_{\mu,\alpha}f$ and $f \to M^*_{\mu,\alpha}f$ are a continuous linear mapping of $\mathcal{H}'_\mu$ into $\mathcal{H}'_{\mu+1}$.

Similary, for $\phi \in \mathcal{H}_\mu$, we define $Q^*_{\mu,\alpha}$ and $P^*_{\mu,\alpha}$ on $\mathcal{H}'_{\mu+1}$ as the adjoints of $-M_{\mu,\alpha}$ and $-N_{\mu,\alpha}$ respectively by,

$$\langle Q^*_{\mu,\alpha} f, \phi\rangle = \langle f, -M_{\mu,\alpha}\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_{\mu+1}.$$

$$\langle P^*_{\mu,\alpha} f, \phi\rangle = \langle f, -N_{\mu,\alpha}\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_{\mu+1}. \tag{4.3}$$

It is clear that $M_{\mu,\alpha}\phi, N_{\mu,\alpha}\phi \in \mathcal{H}_{\mu+1}$ and by Lemma 2.4, $f \to Q^*_{\mu,\alpha} f$ and $f \to P^*_{\mu,\alpha} f$ are a continous linear mapping of $\mathcal{H}'_{\mu+1}$ into $\mathcal{H}'_\mu$.

From (4.2) and (4.3), we define the generalized differential operator $P^*_{\mu,\alpha} M^*_{\mu,\alpha}$ by

$$\langle P^*_{\mu,\alpha} M^*_{\mu,\alpha} f, \phi\rangle = \langle f, Q_{\mu,\alpha} N_{\mu,\alpha}\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_\mu.$$

It follow that $P^*_{\mu,\alpha} M^*_{\mu,\alpha}$ is a continuous linear mapping of $\mathcal{H}'_\mu$ into $\mathcal{H}'_\mu$.

In [19], the operation $f(x) \to x^k f(x)$ was defined as the adjoint of $\phi(x) \longrightarrow x^k\phi(x)$, for any positive or negative integer $k$ and $\phi \in \mathcal{H}_\mu$, $f(x) \to x^k f(x)$ is an isomorphism from $\mathcal{H}'_{\mu+k}$ onto $\mathcal{H}'_\mu$.

Now, we use these equalities and the properties of Proposition 3.2 and Propositio 3.3 to prove the following theorem

**Theorem 4.3.** *Let $\mu$ be any fixed real number and $k$ be any positive integer such that $\mu + k \geq -\frac{1}{2}$ and $f \in \mathcal{H}'_{\mu+1}$. Then,*

- **i)** $h^*_{\mu,k,\alpha}(Q^*_{\mu,\alpha} f) = \frac{y}{|\sin\frac{\alpha}{2}|} h^*_{\mu+1,k,\alpha}(f)$.
- **ii)** $Q^*_{\mu,\alpha} h^*_{\mu+1,k,\alpha}(f) = h^*_{\mu,k,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|} f)$.

*If $f \in \mathcal{H}'_\mu$, we get*

- **iii)** $(h^*_{\mu,k,\alpha})^{-1} P^*_{\mu,\alpha} M^*_{\mu,\alpha} f = -\left(\frac{y}{|\sin\frac{\alpha}{2}|}\right)^2 (h^*_{\mu,k,\alpha})^{-1} f$

*Proof.* **i)** Let $\phi \in \mathcal{H}_{\mu+1}$ and $h_{\mu,k,\alpha}(\phi) = \Phi$. Then,

$$\begin{aligned}\langle h^*_{\mu,k,\alpha}(Q^*_{\mu,\alpha} f), \Phi\rangle &= \langle Q^*_{\mu,\alpha} f, \phi\rangle \\ &= \langle f, -M_{\mu,\alpha}\phi\rangle \\ &= \langle h^*_{\mu+1,k,\alpha}(f), \frac{y}{|\sin\frac{\alpha}{2}|)}\Phi\rangle \\ &= \langle \frac{y}{|\sin\frac{\alpha}{2}|)} h^*_{\mu+1,k,\alpha}(f), \Phi\rangle.\end{aligned}$$

**ii)** Let $F = (h^*_{\mu+1,k,\alpha})^{-1} f$. Then,

$$\begin{aligned}Q^*_{\mu,\alpha} h^*_{\mu+1,k,\alpha}(F) &= h^*_{\mu,k,\alpha}(h^*_{\mu,k,\alpha})^{-1} Q^*_{\mu,\alpha} f \\ &= h^*_{\mu,k,\alpha}(\frac{y}{|\sin\frac{\alpha}{2}|}(h^*_{\mu+1,k,\alpha})^{-1}(f)) \\ &= h^*_{\mu,k,\alpha}(\frac{y}{|\sin\frac{\alpha}{2}|} F).\end{aligned}$$

By replacing $F$ by $f$ and $y$ by $x$, we obtain the result .

**iii)** Let $\phi \in \mathcal{H}_\mu$, and assume moreover that $k$ is restricted as stated in Proposition 3.3. Then, by the definition of $(h^*_{\mu,k,\alpha})^{-1}$, the definition of $P^*_{\mu,\alpha} M^*_{\mu,\alpha}$

when acting on $\mathcal{H}'_\mu$, Proposition 3.3 and we use the fact that the multiplication by $y^2$ in $\mathcal{H}'_\mu$ as the adjoint of multiplication by $y^2$ in $\mathcal{H}_\mu$, we may write

$$\begin{aligned}\langle (h^*_{\mu,k,\alpha})^{-1} P^*_{\mu,\alpha} M^*_{\mu,\alpha} f, \phi\rangle &= \langle P^*_{\mu,\alpha} M^*_{\mu,\alpha} f, h_{\mu,k,\alpha}\phi\rangle = \langle f, Q_{\mu,\alpha} N_{\mu,\alpha} h_{\mu,k,\alpha}\phi\rangle \\ &= \langle f, h_{\mu,k,\alpha}\Big(-\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2\phi\Big)\rangle \\ &= \langle -\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2 (h^*_{\mu,k,\alpha})^{-1} f, \phi\rangle.\end{aligned}$$

□

## 5. Application

Now we shall illustrate, by means of some examples, the advantages of the use of the fractional Hankel transform to solve certain partial differential equations involving the operator $P^*_\mu M^*_\mu$.

**Example 5.1.** Let $P(z)$ be a polynomial having no roots on the nonpositive real axis$-\infty < z \leq 0$. we consider the following problems

$$P(P^*_{\mu,\alpha} M^*_{\mu,\alpha})u = f, \tag{5.1}$$

where $f$ is a given member of $\mathcal{H}'_\mu$, $u$ is unknown but required to be in $\mathcal{H}'_\mu$, and $\mu$ is allowed to assue any real value. Indeed, seeting $U = (h^*_{\mu,k,\alpha})^{-1}u$, $F = (h^*_{\mu,k,\alpha})^{-1}f$ and invoking Theorem 4.3, we may write

$$P(-\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2)U(y) = F(y). \tag{5.2}$$

Now, $1/P(-\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2)$ is a multiplier in $\mathcal{H}'_\mu$ since

$$\Phi(y) \to \Phi(y)/P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)$$

is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. Therefore, we may divide both sidesof (5.2) by $P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)$ and then apply $h^*_{\mu,k,\alpha}$ to obtain the solution

$$u = h^*_{\mu,k,\alpha}\frac{F(y)}{P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)}.$$

If $\mu < -\frac{1}{2}$ and $k$ is chosen as before, if $\phi \in \mathcal{H}_\mu$, and if $\Phi = h^{-1}_{\mu,k,\alpha}\phi$, then $u$ is that member of $\mathcal{H}'_\mu$ that assigns to $\phi$ the mumber

$$\begin{aligned}\langle u, \phi\rangle &= \Big\langle h^*_{\mu,k,\alpha}\frac{F(y)}{P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)}, \phi\Big\rangle \\ &= \Big\langle F(y), \frac{\Phi(y)}{P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)}\Big\rangle \\ &= \Big\langle f, h_{\mu,k,\alpha}\frac{\Phi(y)}{P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)}\Big\rangle \\ &= \Big\langle f(x), (-1)^k\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^{-k} h_{\mu+k,\alpha}M_{\mu+k-1}...M_{\mu+1}M_\mu\frac{\Phi(y)}{P(-(\frac{y}{|\sin\frac{\alpha}{2}|})^2)}\Big\rangle.\end{aligned}$$

By lemma 3.1, the solution $u$ is unique in $\mathcal{H}'_\mu$.

Incidentally if $\mu$ is a negative integer, then each $f \in \mathcal{H}'_\mu$ is also a member of $\mathcal{H}'_\mu$. In view of the fact that

$$P^*_{\mu,\alpha}M^*_{\mu,\alpha} = P^*_{-\mu,\alpha}M^*_{-\mu,\alpha}$$

we may apply $h^*_{-\mu,\alpha,k}$ to (5.1) to obtain a solution in $\mathcal{H}'_{-\mu}$. However, the use of $h^*_{\mu,\alpha,k}$ leads to a stronger result. This is because $\mathcal{H}'_\mu$ contains $\mathcal{H}'_{-\mu}$ as a nondense proper subspace so that equality in $\mathcal{H}'_\mu$ is stronger than equality in $\mathcal{H}'_{-\mu}$. Thus, a solution of (5.1) in $\mathcal{H}'_\mu$ is stronger than a solution in $\mathcal{H}'_{-\mu}$. In other words, if $u$ satisfies (5.1) in the sense of equality in $\mathcal{H}'_\mu$, it certainly does so in $\mathcal{H}'_{-\mu}$, but the converse is not necessarily true. See the last statement of Sec. 5.2, note II[17].

In the case where $\mu$ is negative but not however an integer, there are members of $\mathcal{H}'_\mu$ that are not members of $\mathcal{H}'_{-\mu}$,. For example, the functional $f$ defined on $\mathcal{H}'_\mu$ $(\mu < -\frac{1}{2})$ by

$$\langle f, \phi\rangle = \lim_{x\to 0^+} x^{-1}Dx^{-\mu-\frac{1}{2}}\phi(x) \tag{5.3}$$

is a member of $\mathcal{H}'_\mu$(This limit exists since $x^{-1}Dx^{-\mu-\frac{1}{2}}\phi(x)$ is a smooth and bounded on $0 < x < \infty$). But $f$ is not a member of $\mathcal{H}'_{-\mu}$ if $-1 < \mu < -\frac{1}{2}$ ; indeed,(5.3) doesn't exist if $\phi(x) \in \mathcal{H}'_{-\mu}$ is identical to $\sqrt{x}J_{-\mu}(x)$ on $0 < x < 1$, as is possible. In this case one cannot use $h^*_{-\mu,\alpha,k}$ to get an operational calculus for (5.1).

**Example 5.2.** Find the solution in $\mathcal{H}'_\mu$ of the following differential equation.

$$u - P^*_{\mu,\alpha}M^*_{\mu,\alpha}u = g \tag{5.4}$$

Where $g \in \mathcal{H}'_\mu$, and $u$ is unknown but required to be in $\mathcal{H}'_\mu$ and $\mu$ be real number. Applying $(h^*_{\mu,k,\alpha})^{-1}$ both sides of equations (8), we have

$$(1 + \Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^2)U(y) = G(y),$$

where $U = (h^*_{\mu,k,\alpha})^{-1}u$, and $G = (h^*_{\mu,k,\alpha})^{-1}g$. Then, apply $h^*_{\mu,k,\alpha}$ to obtain the solution

$$u(y) = h^*_{\mu,k,\alpha}\frac{G(y)}{1 + (\frac{y}{|\sin\frac{\alpha}{2}|})^2}.$$

The solution $u$ is in $\mathcal{H}'_\mu$. Indeed, let $\phi \in \mathcal{H}_\mu$ and $\Phi = h^{-1}_{\mu,k,\alpha}\phi$

$$\begin{aligned}
\langle u, \phi\rangle &= \Big\langle h^*_{\mu,k,\alpha}\frac{G(y)}{1+(\frac{y}{|\sin\frac{\alpha}{2}|})^2}, \phi\Big\rangle \\
&= \Big\langle G(y), \frac{\Phi(y)}{1+(\frac{y}{|\sin\frac{\alpha}{2}|})^2}\Big\rangle \\
&= \Big\langle g, h_{\mu,k,\alpha}\frac{\Phi(y)}{1+(\frac{y}{|\sin\frac{\alpha}{2}|})^2}\Big\rangle \\
&= \Big\langle g(x), (-1)^k\Big(\frac{y}{|\sin\frac{\alpha}{2}|}\Big)^{-k} h_{\mu+k,\alpha}M_{\mu+k-1}...M_{\mu+1}M_\mu\frac{\Phi(y)}{1+(\frac{y}{|\sin\frac{\alpha}{2}|})^2}\Big\rangle.
\end{aligned}$$

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