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# CONSTRUCTION OF AN AUTOMORPHISM OF AN ABELIAN GROUP THAT SATISFIES THE PROPERTY OF THE WEAK EXTENSION WITHOUT SATISFYING THE PROPERTY OF THE EXTENSION

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Abstract. In this paper, we construct an automorphism of an abelian group which has the property of the weak extension without possessing the property of the extension.

## 1. Introduction

Throughout this paper, $\mathcal{A}\mathbf{b}$ is a category of abelian groups, $o(x)$ will denote the order of the element $x$ and $p$ is a prime number.
Let $A \in \mathcal{A}\mathbf{b}$, an automorphism $\alpha$ of $A$ has the weak extension property if for all $B \in \mathcal{A}\mathbf{b}$, for all monomorphism $\lambda : A \longrightarrow B$ and if there exists an element $m \in \mathbb{N}^*$ such that the restriction of $\lambda$ to $mA$ is an isomorphism from $mA$ to $mB$, then there exists $\tilde{\alpha} \in Aut(B)$ such that the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \alpha \downarrow & & \downarrow \tilde{\alpha} \\ A & \xrightarrow{\lambda} & B \end{array}$$

is commutative.
In 1987, P. Schupp showed that the extension property in the category of groups characterizes the inner automorphisms, see [11]. Later M. R. Pettet gives a simpler proof of Schupps result and showed that the inner automorphisms of a group are also characterized by the lifting property in the category of groups, see [8]. In [3] L. Ben Yakoub shows that the result of Schupp is not valid in general for Algebras over a commutative ring. It is not yet known whether this result is true for algebras (of finite dimensions) over a field. The automorphisms of abelian groups having the extension property in the category of abelian groups are characterized in [12] and the automorphisms of abelian groups having the weakly extension property in the category of abelian groups are characterized in [10].

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## 2. Counterexample

Lets $A$ an abelian group and $X = \{x_i\}_{i\in I}$ a non-empty family of elements of $A$, and let $p$ a prime integer. We consider $S$ the direct sum of cyclic groups $< x_k >$ with $o(x_k) = p^{k^4}$ for all $k \geq 1$: $S = \bigoplus_{k\geq 1} < x_k >$.
Let $G$ the direct product of the cyclic groups: $G = \prod_{n\geq 1} < x_n >$, We consider $\varphi_k$ the canonical projection of $G$ on $< x_k >$:

$$\begin{array}{rccc} \varphi_k & : & \prod_{i\geq 1} < x_i > & \to & < x_k > \\ & & (\lambda_1 x_1; \lambda_2 x_2; ...) & \mapsto & \lambda_k x_k \end{array}$$

**Lemma 2.1.**

*Let $m$ a positive integer, the element $e_m$ in $G$ is defined by:*

$$\varphi_k(e_m) = \begin{cases} 0 & si \quad k < m \\ p^{k^4-m^4} x_k & si \quad k \geq m \end{cases}$$

(1) $o(e_m) = p^{m^4}$
(2) $\forall m \geq 1 : e_m - p^{(m+1)^4-m^4} e_{m+1} = x_m$

*Proof* 2.2.

1) According to the definition of the element $e_m$, it is easy to see that: $o(e_m) = p^{m^4}$.
2) For the second point of the lemma, we proceed by disjunction cases :
**First case:** $k < m$.
In this case, $\varphi_k(e_m) = 0$ and since $k < m + 1$, then $\varphi_k(e_{m+1}) = 0$,
so $\varphi_k(e_m - p^{(m+1)^4-m^4} e_{m+1}) = \varphi_k(e_m) - p^{(m+1)^4-m^4}\varphi_k(e_{m+1})$.
therefore, $\varphi_k(e_m - p^{(m+1)^4-m^4} e_{m+1}) = 0$.
**Second case:** $k = m$.
In this case, $\varphi_m(e_m) = x_m$ et $\varphi_m(e_{m+1}) = 0$.
Then $\varphi_m(e_m - p^{(m+1)^4-m^4} e_{m+1}) = \varphi_m(e_m) - p^{(m+1)^4-m^4}\varphi_m(e_{m+1})$.
Which is implies that: $\varphi_m(e_m - p^{(m+1)^4-m^4} e_{m+1}) = x_m$.
**Third case:** If $k > m$ then $k \geq m + 1$.
In this case, $\varphi_k(e_m) = p^{k^4-m^4} x_k$ et $\varphi_k(e_{m+1}) = p^{k^4-(m+1)^4} x_k$.
From where $\varphi_k(e_m - p^{(m+1)^4-m^4} e_{m+1}) = p^{k^4-m^4} x_k - p^{(m+1)^4-m^4} p^{k^4-(m+1)^4} x_k$.
Then $\varphi_m(e_m - p^{(m+1)^4-m^4} e_{m+1}) = p^{k^4-m^4} x_k - p^{k^4-m^4} x_k$.
therefore, $\varphi_m(e_m - p^{(m+1)^4-m^4} e_{m+1}) = 0$.
Therefore we have: $\forall m \geq 1 : e_m - p^{(m+1)^4-m^4} e_{m+1} = x_m$.

**Lemma 2.3.**

*Let* $A = \bigoplus_{k\geq 1} < e_k >$.

*If we pose for all* $i \geq 1$: $t_i = e_i - p^{(i+1)^4 - i^4} e_{i+1}$ *and if we consider the homomorphism* $\psi_k$ *of groups defined by:*

$$\begin{array}{rccc} \psi_k & : & A & \to & < e_k > \\ & & \sum\limits_{1\leq i\leq n} \lambda_i e_i & \mapsto & \lambda_k e_k \end{array}$$

*Then:* $< t_1, t_2, ... >= \bigoplus_{k\geq 1} < t_k >$

*Proof* 2.4.

It suffices to show that the family $(t_i)_{i\geq 1}$ is linearly independent.
We suppose that $\sum_{1\leq i\leq n} \lambda_i t_i = 0$.
Let us $k_0 = inf\{k/\lambda_k t_k \neq 0\}$, from where $\psi_{k_0}(\sum_{k_0\leq i\leq n} \lambda_i e_i) = \lambda_{k_0} e_{k_0}$.
Then $\sum_{n_0\leq i\leq n} \lambda_i (e_i - p^{(i+1)^4 - i^4} e_{i+1}) = 0$.
Therefore $\psi_{k_0}(\sum_{n_0\leq i\leq n} \lambda_i (e_i - p^{(i+1)^4 - i^4} e_{i+1})) = 0$.
From where $\lambda_{k_0} e_{k_0} = 0$.
Which is implies: $p^{k_0^4} \mid \lambda_{k_0}$, therefore there is a relative integer $\mu$ such that: $\lambda_{k_0} = \mu p^{k_0^4}$.
Then $\lambda_{k_0} t_{k_0} = \mu p^{k_0^4}(e_{k_0} - p^{(k_0+1)^4 - k_0^4} e_{k_0+1})$.
Which is equivalent to: $\lambda_{k_0} t_{k_0} = \mu(p^{k_0^4} e_{k_0} - p^{(k_0+1)^4} e_{k_0+1})$.
That is to say $\lambda_{k_0} t_{k_0} = 0$, which is absurd.
Therefore, $< t_1, t_2, ... >= \bigoplus_{k\geq 1} < t_k >$.

**Theorem 2.5.**

$\bigoplus_{i\geq 1} < t_i >$ *is a* $p$*-basic subgroup of* $A$.

*Proof* 2.6.

Let $B = A/\bigoplus_{i\geq 1} < t_i >$.
Hence $B =< \overline{e_1}, \overline{e_2}, ... >$.
Then $B =< p^{2^4 - 1^4}\overline{e_2}, p^{3^4 - 2^4}\overline{e_3}, ..., p^{(n+1)^4 - n^4}\overline{e_{n+1}}, ... >$.
which implies that $B$ is $p$-divisible (because $B \subset pB \subset B$).
Furthermore the group $A = \bigoplus_{k\geq 1} < e_k >$ is a direct sum of cyclic groups.
We also have for all $i \geq 1$: $t_i = e_i - p^{(i+1)^4 - i^4} e_{i+1}$.
From where $A = \bigoplus_{1\leq i\leq n} < t_i > \oplus \bigoplus_{i\geq n+1} < e_i >$.
which implies that $\bigoplus_{1\leq i\leq n} < t_i >$ is a direct factor of $A$.
Then $\bigoplus_{1\leq i\leq n} < t_i >$ is a $p$-pure subgroup of $A$.

Therefore $\bigoplus_{i\geq 1} < t_i >$ is also a $p$-pure subgroup of $A$.
Consequently $\bigoplus_{i\geq 1} < t_i >$ is a $p$-basic subgroup of $A$.

**Proposition 2.7.**

*If we pose:* $f_i = e_1 - p^{i^4-1}e_i$ *for all* $i \geq 2$. *Then:*
*(i)* $< f_2, f_3, ..., f_i, f_{i+1}, ... >$ *is a subgroup of* $< t_1, t_2, ..., t_i, ... >$;
*(ii) If we consider the quotient group* $A_1 = (A = \bigoplus_{k\geq 1} < e_k >)/ < f_2, f_3, ..., f_i, f_{i+1}, ... >$,
*then* $\overline{e_1} \neq \overline{0}$.

*Proof* 2.8.

(i) It is easy to see that:
$\forall i \geq 2$: $< f_2, f_3, ..., f_i, f_{i+1}, ... > = < f_2, f_3 - f_2, ..., f_{i+1} - f_i, ... >$.
We also have for all $i \geq 2$: $f_{i+1} - f_i = p^{i^4-1}e_i - p^{(i+1)^4-1}e_{i+1}$.
Which means that $f_{i+1} - f_i = p^{i^4-1}(e_i - p^{(i+1)^4-i^4}e_{i+1})$.
Or $f_{i+1} - f_i = p^{i^4-1}t_i$.
And since $f_2 = e_1 - p^{2^4-1}e_2 = t_1$.
So $< f_2, f_3, ..., f_i, f_{i+1}, ... > = < t_1, p^{2^4-1}t_2, ..., p^{i^4-1}t_i, ... >$ is a subgroup of $< t_1, t_2, ..., t_i, ... >$.
(ii) we Suppose that: $\overline{e_1} = \overline{0}$.
From where $e_1 \in < f_2, f_3, ..., f_i, f_{i+1}, ... > = < t_1, p^{2^4-1}t_2, ..., p^{i^4-1}t_i, ... >$.
Then $e_1 = \lambda_1 t_1 + \sum_{2\leq i\leq n_0} \lambda_i p^{i^4-1}t_i$ with $\lambda_{n_0} p^{n_0^4-1}t_{n_0} \neq 0$.
According to the definition of $\psi_k$, we have: $\psi_{n_0+1}(e_1) = 0$.
Which is equivalent to $\psi_{n_0+1}(\lambda_{n_0} p^{n_0^4-1}t_{n_0}) = 0$.
Which means $\psi_{n_0+1}(\lambda_{n_0} p^{n_0^4-1}e_{n_0} - \lambda_{n_0} p^{(n_0+1)^4-1}e_{n_0+1}) = 0$.
Which means $-\lambda_{n_0} p^{(n_0+1)^4-1}e_{n_0+1} = 0$.
Then $p \mid \lambda_{n_0}$ which implies that there is an integer such that : $\lambda_{n_0} = p\lambda'_{n_0}$.
From where $\lambda_{n_0} p^{n_0^4-1}t_{n_0} = \lambda'_{n_0} p^{n_0^4}t_{n_0}$.
Which is equivalent to $\lambda_{n_0} p^{n_0^4-1}t_{n_0} = \lambda'_{n_0} p^{n_0^4}(e_{n_0} - p^{(n_0+1)^4-n_0^4}e_{n_0+1}) = 0$ which is absurd.

Let now $N \in \mathbb{N}^*$, then $\exists n \in \mathbb{N}^*$ such that: $n^4 - 1 > N$.
Since $f_n = e_1 - p^{n^4-1}e_n$, then:
$\forall N \in \mathbb{N}^*$: $\overline{e_1} = p^{n^4-1}\overline{e_n} \in p^N A_1$ $(n^4 - 1 = N + N' \Rightarrow p^{n^4-1}A_1 = p^N p^{N'} A_1)$.
Which implies that $\overline{e_1} \in \bigcap_{N\in\mathbb{N}^*} p^N A_1$.
And since $o(e_1) = p$, so $\overline{e_1} \in \bigcap_{n\in\mathbb{N}^*} nA_1 = A_1^1$.
On the other hand $\bigoplus_{i\geq 1} < t_i >$ is a $p$-basic subgroup of $A = \bigoplus_{i\geq 1} < e_i >$.
Then $\bigoplus_{i\geq 1} < t_i > / < f_2, f_3, ..., f_i, f_{i+1}, ... > = \bigoplus_{i\geq 1} < t_i > / < t_1 > \oplus \bigoplus_{i\geq 2} p^{i^4-1} < t_i >$
is a $p$-basic subgroup of $\bigoplus_{i\geq 1} < e_i > / < f_2, f_3, ..., f_i, f_{i+1}, ... > = A_1$.
So by theorem 32.4 ([4], p:138) $A_1 = < \overline{t_1} > \oplus (\bigoplus_{i\geq 2} < \overline{t_i} > + pA_1)$.
Finally, we consider the automorphism of $A_1$ defined by:

$$\begin{array}{rcl} \varphi \ : \ <\overline{t_1}> \oplus(\bigoplus_{i\geq 2} <\overline{t_i}> + pA_1) & \to & A_1 \\ \lambda_1\overline{t_1} + \sum_{2\leq i\leq n} \lambda_i\overline{t_i} + p\overline{a} & \mapsto & \lambda_1\overline{t_1} + \sum_{2\leq i\leq n} \lambda_i\overline{t_i} + p\overline{a} + \lambda_1\overline{e_1} \end{array}$$

It's clear that $\varphi = id_{A_1} + \rho$ with $\rho$ an homomorphism of $A_1$ into $A_1^1$ defined by:

$$\begin{array}{rcl} \rho \ : \ <\overline{t_1}> \oplus(\bigoplus_{i\geq 2} <\overline{t_i}> + pA_1) & \to & A_1^1 \\ \lambda_1\overline{t_1} + \sum_{2\leq i\leq n} \lambda_i\overline{t_i} + p\overline{a} & \mapsto & \lambda_1\overline{e_1} \end{array}$$

By theorem 1.1 [10], the automorphism $\varphi$ satisfy the property of the weak extension. By against the automorphism $\varphi$ does not satisfy the property of the extension because the only automorphisms satisfying the property of extension in the category of abelian groups reduced are: $\pm id$, see [12].

## 3. Conclusion

In this work, we will give an example against constructing an automorphism of an abelian group which has the property of low extension without posseder the property of the extension.

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