

ATIYAH MOTIVIC THEOREM FOR THE SPECIAL GROUP SL_d

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ABSTRACT. We exploit calculations of the representations rings of linear group GL_d and the special group SL_d [4] and the various morphisms between them to deduce morphisms between their associated completed rings. From [15], where Riou proves a motivic Atiyah isomorphism for the linear group GL_d wich generalises Atiyah isomorphism, for finite groups [1], between the completed representations ring and the ring of the K-theory of the classifying space, we prove a motivic Atiyah isomorphism for the special group SL_d .

1. INTRODUCTION AND PRELIMINARIES

1.1. Representation theory of groups.

- To study differential equations, S. Lie creates Lie group theory and Lie algebra [12]
- The representations theory of Lie compact groups is used in the resolution of the heat equations [7] and so it's considered as a vast generalisation of Fourier analysis.
- Frobenius inventes the theory of characters of finite groups, Shur developes the representation theory of finite and infinite groups. Cartan, Borel and Serre contribue to the developement of representations theory of groups.
- Representation theory is used in theoretical physics.

1.2. I-adic completion of representations ring.

Atiyah in [1], is the first who defines I-adic completion of the representations ring:

- The C -representations ring $R_C(G)$ is augmented by $R_C(G) \rightarrow Z$ sending each representation $[V]$ to its virtuel dimension. We denote by $I(G)$ the kernel of the augmentation([4])
- Denote the $I(G)$ -adic completion of $R_C(G)$ denoted by $\hat{R}_C(G)$, the inverse limit: $\lim_{inv} R_C(G)/I^n(G)$

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1.3. Topological K-theory.

- Grothendieck defines for any additive category C its Grothendieck group $K_0(C)$ as the $\mathbb{Z}$ -module libre generated by $[V]$ quotiented by the relation $[V + U] = [V] + [U]$
- In order to prove Atiyah Singer indice theorem [5], Atiyah defines the topological K-theory group $K^0(X)$ of any compact space X as the Grothendieck group of the category of vectoriel bundles over X with the Whitney sum [1]
- The tensor product of vector bundles over X gives $K^0(X)$ a commutative ring structure.
- Bott [6] proves that $K^0(X) = [X, BU \times \mathbb{Z}]$, where U is the group of the unitary matrixes. Though $K^0(X)$ stays hard to be calculated.

In the particular case where $X = BG$ i-e X is a classifying space of finite group G and $EG \rightarrow BG$ is the G -universal bundle, Atiyah [1] makes a link between the C -representations ring of G and the K-theory ring of (BG) as described in his fondamental theorem:

Theorem 1.1. *the natural morphism $\alpha^G: R_C(G) \rightarrow K^0(BG)$ sending to $[V]$ the vectoriel bundle $[EG \times_G V]$ factors through the completed:*

$$\hat{\alpha}^G: \hat{R}_C(G) \xrightarrow{\sim} K^0(BG)$$

in an isomorphism.

This fondamental theorem has in topological K-theory many generalisations:

- Atiyah and Hirzebruch[2] extend the theorem 1.1 to compact connexe Lie groups.
- Segal proves, in [17] an analogue theorem for equivariant K -theory.

Theorem 1.2. *Let G be a Lie compact group acting freely on a contractile topological space X and $X_G = X \times EG / G$ then the natural morphism $\alpha_G: K_G(X) \rightarrow K(X_G)$ factors through*

$$\hat{\alpha}^G: \hat{K}_G(X) \xrightarrow{\sim} K(X_G)$$

in an isomorphism

Remark 1.3. *If we take $X = *$ in the theorem 1.2, we get the theorem 1.1*

In the following paragraphs we describe how Atiyah isomorphism was extended to other cohomological theories like, algebraic K-theory, orthogonal K-theory and homotopy theory of schemes.

1.4. Algebraic K-theory.

- D. Quillen [16] defines algebraic K-theory groups $K_n(R)$ of any unitary ring R and $K_n(S)$ of schemes S generalising Grothendieck group. Quillen defines the $+$ construction and the groups $K_n(R) = \pi_n((BGLR)^+ \times K_0(R))$, $K_0(R)$ is the Grothendieck group of the category of R -finite generated projectif modules and $\pi_n(X)$ is the n homotopy group.

- D.Rector [15] defines for any ring R and any group G a natural morphism $\alpha^R: R_R(G) \rightarrow [BG, (BGLR)^+ XK_0(R)]$. Using Adams operations in topological K-theory Rector proves from theorem 1.1:

Theorem 1.4. *If G is a finite group of order prime to q and F_q a finite field then: $\alpha^q: R_{F_q}(G) \rightarrow [BG, (BGLF_q)^+ XZ]$ factors through the completed in an isomorphism*

$$\hat{\alpha}^q: \hat{R}_{F_q} \xrightarrow{\sim} [BG, BGL(F_q)^+ XZ]$$

- The theorem 1.4 is very useful to make calculations in algebraic K-theory. It was used heavily in [9] where the first author showed that the direct summand factor of the algebraic K-theory of non exceptionnal rings is canonical.

1.5. Orthogonal K-theory.

- Let R be a commutatif unitary ring, $O(R)$ its infinite orthogonal group and $BO(R)^+$ the plus construction relative to the $[O(R), O(R)]$. Karoubi defines the orthogonal K-theory groups $KO_n(R) = \pi_n((BO(R)^+ XKO_0(R)))$, $KO_0(R)$ is the Grothendieck group of the category of R -finite quadratic generated projectif modules
- For any group G , we define R representation orthogonal V of G as a representation of G such that the quadratic structure of V si compatible of the G -action.
- with Bruno kahn [10], the first author gives an analogue of the theorem 1.4 to the orthogonal K-theory.

Theorem 1.5. *Let G be a finite group and F_q is a finite field such that cardinal G is prime with q : the natural morphism $\alpha O^q: RO_{F_q} \rightarrow [BG, BO(F_q)^+ XZXZ/2]$ factors*

$$\hat{\alpha} O^q: \widehat{RO}_{F_q}/torsion \xrightarrow{\sim} [BG, BO(F_q)^+ XZ] \text{ on an isomorphism}$$

The theorem 1.5 is very useful to make calculations in orthogonal K-theory. It was used heavily in [8] where the first author constructed a direct summand factor of the orthogonal K-theory of exceptionnal rings.

1.6. Homotopy theory of schemes.

- Veovodsky and Morel in [13] define the homotopy theory of schemes, for any algebraic group G the object BG and the K- algebraic theory group $K_0(BG)$.
- Riou[15] in his thesis, about homotopy theory of shemes, constructs for any field k and any k - linear algebraic group G : $\alpha^G: R_k(G) \rightarrow K_0(BG)$

In the particular case of the group GL_k , Riou proves "Atiyah motivic isomorphism":

Theorem 1.6. *Let k be a field and $GL_d(k)$ the linear algebraic group of order d , and BGL_d its classifying space then the natural morphism*

$$\alpha^{GL_d}: R_k(GL_d) \rightarrow K_0(BGL_d) \text{ factors through the completed in an isomorphism } \hat{\alpha}^{GL_d}: R_k(GL_d) \xrightarrow{\sim} K_0(BGL_d)$$

In this article we prove using our representations rings calculations of GL_d , SL_d [4] and their associated completed rings that theorem 1.6 is still true if we replace GL_d by SL_d .

2. ATIYAH MOTIVIC ISOMORPHISM FOR SPECIAL GROUP SL_d

In [4] we define:

$$k_d : GL_d \longrightarrow GL_{d+1}$$

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

$$s_d : GL_d \longrightarrow SL_{d+1}$$

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & \det A^{-1} \end{bmatrix}$$

Proposition 2.1. *The morphism s_d induces an isomorphism*

$$s_d : RSL_{d+1} \xrightarrow{\sim} RGL_d$$

Proof. See the proposition 4.1 in [4] □

Hence:

Corollary 2.2. *The morphism*

$$\hat{s}_d : \hat{R}(GL_d) \xrightarrow{\sim} \hat{R}(SL_{d+1})$$

is an isomorphism.

Proposition 2.3. *The following diagram is commutatif:*

$$\begin{array}{ccc} \hat{R}(GL_{d+1}) & \xrightarrow{\hat{\alpha}_{d+1}^{GL}} & K_0(BGL_{d+1}) \\ \hat{k}_{d+1} \downarrow & & Bk_{d+1}^* \downarrow \\ \hat{R}(SL_{d+1}) & \xrightarrow{\hat{\alpha}_{d+1}^{SL}} & K_0(BSL_{d+1}) \\ \hat{s}_d \downarrow & & Bs_d^* \downarrow \\ \hat{R}(GL_d) & \xrightarrow{\hat{\alpha}_d^{GL}} & K_0(BGL_d) \end{array}$$

where $\hat{\alpha}_{d+1}^{GL}$, $\hat{\alpha}_d^{GL}$ and $\hat{s}_d$ are isomorphisms.

Proof.

- By the functoriality, the digramm is commutatif
- the morphisms $\hat{\alpha}_{d+1}^{GL}$, $\hat{\alpha}_d^{GL}$ are isomorphisms from Riou's theorem
- the morphisms $\hat{s}_d$ is an isomorphism

□

Corollary 2.4. *The morphism*

$$\hat{\alpha}_{d+1} : \hat{R}(SL_{d+1}) \longrightarrow K_0(BSL_{d+1})$$

is a monomorphism.

Proof. We have $\hat{\alpha}_d$, $\hat{\alpha}_{d+1}$, $\hat{s}_d$ isomorphisms and $\hat{\alpha}_d \circ \hat{s}_d = Bs_d^* \circ \hat{\alpha}_{d+1}$

□

Theorem 2.5. *The morphism*

$$\hat{\alpha}_{d+1}^{SL_d} : \widehat{R}(SL_{d+1}) \xrightarrow{\sim} K_0(BSL_{d+1})$$

is an isomorphism

Proof. • : The morphism $\hat{\alpha}_{d+1}$ is injectif by the previous corollary.

- To show that the morphism $\hat{\alpha}_{d+1}^{SL_d}$ is surjectif, we have from [13] $BGL_d = \varinjlim U_n/GL_d$ and $BSL_d = \varinjlim U_n/SL_d$ where U_n is an open subscheme of $\mathbb{A}^{nd}$ with GL_d free action.

Denote G_m the multiplicatif group GL_1 .

Lemma 2.6. $U_n/SL_d \rightarrow U_n/GL_d$ is a G_m -torsor

Corollary 2.7. *There exists a long exact sequence:*

$$\dots \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/SL_d) \rightarrow 0$$

the morphism $K_0(U_n/GL_d) \rightarrow K_0(U_n/GL_d)$ is denoted by t_* .

Lemma 2.8. *The previous long exact sequence*

gives an exact sequence of projectif system in n :

$$0 \rightarrow K_0(U_n/GL_d)/\ker(t_*) \rightarrow K_0(U_n/GL_d) \rightarrow K_0(U_n/SL_d) \rightarrow 0$$

As $\lim^1 K_0(U_n/GL_d) = 0$ because $K_0(U_{n+1}/GL_d) \rightarrow K_0(U_n/GL_d)$ is surjectif

Lemma 2.9. *We have the following commutatif diagram :*

$$\begin{array}{ccc} R(SL_d) & \longrightarrow & K_0(BSL_d) \\ \uparrow & & \uparrow \\ R(GL_d)/(1 - \det) & \longrightarrow & K_0(BGL_d)/(1 - \det) \end{array}$$

Proposition 2.10. *After completing we get the following commutatif diagram :*

$$\begin{array}{ccc} \widehat{R_k(GL_d)/(1 - \det)} & \xrightarrow{h} & K_0(BGL_d)/(1 - \det) \\ \downarrow & & \downarrow \\ \widehat{R_k(SL_d)} & \xrightarrow{\hat{\alpha}^{SL}} & K_0(BSL_d) \end{array}$$

Denote $v : \widehat{R_k(GL_d)/(1 - \det)} \rightarrow \widehat{R_k(SL_d)}$ the isomorphism and $g : K_0(BGL_d)/(1 - \det) \rightarrow K_0(BSL_d)$ the surjection. The morphism $\hat{\alpha}^{SL} : \widehat{R_k(SL_d)} \rightarrow K_0(BSL_d)$ is surjectif.

□

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