Abstract. We exploit calculations of the representations rings of linear group $GL_d$ and the special group $Sl_d$ [4] and the various morphisms between them to deduce morphisms between their associated completed rings. From [15], where Riou proves a motivic Atiyah isomorphism for the linear group $GL_d$ which generalises Atiyah isomorphism, for finite groups [1], between the completed representations ring and the ring of the K-theory of the classifying space, we prove a motivic Atiyah isomorphism for the special group $SL_d$.

1. Introduction and preliminaries

1.1. Representation theory of groups.

- To study differential equations, S. Lie creates Lie group theory and Lie algebra [12]
- The representations theory of Lie compact groups is used in the resolution of the heat equations [7] and so it’s considered as a vast generalisation of Fourier analysis.
- Frobenius inventes the theory of characters of finite groups, Shur develops the representation theory of finite and infinite groups. Cartan, Borel and Serre contribute to the development of representations theory of groups.
- Representation theory is used in theoretical physics.

1.2. I-adic completion of representations ring.

Atiyah in [1], is the first who defines I-adic completion of the representations ring:

- The $C$-representations ring $R_C(G)$ is augmented by $R_C(G)\rightarrow \mathbb{Z}$ sending each representation $[V]$ to its virtual dimension. We denote by $I(G)$ the kernel of the augmentation([4])
- Denote the $I(G)$-adic completion of $R_C(G)$ denoted by $\hat{R}_C(G)$, the inverse limit: $\lim_{\text{inv}} R_C(G)/I^n(G)$
1.3. Topological K-theory.
- Grothendieck defines for any additive category $C$ its Grothendieck group $K_0(C)$ as the $\mathbb{Z}$-module libre generated by $[V]$ quotiented by the relation $[V + U] = [V] + [U]$
- In order to prove Atiyah Singer indice theorem [5], Atiyah defines the topological K-theory group $K^0(X)$ of any compact space $X$ as the Grothendieck group of the category of vectoriel bundles over $X$ with the Whitney sum [1]
- The tensor product of vector bundles over $X$ gives $K^0(X)$ a commutative ring structure.
- Bott [6] proves that $K^0(X) = [X, BUXZ]$, where $U$ is the group of the unitary matrixes. Though $K^0(X)$ stays hard to be calculated.

In the particular case where $X = BG$ i-e $X$ is a classifying space of finite group $G$ and $EG \to BG$ is the $G$-universal bundle, Atiyah [1] makes a link between the $C$-representations ring of $G$ and the K-theory ring of $(BG)$ as described in his fundamental theorem:

**Theorem 1.1.** the natural morphism $\alpha^G: R_C(G) \to K^0(BG)$ sending to $[V]$ the vectoriel bundle $[EGX_GV]$ factors through the completed:

$$\hat{\alpha}^G: \hat{R}_C(G) \sim \to K^0(BG)$$

in an isomorphism.

This fundamental theorem has in topological K-theory many generalisations:
- Atiyah and Hirzebruch[2] extend the theorem 1.1 to compact connexe Lie groups.
- Segal proves, in [17] an analogue theorem for equivariant K-theory.

**Theorem 1.2.** Let $G$ be a Lie compact group acting freely on a contractile topological space $X$ and $X_G = XXE_G/G$ then the natural morphism $\alpha^G: K_G(X) \to K(X_G)$ factors through

$$\hat{\alpha}^G: \hat{K}_G(X) \sim \to K(X_G)$$

in an isomorphism

**Remark 1.3.** If we take $X = *$ in the theorem1.2, we get the theorem 1.1

In the following paragraphs we describe how Atiyah isomorphism was extended to other cohomological theories like, algebraic K-theory, orthogonal K-theory and homotopy theory of schemes.

1.4. Algebraic K-theory.
- D. Quillen [16] defines algebraic K-theory groups $K_n(R)$ of any unitary ring $R$ and $K_n(S)$ of schemes $S$ generalising Grothendieck group. Quillen defines the $+$ construction and the groups $K_n(R) = \pi_n((BGLR)^+XK_0(R))$, $K_0(R)$ is the Grothendieck group of the category of $R$-finite generated projectif modules and $\pi_n(X)$ is the $n$ homotopy group.
• D.Rector [15] defines for any ring $R$ and any group $G$ a natural morphism $\alpha^R: R_G(G) \to [BG, (BGLR)^+XK_0(R)]$. Using Adams operations in topological K-theory Rector proves from theorem 1.1:

**Theorem 1.4.** If $G$ is a finite group of order prime to $q$ and $F_q$ a finite field then: $\alpha^q: R_{F_q}(G) \to [BG, (BGL_{F_q})^+XZ]$ factors through the completed in an isomorphism

$$\hat{\alpha}^q: \hat{R}_{F_q} \sim \to [BG, BGL_{F_q})^+XZ]$$

• The theorem 1.4 is very useful to make calculations in algebraic K-theory. It was used heavily in [9] where the first author showed that the direct summand factor of the algebraic K-theory of non exceptional rings is canonical.

1.5. Orthogonal K-theory.

• Let $R$ be a commutative unitary ring, $O(R)$ its infinite orthogonal group and $BO(R)^+$ the plus construction relative to the $[O(R), O(R)]$. Karoubi defines the orthogonal K-theory groups $KO_n(R) = \pi_n((BO(R)^+XKO_0(R)))$, $KO_0(R)$ is the Grothendieck group of the category of $R$-finite quadratic generated projective modules

• For any group $G$, we define $R$ representation orthogonal $V$ of $G$ as a representation of $G$ such that the quadratic structure of $V$ is compatible with the $G$-action.

• with Bruno Kahn [10], the first author gives an analogue of the theorem 1.4 to the orthogonal K-theory.

**Theorem 1.5.** Let $G$ be a finite group and $F_q$ is a finite field such that cardinal $G$ is prime with $q$ : the natural morphism $\alpha^O^*: R_{O_q}(G) \to [BG, BO(F_q)^+XZ]/2$ factors

$$\hat{\alpha}^O*: \hat{R}_{O_q}/\text{torsion} \sim \to [BG, BO(F_q)^+XZ]$$

The theorem 1.5 is very useful to make calculations in orthogonal K-theory. It was used heavily in [8] where the first author constructed a direct summand factor of the orthogonal K-theory of exceptional rings.

1.6. Homotopy theory of schemes.

• Voevodsky and Morel in [13] define the homotopy theory of schemes, for any algebraic group $G$ the object $BG$ and the $K$-algebraic theory group $K_0(BG)$.

• Riou [15] in his thesis, about homotopy theory of schemes, constructs for any field $k$ and any $k$-linear algebraic group $G$: $\alpha^G: R_k(G) \to K_0(BG)$

In the particular case of the group $GL_k$, Riou proves "Atiyah motivic isomorphism":

**Theorem 1.6.** Let $k$ be a field and $GL_d(k)$ the linear algebraic group of order $d$, and $BGL_d$ its classifying space then the natural morphism

$$\alpha^{GL_d}: R_k(GL_d) \to K_0(BGL_d)$$

factors through the completed in an isomorphism $\hat{\alpha}^{GL_d}: R_k(GL_d) \sim \to K_0(BGL_d)$
In this article we prove using our representations rings calculations of $GL_d$, $SL_d$ [4] and their associated completed rings that theorem 1.6 is still true if we replace $GL_d$ by $SL_d$.

2. Atiyah motivic isomorphism for special group $SL_d$

In [4] we define:

$$k_d : GL_d \rightarrow GL_{d+1}$$

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

$$s_d : GL_d \rightarrow SL_{d+1}$$

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & \det A^{-1} \end{bmatrix}$$

**Proposition 2.1.** The morphism $s_d$ induces an isomorphism

$$s_d : RSL_{d+1} \sim RGL_d$$

**Proof.** See the proposition 4.1 in [4] \hfill \square

Hence:

**Corollary 2.2.** The morphism

$$\hat{s}_d : \hat{R}(GL_d) \sim \hat{R}(SL_{d+1})$$

is an isomorphism.

**Proposition 2.3.** The following diagram is commutatif:

$$\begin{array}{ccc}
\hat{R}(GL_{d+1}) & \xrightarrow{\hat{\alpha}_{d+1}^{GL}} & K_0(BGL_{d+1}) \\
\hat{k}_{d+1} & & \downarrow Bk_{d+1} \\
\hat{R}(SL_{d+1}) & \xrightarrow{\hat{\alpha}_{d+1}^{SL}} & K_0(BSL_{d+1}) \\
\hat{s}_d & & \downarrow Bs_d \\
\hat{R}(GL_d) & \xrightarrow{\hat{\alpha}_d^{GL}} & K_0(BGL_d) \\
\end{array}$$

where $\hat{\alpha}_{d+1}^{GLd}$, $\hat{\alpha}_d^{GLd}$ and $\hat{s}_d$ are isomorphisms.

**Proof.**

- By the fonctoriality, the digramm is commutatif
- the morphisms $\hat{\alpha}_{d+1}^{GLd}$, $\hat{\alpha}_d^{GLd}$ are isomorphisms from Riou’s theorem
- the morphisms $\hat{s}_d$ is an isomorphism \hfill \square

**Corollary 2.4.** The morphism

$$\hat{\alpha}_{d+1} : \hat{R}(SL_{d+1}) \rightarrow K_0(BSL_{d+1})$$

is a monomorphism.
Proof. We have $\hat{\alpha}_d$, $\hat{\alpha}_{d+1}$, $\hat{s}_d$ isomorphisms and $\hat{\alpha}_d \circ \hat{s}_d = B\hat{s}_d^* \circ \hat{\alpha}_{d+1}$. □

**Theorem 2.5.** The morphism

$$\hat{\alpha}_{d+1}^{SLd}: \hat{R}(SL_{d+1}) \simto K_0(BSL_{d+1})$$

is an isomorphism.

**Proof.**

- : The morphism $\hat{\alpha}_{d+1}$ is injective by the previous corollary.
- To show that the morphism $\hat{\alpha}_d^{SLd}$ is surjective, we have from [13] $BGL_d = \lim\limits_{\longrightarrow} U_n/GL_d$ and $BSL_d = \lim\limits_{\longrightarrow} U_n/SL_d$ where $U_n$ is an open subscheme of $\mathbb{A}_d^{nd}$ with $GL_d$ free action.

Denote $G_m$ the multiplicative group $GL_1$.

**Lemma 2.6.** $U_n/SL_d \rightarrow U_n/GL_d$ is a $G_m$-torsor

**Corollary 2.7.** There exists a long exact sequence:

$$\ldots \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/SL_d) \rightarrow 0$$

the morphism $K_0(U_n/GL_d) \rightarrow K_0(U_n/GL_d)$ is denoted by $t_*$. 

**Lemma 2.8.** The previous long exact sequence gives an exact sequence of projective system in $n$:

$$0 \rightarrow K_0(U_n/GL_d)/\ker(t_*) \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/SL_d) \rightarrow 0$$

As $\lim^1 K_0(U_n/GL_d) = 0$ because $K_0(U_{n+1}/GL_d) \rightarrow K_0(U_n/GL_d)$ is surjective.

**Lemma 2.9.** We have the following commutative diagram:

$$R(SL_d) \rightarrow K_0(BSL_d)$$

$$\begin{array}{c}
\uparrow \\
R(GL_d)/(1-\text{det}) \rightarrow K_0(BGL_d)/(1-\text{det})
\end{array}$$

**Proposition 2.10.** After completing we get the following commutative diagram:

$$\begin{array}{c}
R_k(GL_d)/(1-\text{det}) \rightarrow K_0(BGL_d)/(1-\text{det}) \\
\downarrow \\
R_k(SL_d) \rightarrow K_0(BSL_d)
\end{array}$$

Denote $v : R(GL_d)/(1-\text{det}) \rightarrow R(SL_d)$ the isomorphism and $g : K_0(BGL_d)/(1-\text{det}) \rightarrow K_0(BSL_d)$ the surjection. The morphism $\hat{\alpha}_{SL} : R_k(SL_d) \rightarrow K_0(BSL_d)$ is surjective. □

**Acknowledgement.** The authors would like to thank Tom De Liso, Pierre Vogel, Christophe Deninger, Joel Riou, Perre Pascual and Fabien Morel for the interesting discussions.
ATIYAH MOTIVIC THEOREM FOR THE SPECIAL GROUP $SL_d$

**References**

3. K.Azi , théorèmes d Atiyah motiviques, thèse nationale, 2014, faculète des sciences 1, universite hassan II
4. K.Azi, H. Hamraoui Calculs d anneaux de representations des groupes $GL_d$ et $SL_d$, communication au colloque international de Taza/Maroc
7. M. Edraoui, H.Hamraoui, Representations de groupes SU(2) et Equations de la chaleur, PFE, juin 2016 , Universite Hassan II
12. Kosmann- Schwarzbach: Groups and symetries. Springer

1 Department of Mathematics, University Hassan II,Csablancai, Morocco

E-mail address: hindahamraoui@yahoo.fr

2 CRESC, EGE, Rabat, Morocco.

E-mail address: azi_macquart@hotmail.com