

A NUMERICAL APPROACH FOR SOLVING A SEMILINEAR OBSTACLE PROBLEM ON THE BOUNDARY

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ABSTRACT. This paper presents an iterative method for solving a semilinear obstacle problem on the boundary. The approach begins by approximating the problem using a penalty method, transforming it from a problem of the first kind to a sequence of problems of the second kind. The convergence of these approximations is then proven. The approximate problem is reformulated as a set-valued problem and expressed in a mixed formulation, leveraging the sub-differential of a continuous convex function to characterize the contact domain with the obstacle. Finally, we show that both the approximate solution and sub-differential can be effectively computed using an appropriate projection/fixed-point algorithm.

1. INTRODUCTION

Numerous studies have explored semilinear variational inequalities, including [4, 6, 14, 18, 21]. Recent studies have focused on various approaches to solving obstacle problems, emphasizing the theoretical framework and numerical validation. For instance, [5] proposed a regularization method along with a posteriori error estimates to address the bilateral obstacle problem, providing valuable insights into error control and solution accuracy in such settings.

This work primarily aims to provide a numerical approach for solving a semilinear obstacle problem at the boundary. Specifically, we address a unilateral obstacle problem expressed as a first-kind semilinear variational inequality. In the linear case, this formulation corresponds to a weak form of the Simplified Signorini problem in elasticity, which can be viewed as a free boundary problem (see [10]). The literature includes various studies on the analysis and numerical methods for the Signorini problem as a linear variational inequality, such as [7, 9, 10, 11, 20].

We consider a bounded and open domain Ω in $\mathbb{R}^n$ (where $n \geq 1$) with a smooth boundary $\partial\Omega$ and h as an element of $H^{1/2}(\partial\Omega)$. Our focus is on addressing the semilinear obstacle problem on the boundary:

Date: Received: Jan 18, 2025; Accepted: Feb 5, 2025.

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2020 *Mathematics Subject Classification.* Primary 49J40 ; Secondary 65K15.

Key words and phrases. Variational inequality, obstacle problem, semilinear, numerical method, approximation.

$$\left\{ \begin{array}{l} \text{Find } u \in K := \{v \in H^1(\Omega) \mid \text{Tr}(v) \geq h \text{ on } \partial\Omega\} \text{ such that} \\ \int_{\Omega} \nabla u \nabla (v - u) dx + \int_{\Omega} u(v - u) dx + \int_{\Omega} g(u)(v - u) dx + \int_{\Omega} f(v - u) dx \geq 0 \quad \forall v \in K, \end{array} \right. \quad (1.1)$$

where f belongs to $L^2(\Omega)$ and $g : L^2(\Omega) \rightarrow L^2(\Omega)$ is a suitable function ensuring the existence of a unique solution to the problem.

We approximate this first-kind problem by employing a penalty method, converting it into a sequence of second-kind semilinear variational inequalities. We then prove the convergence of the approximate solutions u_ε to the solution of the original problem. Following this, we reformulate the problem as a set-valued semilinear partial differential equation involving the sub-differential μ_ε of a convex continuous function. The task is to find a solution pair $(u_\varepsilon, \mu_\varepsilon)$, where μ_ε describes the contact domain (or coincidence set) of the solution u_ε with the obstacle h , one of the unknowns in the problem. Finally, we establish the equivalence between the reformulated problem and a projection/fixed-point problem, where an appropriate iterative method yields the solution $(u_\varepsilon, \mu_\varepsilon)$.

This study builds upon the methodology introduced in [2] (also referred to in [17]) to transform a unilateral obstacle problem into a set-valued equation problem, which can be solved by an appropriate projection method [8] (see also [1, 17]), then developed in [3, 1, 16] for other obstacle problems.

The paper is organized as follows: Section 2 provides an introduction to the semilinear obstacle problem on the boundary. Section 3 introduces a penalty method for approximating the first-kind variational inequality and establishes the convergence of the approximate solutions. In Section 4, the problem is expressed as a set-valued semilinear equation, which can be considered as mixed formulation of the approximate problem. Section 5 demonstrates the equivalence between the reformulated problem and a projection/fixed-point problem, and presents the algorithms used to solve the iterative problem. Lastly, Section 6 concludes the paper with final concluding remarks.

2. THE SEMILINEAR OBSTACLE PROBLEM ON THE BOUNDARY

Let $\Omega \subset \mathbb{R}^n$ (where $n \geq 1$) be an open bounded domain with a smooth boundary $\partial\Omega$. In this context, let $(\cdot, \cdot)_{\partial\Omega}$, $(\cdot, \cdot)$, and $a(\cdot, \cdot)$ denote the inner products of $L^2(\partial\Omega)$, $L^2(\Omega)$, and $H^1(\Omega)$, respectively, defined by

$$\begin{aligned} (u, v)_{\partial\Omega} &:= \int_{\partial\Omega} uv d\sigma \quad \forall u, v \in L^2(\partial\Omega), \\ (u, v) &:= \int_{\Omega} uv dx \quad \forall u, v \in L^2(\Omega), \\ a(u, v) &:= (\nabla u, \nabla v) + (u, v) \quad \forall u, v \in H^1(\Omega). \end{aligned}$$

The norms of $L^2(\partial\Omega)$, $L^2(\Omega)$, and $H^1(\Omega)$ are denoted by $\|\cdot\|_{\partial\Omega}$, $\|\cdot\|_{L^2}$, and $\|\cdot\|$, respectively, and are defined as follows:

$$\|v\|_{\partial\Omega}^2 = (v, v)_{\partial\Omega}, \quad \|v\|_{L^2}^2 = (v, v), \quad \text{and} \quad \|v\|^2 = a(v, v).$$

Let Tr represent the trace operator:

$$Tr : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega), \quad v \mapsto Tr(v),$$

which is surjective and continuous on $H^1(\Omega)$, $g : L^2(\Omega) \rightarrow L^2(\Omega)$ a functional that satisfies the assumptions:

$$(g(u) - g(v), u - v) \geq 0 \quad \text{for all } u, v \in L^2(\Omega), \quad (2.1)$$

$$\|g(u) - g(v)\|_{L^2} \leq k\|u - v\|_{L^2} \quad \forall u, v \in L^2(\Omega), \quad \text{with } 0 < k < 1, \quad (2.2)$$

and h in $H^{1/2}(\partial\Omega)$. We set

$$K := \{v \in H^1(\Omega) \mid Tr(v) \geq h \text{ on } \partial\Omega\}.$$

Problem: Let f belong to $L^2(\Omega)$, we consider the first-kind semilinear variational inequality problem:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v - u) + (g(u), v - u) + (f, v - u) \geq 0 \quad \forall v \in K. \end{cases} \quad (2.3)$$

Problem (2.3) is called a semilinear obstacle problem on the boundary. We show that this problem has a unique solution.

Consider the operator $B : H^1(\Omega) \rightarrow H^1(\Omega)^*$ defined as follows:

$$u \in H^1(\Omega), \quad \langle B(u), v \rangle = a(u, v) + (g(u), v) \quad \forall v \in H^1(\Omega), \quad (2.4)$$

where $H^1(\Omega)^*$ represents the dual space of $H^1(\Omega)$, and $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$. The operator B possesses the following properties: it is bounded, strictly monotone, hemicontinuous, and satisfies the following condition of coercivity:

$$\frac{\langle Bv, v \rangle}{\|v\|} \rightarrow \infty \quad \text{as } \|v\| \rightarrow \infty, \quad v \in H^1(\Omega).$$

Indeed, for $v \in H^1(\Omega)$, we obtain:

$$\begin{aligned} \|B(v)\| &= \sup_{\|w\|=1} \langle Bv, w \rangle \\ &\leq \|v\| + \|g(v)\|_{L^2} \\ &\leq \|v\| + \|g(0)\|_{L^2} + k\|v\| \quad (\text{by hypothesis (2.2)}) \\ &\leq (1 + k)\|v\| + \|g(0)\|_{L^2}. \end{aligned}$$

Consequently, B maps bounded sets to bounded sets. Hypothesis (2.1) and the strict monotonicity of $a(\cdot, \cdot)$ imply the strict monotonicity of B . Moreover, the functions $t \mapsto a(u + tv, \cdot)$ and $t \mapsto g(u + tv)$, with u and v belonging to $H^1(\Omega)$, are continuous on $\mathbb{R}$. Thus, we conclude that B is hemicontinuous.

Let now $v \in H^1(\Omega)$, then we obtain the following

$$\begin{aligned} \frac{\langle Bv, v \rangle}{\|v\|} &= \frac{a(v, v)}{\|v\|} + \frac{(g(v), v)}{\|v\|} \\ &\geq \|v\| + \frac{(g(0), v)}{\|v\|} \quad (\text{by hypothesis (2.1)}). \end{aligned}$$

Given that $\frac{(g(0), v)}{\|v\|}$ is bounded, we deduce that

$$\frac{\langle Bv, v \rangle}{\|v\|} \rightarrow \infty \quad \text{as } \|v\| \rightarrow \infty, v \in H^1(\Omega).$$

Consequently, based on Theorem 2.1 and Remark 2.3 below, we can conclude that problem (2.3) possesses a unique solution.

Consider a separable reflexive Banach space V and its dual space V^* . The norm in V is denoted by $\|\cdot\|_V$, and the duality pairing between V^* and V is represented by $\langle \cdot, \cdot \rangle_V$. Consider an operator $\mathcal{B} : V \rightarrow V^*$ that is pseudo monotone, demi-continuous, and bounded. We have the subsequent theorems.

Theorem 2.1. ([13], p.247) *Let K be a convex, closed, and nonempty subset of V . In the case where K is unbounded, we suppose that $\mathcal{B}$ satisfies the coercivity condition:*

$$\exists v_0 \in K \text{ such that } \frac{\langle \mathcal{B}v, v - v_0 \rangle_V}{\|v\|_V} \rightarrow \infty, \quad \text{as } \|v\|_V \rightarrow \infty, v \in K. \quad (2.5)$$

Thus, for any element f in V^* , the following variational inequality admits at least one solution

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \langle \mathcal{B}(u) + f, v - u \rangle_V \geq 0 \quad \forall v \in K. \end{cases} \quad (2.6)$$

Theorem 2.2. ([13], p. 251) *Let $\varphi : V \rightarrow]-\infty, +\infty]$ be a convex, lower semicontinuous (l.s.c.), and proper. Suppose that there exists $v_0 \in V$ where $\varphi(v_0) < \infty$ and $\mathcal{B}$ satisfies the coercivity condition:*

$$\frac{\langle \mathcal{B}v, v - v_0 \rangle_V + \varphi(v)}{\|v\|_V} \rightarrow \infty, \quad \text{as } \|v\|_V \rightarrow \infty, v \in V. \quad (2.7)$$

Thus, for any element f in V^* , the following variational inequality admits at least one solution

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \langle \mathcal{B}(u) + f, v - u \rangle_V + \varphi(v) - \varphi(u) \geq 0 \quad \forall v \in V. \end{cases} \quad (2.8)$$

Remark 2.3.

- (1) In the case where the operator $\mathcal{B}$ is strictly monotone, the solutions of the variational inequalities (2.6) and (2.8) are unique.
- (2) The hypothesis that the operator $\mathcal{B} : V \rightarrow V^*$ is pseudo monotone, demi-continuous, and bounded, is satisfied, for example, if the operator $\mathcal{B}$ is monotone, hemicontinuous and bounded (refer to [19]).

- (3) An abstract existence result of problem (2.6), for a monotone operator that is continuous on finite-dimensional subspaces, is given in [12].

3. APPROXIMATION OF PROBLEM (2.3) BY A PENALTY METHOD

Let $j : H^1(\Omega) \rightarrow]-\infty, +\infty]$ be a convex, proper, and lower semicontinuous (l.s.c.) function such that

$$j(v) = 0 \quad \text{if } v \in K, \quad (3.1)$$

$$j(v) \geq 0 \quad \forall v \in H^1(\Omega). \quad (3.2)$$

Let $\varepsilon > 0$, we consider the penalty function:

$$j_\varepsilon : H^1(\Omega) \rightarrow]-\infty, +\infty], \quad v \mapsto j_\varepsilon(v) = \frac{1}{\varepsilon} j(v), \quad (3.3)$$

given in [10] to approximate a first-kind linear variational inequality problem by a sequence of second-kind variational inequalities.

We take into consideration the second-kind semilinear variational inequality problem defined as follows:

$$\begin{cases} \text{Find } u_\varepsilon \in H^1(\Omega) \text{ such that} \\ a(u_\varepsilon, v - u_\varepsilon) + (g(u_\varepsilon), v - u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) + (f, v - u_\varepsilon) \geq 0 \quad \forall v \in H^1(\Omega). \end{cases} \quad (3.4)$$

Considering the positivity of the function j_ε , we can observe that as the norm $\|v\|$ tends to infinity for $v \in H^1(\Omega)$, the expression $\frac{\langle Bv, v \rangle + j_\varepsilon(v)}{\|v\|}$ diverges towards infinity, where B is the operator given in (2.4). This can be mathematically expressed as follows:

$$\frac{\langle Bv, v \rangle + j_\varepsilon(v)}{\|v\|} \rightarrow \infty \quad \text{as } \|v\| \rightarrow \infty, v \in H^1(\Omega).$$

Consequently, problem (3.4) has a unique solution according to Theorem 2.2. Therefore, we can conclude with the next theorem:

Theorem 3.1. *Consider u and u_ε to be, respectively, the solutions of problems (2.3) and (3.4), then we have the strong convergences $u_\varepsilon \rightarrow u$ and $j_\varepsilon(u_\varepsilon) \rightarrow j(u)$.*

Proof. From problem (3.4) we have

$$a(u_\varepsilon, u_\varepsilon) + j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon, v) + (g(u_\varepsilon), v - u_\varepsilon) + j_\varepsilon(v) + (f, v - u_\varepsilon) \quad \forall v \in H^1(\Omega).$$

Since $j_\varepsilon(v) = \frac{1}{\varepsilon} j(v) = 0$ for any $v \in K$, by (3.1), we have

$$a(u_\varepsilon, u_\varepsilon) + j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon, v) + (g(u_\varepsilon), v - u_\varepsilon) + (f, v - u_\varepsilon) \quad \forall v \in K. \quad (3.5)$$

Using the monotonicity of g given by (2.1) and the fact that $j_\varepsilon(u_\varepsilon) \geq 0$ given by (3.2), we get

$$a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, v) + (g(v), v - u_\varepsilon) + (f, v - u_\varepsilon) \quad \forall v \in K. \quad (3.6)$$

Let $v = v_0 \in K$ in the above inequality, then we have

$$\begin{aligned} \|u_\varepsilon\|^2 &\leq \|u_\varepsilon\| \|v_0\| + \|g(v_0)\|_{L^2} (\|v_0\| + \|u_\varepsilon\|) + \|f\|_{L^2} (\|v_0\| + \|u_\varepsilon\|) \\ &= \|u_\varepsilon\| (\|v_0\| + \|g(v_0)\|_{L^2} + \|f\|_{L^2}) + \|g(v_0)\|_{L^2} \|v_0\| + \|f\|_{L^2} \|v_0\|. \end{aligned} \quad (3.7)$$

Hence (3.7) implies that the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $H^1(\Omega)$, that is

$$\|u_\varepsilon\| \leq c_1 \quad \forall \varepsilon > 0, \quad (3.8)$$

where c_1 is a constant independent of ε . Inequality (3.5) implies that

$$j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon, v_0) + (g(u_\varepsilon), v_0 - u_\varepsilon) + (f, v_0 - u_\varepsilon) \quad \forall v_0 \in K. \quad (3.9)$$

Using the monotonicity of g , we obtain

$$j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon, v_0) + (g(v_0), v_0 - u_\varepsilon) + (f, v_0 - u_\varepsilon) \quad \forall v_0 \in K. \quad (3.10)$$

Hence

$$\begin{aligned} j(u_\varepsilon) &\leq \varepsilon [\|u_\varepsilon\| \|v_0\| + \|g(v_0)\|_{L^2} (\|v_0\| + \|u_\varepsilon\|) + \|f\|_{L^2} (\|v_0\| + \|u_\varepsilon\|)] \\ &\leq \varepsilon [\|u_\varepsilon\| (\|v_0\| + \|g(v_0)\|_{L^2} + \|f\|_{L^2}) + \|g(v_0)\|_{L^2} \|v_0\| + \|f\|_{L^2} \|v_0\|] \end{aligned} \quad (3.11)$$

which combined with (3.2) and (3.8) gives

$$0 \leq j(u_\varepsilon) \leq c_2 \varepsilon, \quad (3.12)$$

where c_2 is a constant independent of ε . As a consequence of (3.8), there exists a subsequence denoted also by $(u_\varepsilon)_\varepsilon$ which converges weakly in $H^1(\Omega)$:

$$u_\varepsilon \rightharpoonup u^*, \quad u^* \in H^1(\Omega). \quad (3.13)$$

Since j is convex and l.s.c. on $H^1(\Omega)$, j is weakly l.s.c., which combined with (3.12) and (3.13) implies that

$$j(u_\varepsilon) \rightarrow j(u^*) = 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.14)$$

Hence $u^* \in K$, by hypothesis (3.1).

Now, we will show that $u^* = u$. From (3.6), we have

$$a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, v) + (g(v), v - u_\varepsilon) + (f, v - u_\varepsilon) \quad \forall v \in K. \quad (3.15)$$

We know that $u_\varepsilon \rightharpoonup u^*$, hence we get

$$\|u^*\| \leq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|.$$

Then, from (3.15), we have

$$a(u^*, u^*) \leq \liminf_{\varepsilon \rightarrow 0} a(u_\varepsilon, u_\varepsilon) \leq a(u^*, v) + (g(v), v - u^*) + (f, v - u^*). \quad (3.16)$$

Hence we obtain

$$u^* \in K, \quad a(u^*, v - u^*) + (g(v), v - u^*) + (f, v - u^*) \geq 0 \quad \forall v \in K. \quad (3.17)$$

Therefore, from the uniqueness of the solution of (2.3), we have $u^* = u$. Hence the whole solution $(u_\varepsilon)_\varepsilon$ converges weakly to u .

Now, we show the strong convergence. We have

$$a(u_\varepsilon, v - u_\varepsilon) = a(u_\varepsilon - u, u - u_\varepsilon) + a(u_\varepsilon - u, v - u) + a(u, v - u_\varepsilon).$$

Using this equality in (3.4), we get

$$\|u - u_\varepsilon\|^2 + j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon - u, v - u) + a(u, v - u_\varepsilon) + (g(u_\varepsilon), v - u_\varepsilon) + j_\varepsilon(v) + (f, v - u_\varepsilon) \quad \forall v \in H^1(\Omega).$$

Hence, using hypotheses (3.1) and (3.2) and the monotonicity of g yields

$$0 \leq \|u - u_\varepsilon\|^2 + j_\varepsilon(u_\varepsilon) \leq a(u_\varepsilon - u, v - u) + a(u, v - u_\varepsilon) + (g(v), v - u_\varepsilon) + (f, v - u_\varepsilon) \quad \forall v \in K.$$

Passing to the limit, we obtain

$$0 \leq \limsup_{\varepsilon \rightarrow 0} (\|u - u_\varepsilon\|^2 + j_\varepsilon(u_\varepsilon)) \leq a(u, v - u) + (g(v), v - u) + (f, v - u) \quad \forall v \in K. \quad (3.18)$$

As $u = u^* \in K$, we take $v = u$ in (3.18). Then we get

$$\lim_{\varepsilon \rightarrow 0} (\|u - u_\varepsilon\|^2 + j_\varepsilon(u_\varepsilon)) = 0,$$

which proves the strong convergences $u_\varepsilon \rightarrow u$ and $j_\varepsilon(u_\varepsilon) \rightarrow j(u)$. □

4. EQUIVALENT FORMULATION

We use the following notation, for every element $v \in L^2(\partial\Omega)$:

$$v^+ := \max\{v, 0\} \quad \text{and} \quad v^- := \min\{v, 0\}. \quad (4.1)$$

We define the function $j : H^1(\Omega) \rightarrow \mathbb{R}$ as follows

$$v \mapsto j(v) = - \int_{\partial\Omega} (Tr(v) - h)^- d\sigma.$$

Since $Tr : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is continuous, the function j is continuous on $H^1(\Omega)$. Moreover, j is convex and satisfies hypotheses (3.1) and (3.2).

Let φ_ε be the function defined by

$$\varphi_\varepsilon : L^2(\partial\Omega) \rightarrow \mathbb{R} \quad v \mapsto \varphi_\varepsilon(v) = -\frac{1}{\varepsilon} \int_{\partial\Omega} (v - h)^- d\sigma. \quad (4.2)$$

We have

$$\varphi_\varepsilon(Tr(v)) = j_\varepsilon(v) = \frac{1}{\varepsilon} j(v) \quad \forall v \in H^1(\Omega).$$

Let $\partial\varphi_\varepsilon(u)$ represent the sub-differential of φ_ε at u in $L^2(\partial\Omega)$. It is defined as follows:

$$\partial\varphi_\varepsilon(u) := \{\mu \in L^2(\partial\Omega) \mid \varphi_\varepsilon(v) - \varphi_\varepsilon(u) \geq (\mu, v - u)_{\partial\Omega} \quad \forall v \in L^2(\partial\Omega)\}.$$

Consider the problem:

$$\begin{cases} \text{Find } (u_\varepsilon, \mu_\varepsilon) \in H^1(\Omega) \times L^2(\partial\Omega) \text{ such that} \\ a(u_\varepsilon, v) + (g(u_\varepsilon), v) + (\mu_\varepsilon, Tr(v))_{\partial\Omega} + (f, v) = 0 \quad \forall v \in H^1(\Omega), \\ \mu_\varepsilon \in \partial\varphi_\varepsilon(Tr(u_\varepsilon)). \end{cases} \quad (4.3)$$

As a result of the aforementioned problem formulation, the following theorem holds.

Theorem 4.1. *Let $u_\varepsilon \in H^1(\Omega)$. The function u_ε is a solution to problem (3.4) if and only if there exists $\mu_\varepsilon \in L^2(\partial\Omega)$ for which the pair $(u_\varepsilon, \mu_\varepsilon) \in H^1(\Omega) \times L^2(\partial\Omega)$ satisfies problem (4.3).*

Proof. Suppose that u_ε is a solution of problem (3.4) and let w be an element of $H^1(\Omega)$. We set $v = u_\varepsilon + w$ in the inequality of (3.4), we have

$$a(u_\varepsilon, w) + (g(u_\varepsilon), w) + \varphi_\varepsilon(Tr(u_\varepsilon + w)) - \varphi_\varepsilon(Tr(u_\varepsilon)) + (f, w) \geq 0. \quad (4.4)$$

Using that $(a + b)^- \geq a^- + b^-$ for any reals a and b , we get:

$$\varphi_\varepsilon(Tr(u_\varepsilon + w)) - \varphi_\varepsilon(Tr(u_\varepsilon)) \leq -\frac{1}{\varepsilon} \int_{\partial\Omega} Tr(w)^- d\sigma.$$

Hence, from (4.4), we obtain

$$a(u_\varepsilon, w) + (g(u_\varepsilon), w) + (f, w) \geq \frac{1}{\varepsilon} \int_{\partial\Omega} Tr(w)^- d\sigma \geq -\frac{1}{\varepsilon} mes(\partial\Omega) \|Tr(w)\|_{L^2(\partial\Omega)},$$

where $mes(\partial\Omega)$ is the measure of $\partial\Omega$. If we substitute w by $-w$ in the above inequality, we get

$$a(u_\varepsilon, w) + (g(u_\varepsilon), w) + (f, w) \leq \frac{1}{\varepsilon} mes(\partial\Omega) \|Tr(w)\|_{L^2(\partial\Omega)}.$$

Hence

$$|a(u_\varepsilon, w) + (g(u_\varepsilon), w) + (f, w)| \leq \frac{1}{\varepsilon} mes(\partial\Omega) \|Tr(w)\|_{L^2(\partial\Omega)}. \quad (4.5)$$

The trace operator $Tr : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is surjective, then the operator $A_\varepsilon : H^{1/2}(\partial\Omega) \rightarrow \mathbb{R}$ defined by

$$A_\varepsilon(Tr(w)) = a(u_\varepsilon, w) + (g(u_\varepsilon), w) + (f, w), \quad w \in H^1(\Omega),$$

is well-defined. Indeed, let $w_1, w_2 \in H^1(\Omega)$ such that $w_1 = w_2$ on $\partial\Omega$. Taking $w = w_1 - w_2$ in (4.5), we get

$$|a(u_\varepsilon, w_1 - w_2) + (g(u_\varepsilon), w_1 - w_2) + (f, w_1 - w_2)| \leq \frac{1}{\varepsilon} mes(\partial\Omega) \|Tr(w_1 - w_2)\|_{L^2(\partial\Omega)} = 0.$$

Hence, $w_1 = w_2$ on $\partial\Omega$, and therefore A_ε is a well-defined application. From (4.5), we observe that A_ε is continuous on $H^{1/2}(\partial\Omega)$ with respect to the $L^2(\partial\Omega)$ norm. Given that $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$, this linear form can be extended from $H^{1/2}(\partial\Omega)$ to $L^2(\partial\Omega)$. Consequently, there exists a unique μ_ε^* in $L^2(\partial\Omega)$ satisfying:

$$(\mu_\varepsilon^*, Tr(w))_{\partial\Omega} = A_\varepsilon(Tr(w)) = a(u_\varepsilon, w) + (g(u_\varepsilon), w) + (f, w) \quad \forall w \in H^1(\Omega). \quad (4.6)$$

If we set $\mu_\varepsilon = -\mu_\varepsilon^*$, then the pair $(\mu_\varepsilon, u_\varepsilon)$ satisfies equation (4.3). To see this, consider that since u_ε is a solution of the inequality in (3.4), we get

$$a(u_\varepsilon, v - u_\varepsilon) + (g(u_\varepsilon), v - u_\varepsilon) + j_\varepsilon(v) - j(u_\varepsilon) + (f, v - u_\varepsilon) \geq 0 \quad \forall v \in H^1(\Omega).$$

Combining this inequality with (4.6), we obtain

$$\begin{aligned} j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) &\geq -a(u_\varepsilon, v - u_\varepsilon) - (g(u_\varepsilon), v - u_\varepsilon) - (f, v - u_\varepsilon) \\ &= (\mu_\varepsilon, Tr(v) - Tr(u_\varepsilon))_{\partial\Omega} \quad \forall v \in H^1(\Omega). \end{aligned}$$

Since $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$ and $j_\varepsilon(v) = \varphi_\varepsilon(Tr(v))$ for any $v \in H^1(\Omega)$, we have

$$\varphi_\varepsilon(v) - \varphi_\varepsilon(Tr(u_\varepsilon)) \geq (\mu_\varepsilon, v - Tr(u_\varepsilon))_{\partial\Omega} \quad \forall v \in L^2(\partial\Omega).$$

Hence, we can conclude that $\mu_\varepsilon \in \partial\varphi_\varepsilon(Tr(u_\varepsilon))$.

Conversely, let $(\mu_\varepsilon, u_\varepsilon) \in L^2(\partial\Omega) \times H^1(\Omega)$ be a solution of problem (4.3). Since $\mu_\varepsilon \in \partial\varphi_\varepsilon(Tr(u_\varepsilon))$, in particular we have

$$j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq (\mu_\varepsilon, Tr(v) - Tr(u_\varepsilon))_{\partial\Omega} \quad \forall v \in H^1(\Omega).$$

Hence

$$\begin{aligned} & a(u_\varepsilon, v - u_\varepsilon) + (g(u_\varepsilon), v - u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) + (f, v - u_\varepsilon) \\ & \geq a(u_\varepsilon, v - u_\varepsilon) + (g(u_\varepsilon), v - u_\varepsilon) + (\mu_\varepsilon, Tr(v) - Tr(u_\varepsilon))_{\partial\Omega} + (f, v - u_\varepsilon) \\ & = 0, \end{aligned} \quad \forall v \in H^1(\Omega).$$

It shows that u_ε is a solution of (3.4) which completes the proof. $\square$

Let $v \in H_0^1(\Omega)$ in the equation of (4.3), we have then

$$a(u_\varepsilon, v) + (g(u_\varepsilon), v) + (f, v) = 0.$$

It follows that

$$-\Delta u_\varepsilon + u_\varepsilon + g(u_\varepsilon) + f = 0 \quad \text{in } \Omega. \quad (4.7)$$

Multiplying (4.7) by v and using Green's formula, we obtain

$$a(u_\varepsilon, v) + \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} Tr(v) d\sigma + (g(u_\varepsilon), v) + (f, v) = 0 \quad \forall v \in H^1(\Omega). \quad (4.8)$$

Hence, from the equation in (4.3) and (4.8), we get:

$$\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} Tr(v) d\sigma = \int_{\partial\Omega} \mu_\varepsilon Tr(v) d\sigma \quad \forall v \in H^1(\Omega). \quad (4.9)$$

Given that $L^2(\partial\Omega)$ is dense in $H^{1/2}(\Omega)$ and based on equation (4.9), we can deduce that the following equality holds on the boundary of Ω :

$$\frac{\partial u_\varepsilon}{\partial n} = \mu_\varepsilon \quad \text{on } \partial\Omega. \quad (4.10)$$

Then, from (4.7) and (4.10), we can see that the solution of problem (4.3) satisfies the following Neumann problem:

$$\begin{cases} \Delta u_\varepsilon - u_\varepsilon = g(u_\varepsilon) + f & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial n} = \mu_\varepsilon & \text{on } \partial\Omega. \end{cases} \quad (4.11)$$

The function μ_ε can be obtained through a projection algorithm.

To conclude, we relate the function μ_ε to the contact domain of the solution u_ε of problem (3.4) with the boundary obstacle h , defined as:
 $\partial\Omega_0 := \{x \in \partial\Omega; u_\varepsilon(x) = h(x)\}.$

Lemma 4.2. *Consider the functional ϕ defined from $L^2(\partial\Omega)$ to $\mathbb{R}$ as follows:*

$$\phi(v) := (F, v^-)_{\partial\Omega},$$

where F is a negative function of $L^2(\partial\Omega)$. Thus, the following holds for any elements u and ν belonging to $L^2(\partial\Omega)$

$$\nu \in \partial\phi(u) \Leftrightarrow \nu(x) \in \begin{cases} \{F(x)\} & \text{if } u(x) < 0, \\ [F(x), 0] & \text{if } u(x) = 0, \\ \{0\} & \text{if } u(x) > 0. \end{cases}$$

Proof. The proof follows that provided in [2, 16] for a function $\phi : L^2(\Omega) \rightarrow \mathbb{R}$ given by $\phi(v) = (F, v^+)$, where F is a positive function of $L^2(\Omega)$. $\square$

Hence, using the definition of the function φ_ε in (4.2) and Lemma 4.2, the pointwise characterization of the function μ_ε in (4.3) is

$$\mu_\varepsilon \in \partial\varphi_\varepsilon(Tr(u_\varepsilon)) \Leftrightarrow \mu_\varepsilon(x) \in \begin{cases} \{-\frac{1}{\varepsilon}\} & \text{if } Tr(u_\varepsilon(x)) < h(x) \text{ on } \partial\Omega, \\ [-\frac{1}{\varepsilon}, 0] & \text{if } Tr(u_\varepsilon(x)) = h(x) \text{ on } \partial\Omega, \\ \{0\} & \text{if } Tr(u_\varepsilon(x)) > h(x) \text{ on } \partial\Omega. \end{cases} \quad (4.12)$$

5. PROJECTION/FIXED-POINT METHOD

To begin this section, we provide a characterization of the sub-differential of φ_ε . We consider a Hilbert space denoted by H with its inner product denoted by $(\cdot, \cdot)$. We recall that $\phi : H \rightarrow \mathbb{R}$ is said to be a positively homogeneous function of degree 1 if it satisfies $\phi(\lambda v) = \lambda\phi(v)$ for all $\lambda \geq 0$. Now, consider a convex function $\phi : H \rightarrow \mathbb{R}$ and define the following convex set:

$$C := \{\nu \in H \mid (\nu, v) \leq \phi(v) \quad \forall v \in H\}. \quad (5.1)$$

Lemma 5.1. (see [2]) *The functional ϕ is positively homogeneous of degree 1, if and only if, for any element v in H , the statements bellow are equivalent:*

- (a) $\mu \in \partial\phi(v)$.
- (b) $\mu \in C$ and $(\mu, v) = \phi(v)$.
- (c) $\mu \in C$ and $(\nu - \mu, v) \leq 0 \quad \forall \nu \in C$.

The proof of this Lemma can also be found in [16].

Let $H := L^2(\partial\Omega)$ and define the function $\phi_\varepsilon : L^2(\partial\Omega) \rightarrow \mathbb{R}$ as

$$v \mapsto \phi_\varepsilon(v) = -\frac{1}{\varepsilon}(1, v^-)_{\partial\Omega} = -\frac{1}{\varepsilon} \int_{\partial\Omega} v^- d\sigma. \quad (5.2)$$

Let

$$C_\varepsilon := \{\nu \in L^2(\partial\Omega) \mid (\nu, v)_{\partial\Omega} \leq \phi_\varepsilon(v) \quad \forall v \in L^2(\partial\Omega)\}, \quad (5.3)$$

and

$$\varphi_\varepsilon : L^2(\partial\Omega) \rightarrow \mathbb{R}, \quad v \mapsto \varphi_\varepsilon(v) = \phi_\varepsilon(v - h). \quad (5.4)$$

We have then

$$\mu_\varepsilon \in \partial\varphi_\varepsilon(v) \Leftrightarrow \mu_\varepsilon \in \partial\phi_\varepsilon(v - h).$$

According to Lemma 5.1, problem (4.3) can be written as follows

$$\begin{cases} \text{Find } (u_\varepsilon, \mu_\varepsilon) \in H^1(\Omega) \times C_\varepsilon \text{ such that} \\ a(u_\varepsilon, v) + (g(u_\varepsilon), v) + (\mu_\varepsilon, \text{Tr}(v))_{\partial\Omega} + (f, v) = 0 \quad \forall v \in H^1(\Omega), \\ (\nu - \mu_\varepsilon, \text{Tr}(u_\varepsilon) - h)_{\partial\Omega} \leq 0 \quad \forall \nu \in C_\varepsilon. \end{cases} \quad (5.5)$$

We set

$$M_\varepsilon := \{w \in H^1(\Omega) \mid \exists \nu \in C_\varepsilon : a(w, v) = (\nu, \text{Tr}(v))_{\partial\Omega} \quad \forall v \in H^1(\Omega)\}. \quad (5.6)$$

Proposition 5.2. *The set M_ε is convex and closed in $H^1(\Omega)$, and each element w in M_ε is represented by a unique element in C_ε .*

Proof. It is clear that M_ε is a convex set, we show that M_ε is closed in $H^1(\Omega)$. Let $(w_n)_n$ be a sequence of M_ε such that $w_n \rightarrow w$ in $H^1(\Omega)$. For each w_n there exists a $\nu_n \in C_\varepsilon$ such that

$$a(w_n, v) = (\nu_n, \text{Tr}(v))_{\partial\Omega} \leq \phi_\varepsilon(\text{Tr}(v)) \leq \text{mes}(\Omega)\|v\|_{L^2(\partial\Omega)} \quad \forall v \in H^1(\Omega). \quad (5.7)$$

Taking the limit as n approaches infinity yields

$$a(w, v) \leq \text{mes}(\Omega)\|v\|_{L^2(\partial\Omega)} \quad \forall v \in H^1(\Omega).$$

Let $L : H^{1/2}(\partial\Omega) \rightarrow \mathbb{R}$ be the operator defined by

$$L(\text{Tr}(v)) = a(w, v), \quad v \in H^1(\Omega).$$

Similarly to the proof of Theorem 5.6 (see (4.5)-(4.6)), we show that the operator L is well defined and that there exists a unique ν in $L^2(\partial\Omega)$ satisfying

$$(\nu, \text{Tr}(v))_{\partial\Omega} = L(\text{Tr}(v)) = a(w, v) \quad \forall v \in H^1(\Omega).$$

Now, we show that $\nu \in C_\varepsilon$. From (5.7), we have

$$a(w_n, v) = (\nu_n, \text{Tr}(v))_{\partial\Omega} \leq \phi_\varepsilon(\text{Tr}(v)) \quad \forall v \in H^1(\Omega).$$

Passing to the limit on n , we obtain

$$(\nu, \text{Tr}(v))_{\partial\Omega} = a(w, v) \leq \phi_\varepsilon(\text{Tr}(v)) \quad \forall v \in H^1(\Omega).$$

Hence $\nu \in C_\varepsilon$. Consequently, the set M_ε is closed in $H^1(\Omega)$. □

Let us define the function l as follows:

$$l : H^1(\Omega) \rightarrow H^1(\Omega), \quad u \mapsto l(u) = w, \quad (5.8)$$

where w satisfies:

$$a(w, v) = (g(u), v) \quad \forall v \in H^1(\Omega). \quad (5.9)$$

Furthermore, let z_ε and t be the solutions to the respective equations below:

$$z_\varepsilon \in H^1(\Omega), \quad a(z_\varepsilon, v) = (\mu_\varepsilon, \text{Tr}(v))_{\partial\Omega} \quad \forall v \in H^1(\Omega), \quad (5.10)$$

$$t \in H^1(\Omega), \quad a(t, v) = (f, v) \quad \forall v \in H^1(\Omega). \quad (5.11)$$

Hence, using the definition of the convex set M_ε and relations (5.8)-(5.11), problem (5.5) can be written as

$$\begin{cases} \text{Find } (u_\varepsilon, z_\varepsilon) \in H^1(\Omega) \times M_\varepsilon \text{ such that} \\ a(u_\varepsilon + l(u_\varepsilon) + z_\varepsilon + t, v) = 0 \quad \forall v \in H^1(\Omega), \\ a(w - z_\varepsilon, u_\varepsilon - \tilde{h}) \leq 0 \quad \forall w \in M_\varepsilon, \end{cases} \quad (5.12)$$

where $\tilde{h}$ is an element of $H^1(\Omega)$ such that $Tr(\tilde{h}) = h$ on $\partial\Omega$. Then based on (5.12), we deduce that

$$a(w - z_\varepsilon, -l(u_\varepsilon) - t - \tilde{h} - z_\varepsilon) \leq 0 \quad \forall w \in M_\varepsilon. \quad (5.13)$$

Which implies that

$$z_\varepsilon = \Pi_{M_\varepsilon}(-l(u_\varepsilon) - t - \tilde{h}), \quad (5.14)$$

where Π_{M_ε} denotes the projection operator onto M_ε with respect to the inner product $a(\cdot, \cdot)$. Hence, the solution $(u_\varepsilon, z_\varepsilon)$ of (5.12) can be described as:

$$\begin{cases} a(l(u_\varepsilon), v) = (g(u_\varepsilon), v) \quad \forall v \in H^1(\Omega), \\ z_\varepsilon = \Pi_{M_\varepsilon}(-l(u_\varepsilon) - t - \tilde{h}), \\ u_\varepsilon = -l(u_\varepsilon) - z_\varepsilon - t. \end{cases} \quad (5.15)$$

The solution $(u_\varepsilon, z_\varepsilon)$ in (5.15) can be approximated by an iterative method based on the combination of a projection and fixed-point methods.

Algorithm 5.3. (Projection/Fixed-point algorithm)

Step 0: Let $u_{\varepsilon,0}$ be an arbitrary function of $H^1(\Omega)$, and then proceed with the subsequent steps.

Step 1: Given $u_{\varepsilon,k} \in H^1(\Omega)$ for $k \geq 1$, calculate $l(u_{\varepsilon,k}) \in H^1(\Omega)$, which is the solution of:

$$a(l(u_{\varepsilon,k}), v) = (g(u_{\varepsilon,k}), v) \quad \forall v \in H^1(\Omega). \quad (5.16)$$

Step 2: Determine $z_{\varepsilon,k+1}$ and $u_{\varepsilon,k+1}$ as the solutions to the following equations:

$$z_{\varepsilon,k+1} = \Pi_{M_\varepsilon}(-l(u_{\varepsilon,k}) - t - \tilde{h}), \quad (5.17)$$

$$u_{\varepsilon,k+1} = -l(u_k) - z_{\varepsilon,k+1} - t. \quad (5.18)$$

Step 3: Repeat steps 1 and 2 until satisfying a stopping criterion.

To establish the convergence of the algorithm, we show that the function Φ defined on $H^1(\Omega)$ by:

$$\begin{aligned} \Phi : H^1(\Omega) &\rightarrow H^1(\Omega), \\ v &\mapsto -l(v) - \Pi_{M_\varepsilon}(-l(v) - t - \tilde{h}) - t, \end{aligned} \quad (5.19)$$

is a strict contraction, which implies that Φ admits a unique fixed point. Hence, for a given element u_0 in $H^1(\Omega)$, the iterative solution

$$u_{k+1} := \Phi(u_k), \quad k \geq 0, \quad (5.20)$$

converges to the fixed point of Φ .

Lemma 5.4. *Let $l : H^1(\Omega) \rightarrow H^1(\Omega)$ be the function defined by (5.8)-(5.9). We establish that l is a k -Lipschitz function on $H^1(\Omega)$:*

$$\|l(v_1) - l(v_2)\| \leq k\|v_1 - v_2\| \quad \forall v_1, v_2 \in H^1(\Omega),$$

with $0 < k < 1$ is the constant specified in assumption (2.2).

Proof. We find:

$$\begin{aligned} \|l(v_1) - l(v_2)\|^2 &= a(l(v_1) - l(v_2), l(v_1) - l(v_2)) \\ &= (g(v_1) - g(v_2), l(v_1) - l(v_2)) \quad (\text{by (5.9)}) \\ &\leq \|g(v_1) - g(v_2)\|_{L^2} \|l(v_1) - l(v_2)\|_{L^2} \\ &\leq k\|v_1 - v_2\| \|l(v_1) - l(v_2)\|, \end{aligned}$$

where $0 < k < 1$, in (2.2). Consequently, we get

$$\|l(v_1) - l(v_2)\| \leq k\|v_1 - v_2\|.$$

□

Using Lemma 5.4 we show that the function Φ given by (5.19) is a strict contraction and then it admits a unique fixed point, which can be approximated by the iteration (5.20).

Theorem 5.5. *Let $\Phi : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ be the function given in (5.19). Consequently, Φ has a unique fixed point. As a consequence, the iterative solutions of Algorithm 5.3 converge as follows: $u_{\varepsilon,k} \rightarrow u_\varepsilon$, $z_{\varepsilon,k} \rightarrow z_\varepsilon$, and $l(u_{\varepsilon,k}) \rightarrow l(u_\varepsilon)$, where u_ε , z_ε , and $l(u_\varepsilon)$ are solutions to problem (5.15).*

Proof. The proof follows a similar approach to the one presented in [15], Theorem 4.3.

□

To solve the projection problems (5.14) and (5.17), as well as to compute the functional μ_ε (or z_ε), we apply a projection algorithm proposed by Degueil in his doctoral thesis [8] (see also [1, 17]), where it was developed to solve the Stefan problem in terms of the temperature and the content of unfrozen material.

Consider the closed convex set M_ε in $H^1(\Omega)$ given by (5.6) and let s belongs to $H^1(\Omega)$. The orthogonal projection of s onto M_ε with respect to the inner product $a(\cdot, \cdot)$ of $H^1(\Omega)$, $z := \Pi_{M_\varepsilon}(s)$, can be obtained by Algorithm 5.6.

Algorithm 5.6. [8] (Projection algorithm)

Step 0: Choose any element μ_0 from the convex set C_ε defined by (5.3). Find $z_0 \in M_\varepsilon$ as the solution to equation (5.10) with $\mu_\varepsilon = \mu_0$.

Step 1: Given z_n for $(n \geq 0)$ in M_ε , construct w_n in M_ε such that

$$a(w - w_n, z_n - s) \geq 0 \quad \forall w \in M_\varepsilon. \quad (5.21)$$

Step 2: Compute

$$z_{n+1} = \Pi_{[z_n, w_n]}(s), \quad (5.22)$$

where $\Pi_{[z_n, w_n]}$ represents the projection onto the line segment $[z_n, w_n]$ defined as:

$$[z_n, w_n] := \{w \in H^1(\Omega) \mid w = \beta z_n + (1 - \beta)w_n, \beta \in [0, 1]\}.$$

Step 3: Repeat steps 1 and 2 until satisfying a stopping criterion.

The algorithm terminates with a sequence z_n that strongly converges to z in $H^1(\Omega)$. This sequence is characterized by the relations described in [8] as follows (see also [1, 17]):

$$\begin{cases} z_{n+1} &= \beta_n w_n + (1 - \beta_n)z_n, \\ \beta_n &= \min \left(1, \frac{a(z_n - w_n, z_n - s)}{\|w_n - z_n\|^2} \right). \end{cases}$$

Now, we will describe the construction of the element w_n in step 1 of Algorithm 5.6. According to the definition of the convex set M_ε given by (5.6), inequality (5.21) can be rewritten as follows:

$$\nu_n \in C_\varepsilon, \quad (\nu - \nu_n, Tr(z_n - s))_{\partial\Omega} \geq 0 \quad \forall \nu \in C_\varepsilon,$$

Lemma 5.1 implies that $\nu_n \in \partial\phi_\varepsilon(Tr(s - z_n))$. Hence w_n is the solution of the equation:

$$w_n \in H^1(\Omega), \quad a(w_n, v) = (\nu_n, Tr(v))_{\partial\Omega} \quad \forall v \in H^1(\Omega).$$

From Lemma 4.2 we may take

$$\nu_n(x) = -\frac{1}{\varepsilon} \chi_{[s - z_n < 0]_{\partial\Omega}} = \begin{cases} -\frac{1}{\varepsilon} & \text{if } s(x) - z_n(x) < 0 \text{ on } \partial\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.7. During each iteration k of Algorithm 5.3, we update s and z_n in Algorithm 5.6 with $-l(u_{\varepsilon, k}) - t - \tilde{h}$ and $(z_{k+1})_n := z_n^{k+1}$, respectively. In this case, the function ν_n is given by

$$\nu_n(x) = -\frac{1}{\varepsilon} \chi_{[-l(u_{\varepsilon, k}) - t - z_n^{k+1} < h]_{\partial\Omega}} = \begin{cases} -\frac{1}{\varepsilon} & \text{if } -l(u_{\varepsilon, k}) - t - z_n^{k+1}(x) < h(x) \text{ on } \partial\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

6. CONCLUSION

In this work, we have treated a semilinear unilateral obstacle problem on the boundary. We have proposed a numerical method to solve this problem by reformulating it as a set-valued equation problem, which can be considered as mixed formulation problem. This reformulation introduces a sub-differential μ_ε of a continuous function representing the contact domain, an unknown element of the problem. We have established that the reformulated problem is equivalent to a projection/fixed-point problem. We have demonstrated that the reformulated problem is equivalent to a projection/fixed-point problem. To approximate the solution numerically $(u_\varepsilon, \mu_\varepsilon)$, we have developed iterative algorithms that are computationally efficient. These algorithms involve solving linear systems and performing simple computations, making them easy to understand and implement.

Furthermore, we note that the reformulated problem can be approached numerically as a mixed formulation problem. In our future work, we plan to investigate the numerical analysis of the reformulated problem using the finite element method. This will involve approximating the problem by a sequence of finite-dimensional problems and studying the convergence properties of the numerical solutions.

Declarations of interest: none.

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