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SOME RESULTS OF NEW DEFINITION OF $\mathcal{N}_F^\alpha$-FOURIER TRANSFORM AND THEIR APPLICATIONS

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Abstract. Based on a new definition of fractional derivative, called $\mathcal{N}_F^\alpha$-derivative, we introduce a new definition and present some results on N_α^F-periodic functions and $\mathcal{N}_F^\alpha$-Fourier transform related to this type of functions. We also use these results to study the $\mathcal{N}_F^\alpha$-partial differential equation $\mathcal{N}_F^\alpha x(t) = Ax(t) + \mathcal{I}_F^\alpha x(t) + f(t)$, where A is a linear closed operator on a Banach space X.

1. Introduction

The N_F^α-derivative operator $\mathcal{N}_F^\alpha(f)(t) := \lim_{h\to 0} \frac{f(t+\frac{h}{F(t,\alpha)})-f(t)}{h}$, $F(t,\alpha)\neq 0$, for all $t\in[0,+\infty)$, has been recently introduced in [5]. This definition generalizes the definitions given earlier in [13], defined by $T^{(\alpha)}f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\varepsilon})-f(t)}{\varepsilon}$.

Motivated by the fact that functional differential equations arise in many areas of applied mathematics, this type of equations has received much attention in recent years. We refer the readers to [6]. In particular, the existence problem of periodic solutions has been considered by several authors, as one can see in [2, 1], and the references therein.

In this work, we define the $\mathcal{N}_F^\alpha$-Fourier transform, and use it to study the existence of $\mathcal{N}_F^\alpha$-periodic solutions for the $\mathcal{N}_F^\alpha$-differential equation

$$\mathcal{N}_F^\alpha x(t) = Ax(t) + \mathcal{I}_F^\alpha x(t) + f(t), (0<\alpha\leq 1) \tag{1.1}$$

with the periodic condition $x(p) = x(0)$.

This work is organized as follows: in Section 2, we collect some preliminary results and definitions related to $\mathcal{N}_F^\alpha$-derivative. In section 3, we prove some results about $\mathcal{N}_F^\alpha$-periodic functions. Then we provide some findings and a novel definition of some integral transforms, which are important for the rest of our workt. In section 5, we study the existence and uniqueness of $\mathcal{N}_F^\alpha$-periodic strong solutions of (1.1) solely in terms of a property of R-boundedness for the sequence of operators $\{\delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})^{-1},\ k\in\mathbb{Z}\}$.

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2. Fundamental definitions

The definition of the $\mathcal{N}_F^\alpha$-derivative, and its significant characteristics are presented in this section.

Definition 2.1. [5] Assume that $f : [0,+\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$. If $F(\cdot,\alpha)$ is a function defined on $[0,+\infty)$ such that $F(t,\alpha) \neq 0$, for all $t \in [0,+\infty)$, we define the $\mathcal{N}_F^\alpha$-derivative of the function f of order α at the point t by

$$\mathcal{N}_F^\alpha(f)(t) := \lim_{h\to 0} \frac{f(t + \frac{h}{F(t,\alpha)}) - f(t)}{h}.$$

If this limit exists, we say that f is $\mathcal{N}_F^\alpha$-differentiable at t.

For the rest of the paper, F is a function as in the above definition, and $\alpha \in (0,1]$.

Theorem 2.2. [5] *Let $\alpha \in (0,1]$ and let f,g be $\mathcal{N}_F^\alpha$-differentiable at a point $t > 0$. If a,b are scalars, then*

(1) $\mathcal{N}_F^\alpha(af + bg)(t) = a\mathcal{N}_F^\alpha(f)(t) + b\mathcal{N}_F^\alpha(g)(t)$.
(2) $\mathcal{N}_F^\alpha(a) = 0$.
(3) $\mathcal{N}_F^\alpha(fg)(t) = \mathcal{N}_F^\alpha(f)(t)g(t) + f(t)\mathcal{N}_F^\alpha(g)(t)$.
(4) $\mathcal{N}_F^\alpha(\frac{f}{g})(t) = \frac{g(t)\mathcal{N}_F^\alpha(f)(t) - f(t)\mathcal{N}_F^\alpha(g)(t)}{g^2(t)}$.
(5) *If, in addition, f is differentiable then* $\mathcal{N}_F^\alpha(f)(t) = \frac{f'(t)}{F(t,\alpha)}$.

Example 2.3. For $F(t,\alpha) = 3\alpha^2t^2 + 2\alpha t + 1$, we have

$$\mathcal{N}_F^\alpha\left(3\alpha^2\frac{t^5}{5} + \frac{\alpha t^4}{2} + (1+6\alpha^2)\frac{t^3}{3} + 2\alpha t^2 + 2t\right) = t^2 + 2.$$

Definition 2.4. Let $0 < \alpha \le 1$ and $f : [0,+\infty) \to \mathbb{R}$.

(1) We say that f is $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty)$ if it is $\mathcal{N}_F^\alpha-$differentiable at each $t \in (0,\infty)$, and $\mathcal{N}_F^\alpha f(0) = \lim_{t\to 0^+} \mathcal{N}_F^\alpha f(t)$ exists.
(2) We say that f is continuously $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty)$ if f is $\mathcal{N}_F^\alpha$-differentiable and $\mathcal{N}_F^\alpha f(t)$ is continuous.

Definition 2.5. Let $0 < \alpha \le 1, n \in \mathbb{N}$ and $f : [0,+\infty) \to \mathbb{R}$.

(1) We say that f is n times $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty)$ if, $\forall j \in \{0,...n\}$, $\mathcal{N}_F^{(j\alpha)} f(t) = \mathcal{N}_F^\alpha(\mathcal{N}_F^\alpha ...(\mathcal{N}_F^\alpha(f)))(t)$, j times, exist for all $t \in (0,+\infty)$, and $\mathcal{N}_F^\alpha f(0) = \lim_{t\to 0^+} \mathcal{N}_F^\alpha f(t)$ exists.
(2) We say that f is n times continuously $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty)$, if f is n times $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty)$ and, $\forall j \in \{0,...n\}$, $\mathcal{N}_F^\alpha f(t)$ is continuous on $[0,+\infty)$.

Definition 2.6. Let f and F be as above, and let $\alpha \in (0,1]$. The $\mathcal{N}_F^\alpha$-integral of f is defined by

$$\mathcal{I}_F^\alpha f(t) = \int_0^t F(s,\alpha) f(s)ds, \quad t \in [0,+\infty).$$

Example 2.7. For $F(t,\alpha)=3\alpha^2t^2+\alpha t+1,\ t\in[0,+\infty)$ and $\alpha\in(0,1]$, we have

$$\mathcal{I}_F^\alpha(t^2+1)=3\alpha^2\frac{t^5}{5}+\alpha\frac{t^4}{4}+(3\alpha^2+1)\frac{t^3}{3}+\alpha\frac{t^2}{2}+t.$$

Lemma 2.8. [5] *For any function $f:[0,+\infty)\to\mathbb{R}$,*

$$\mathcal{N}_F^\alpha(\mathcal{I}_F^\alpha f(t))=f(t),\ t>0.$$

Example 2.9. For $F(t,\alpha)=2\alpha t+1$, we have by Example 2.7 and Theorem 2.2 that

$$\mathcal{N}_F^\alpha(\mathcal{I}_F^\alpha(t^2+1))=\frac{\left(3\alpha^2\frac{t^5}{5}+\alpha\frac{t^4}{4}+(3\alpha^2+1)\frac{t^3}{3}+\alpha\frac{t^2}{2}+t\right)'}{3\alpha^2t^2+\alpha t+1}=t^2+1,$$

for all $t>0$.

Lemma 2.10. [5] *With the same parameters as above, if f is $\mathcal{N}_F^\alpha$-differentiable, then*

$$\mathcal{I}_F^\alpha(\mathcal{N}_F^\alpha f(t))=f(t)-f(0),\ t>0.$$

Example 2.11. Consider the function defined in Example 2.7. By Theorem 2.2

$$\mathcal{I}_F^\alpha(\mathcal{N}_F^\alpha f(t))=\mathcal{I}_F^\alpha\left(\frac{2t}{3\alpha^2t^2+\alpha t+1}\right)=\int_0^t 2sds=t^2=f(t)-f(0),$$

where $f(t)=t^2+1$.

3. Special functions

Definition 3.1. Let f,F,α be as discussed earlier, and assume in addition that $F\geq 0$. Let $G_\alpha(t)=\int_0^t F(\tau,\alpha)d\tau$ be a primitive of F. If there is a continuous function $g:[0,+\infty)\to\mathbb{R}$ such that

$$f(t)=g(G_\alpha(t))=g(G_\alpha(t)+G_\alpha(p))$$

then the function f is termed $\mathcal{N}_F^\alpha$ p-periodic.

Remark 3.2. The function $g(t)=f(G_\alpha^{-1}(t))$ is $G_\alpha(p)$-periodic.

Example 3.3. Consider

$$f(t)=\begin{cases}\alpha t^2+2\sqrt{\alpha}t, & 0\leq t<\frac{1}{\alpha}-\frac{1}{\sqrt{\alpha}}\\[2ex] (\alpha-1)t^2+(2\sqrt{\alpha}-\frac{2}{\sqrt{\alpha}})t+\frac{1}{\alpha^2}-\frac{1}{\alpha}, & \frac{1}{\alpha}-\frac{1}{\sqrt{\alpha}}\leq t<\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1}-1)\end{cases},$$

and $F(t,\alpha)=2\alpha t+2\sqrt{\alpha}$. Then $f(t)=g(G_\alpha(t))$, where $G_\alpha(t)=\alpha t^2+2\sqrt{\alpha}t$ and

$$g(t)=\begin{cases}t, & 0\leq t<\frac{1}{\alpha}-1\\[2ex] (1-\frac{1}{\alpha})t+\frac{1}{\alpha^2}-\frac{1}{\alpha}, & \frac{1}{\alpha}-1\leq t<\frac{1}{\alpha}\end{cases}.$$

It is clear that g is continuous, and $\frac{1}{\alpha}$-periodic. Consequently, f is $\mathcal{N}_F^\alpha$ $\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1}-1)$-periodic. .

Theorem 3.4. *For $F \geq 0$, and f, α, G_α as above, if f is $\mathcal{N}_F^\alpha$ p-periodic that satisfies*

$$\mathcal{I}_F^\alpha f(p) = \int_0^{G_\alpha(p)} f(G_\alpha^{-1}(u))du = 0, \tag{3.1}$$

then $\mathcal{I}_\alpha f(t)$ is $\mathcal{N}_F^\alpha$ p-periodic.

Proof. We substitute $u = G_\alpha(s)$, and we use Definition 2.6 to obtain

$$\mathcal{I}_F^\alpha f(t) = \int_0^t F(s,\alpha) f(s) ds = \int_0^{G_\alpha(t)} f(G_\alpha^{-1}(u))du = g_1(G_\alpha(t)), \tag{3.2}$$

where

$$g_1(t) = \int_0^t f(G_\alpha^{-1}(u))du.$$

Alternatively, if g is as in Definition 3.1, we have

$$\begin{aligned} g_1(G_\alpha(t) + G_\alpha(p)) &= \int_0^{G_\alpha(t)+G_\alpha(p)} f(G_\alpha^{-1}(u))du \\ &= \int_0^{G_\alpha(p)} f(G_\alpha^{-1}(u))du + \int_{G_\alpha(p)}^{G_\alpha(t)+G_\alpha(p)} g(u)du \\ &= \int_0^{G_\alpha(t)} g(s + G_\alpha(p))ds = \int_0^{G_\alpha(t)} g(s)ds \\ &= g_1(G_\alpha(t)). \end{aligned}$$

Therefore, g_1 is $G_\alpha(p)$-periodic, and hence, $\mathcal{I}_\alpha f(t)$ is $\mathcal{N}_F^\alpha$ p-periodic. □

Example 3.5. 1. Consider

$$f_1(t) = \begin{cases} \alpha t^2 + 2\sqrt{\alpha} t, & 0 \leq t < \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1} - 1) \\ -\alpha t^2 - 2\sqrt{\alpha} t + \frac{1}{2\alpha}, & \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1} - 1) \leq t < \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{3}{4\alpha}+1} - 1) \\ \alpha t^2 + 2\sqrt{\alpha} t - \frac{1}{\alpha}, & \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{3}{4\alpha}+1} - 1) \leq t < \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1} - 1) \end{cases}.$$

We have $f_1(t) = g(G_\alpha(t))$, where $G_\alpha(t) = \alpha t^2 + 2\sqrt{\alpha} t$ and

$$g(t) = \begin{cases} t, & 0 \leq t < \frac{1}{4\alpha} \\ \frac{1}{2\alpha} - t, & \frac{1}{4\alpha} \leq t < \frac{3}{4\alpha} \\ t - \frac{1}{\alpha}, & \frac{3}{4\alpha} \leq t < \frac{1}{\alpha} \end{cases}$$

for all $t \in [0, \frac{1}{\alpha}]$.

The function f is α $\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1} - 1)$-periodic and by (3.2)

$$\mathcal{I}_F^\alpha f_1(t) = \int_0^{G_\alpha(t)} g(u)du$$

$$\mathcal{I}_F^{\alpha} f_1(t) = \begin{cases} \int_0^{\alpha t^2+2\sqrt{\alpha}t} s ds, \quad 0 \le t < r_1 \\ \\ \int_0^{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1}-1)} s ds + \int_{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1}-1)}^{\alpha t^2+2\sqrt{\alpha}t} (\frac{1}{2\alpha} - s) ds, \\ r_1 \le t < r_2 \\ \\ \int_0^{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1}-1)} s ds + \int_{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1}-1)}^{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{3}{4\alpha}+1}-1)} (\frac{1}{2\alpha} - s) ds + \int_{\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{3}{4\alpha}+1}-1)}^{\alpha t^2+2\sqrt{\alpha}t} (s - \frac{1}{\alpha}) ds, \\ r_2 \le t < r_3 \end{cases}$$

then

$$I_\alpha f_1(t) = \begin{cases} \frac{1}{2}(\alpha t^2 + 2\sqrt{\alpha}t)^2, \quad 0 \le t < r_1 \\ \\ -\frac{1}{2}(\alpha t^2 + 2\sqrt{\alpha}t)^2 + \frac{1}{2\alpha}(\alpha t^2 + 2\sqrt{\alpha}t) + \frac{(\sqrt{4\alpha+1}-2\sqrt{\alpha})(\sqrt{4\alpha+1}-2\sqrt{\alpha}-1)}{4\alpha^2}, \\ r_1 \le t < r_2 \\ \\ \frac{1}{2}(\alpha t^2 + 2\sqrt{\alpha}t)^2 - \frac{1}{\alpha}(\alpha t^2 + 2\sqrt{\alpha}t) + A, r_2 \le t < r_3 \end{cases} \tag{3.3}$$

where

$$r_1 = \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{4\alpha}+1}-1), r_2 = \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{3}{4\alpha}+1}-1), r_3 = \frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1}-1)$$

and

$$A = \frac{-4\sqrt{\alpha} - 4\sqrt{\alpha}\sqrt{4\alpha+1} - \sqrt{4\alpha+1} + 4\sqrt{\alpha}\sqrt{4\alpha+3} + 3\sqrt{4\alpha+3} - 2}{4\alpha^2}$$

and

$$g(t) = I_\alpha f_1((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} \frac{1}{2}t^2, \quad 0 \le t \le \frac{1}{4\alpha} \\ \\ -\frac{1}{2}t^2 + \frac{1}{2\alpha}t + \frac{((4\alpha+1)^{\frac{1}{2}} - 2(\alpha)^{\frac{1}{2}})((4\alpha+1)^{\frac{1}{2}} - 2\alpha^{\frac{1}{2}} - 1)}{4\alpha^2}, \\ \frac{1}{4\alpha} < t \le \frac{3}{4\alpha} \\ \\ \frac{1}{2}t^2 - \frac{1}{\alpha}t + A, \quad \frac{3}{4\alpha} < t \le \frac{1}{\alpha} \end{cases} \tag{3.4}$$

We have $I_\alpha f_1(\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1}-1)) \neq 0$, then $I_\alpha f_1$ isn't α-periodic function with period $\frac{1}{\sqrt{\alpha}}(\sqrt{\frac{1}{\alpha}+1}-1)$.

2. Now, let consider $\alpha = \frac{1}{2}, G_{\frac{1}{2}}(t) = 2\sqrt{t}$ and the function $f_2(t)$ defined by

$$f_2(t) = \begin{cases} 2\sqrt{t}, \quad 0 \le t < \frac{1}{4} \\ \\ 2 - 2\sqrt{t}, \ \frac{1}{4} \le t < \frac{9}{4} \\ \\ 2\sqrt{t} - 4, \frac{9}{4} \le t < 4 \end{cases} \tag{3.5}$$

we have

$$I_{\frac{1}{2}}f_2(t)=\begin{cases}\int_0^{2\sqrt{t}} sds, \quad 0\le t<\frac{1}{4}\\ \int_0^{\frac{1}{4}} sds+\int_{\frac{1}{4}}^{2\sqrt{t}}(2-s)ds, \ \frac{1}{4}\le t<\frac{9}{4}\\ \int_0^{\frac{1}{4}} sds+\int_{\frac{1}{4}}^{\frac{9}{4}}(2-s)ds+\int_{\frac{9}{4}}^{2\sqrt{t}}(s-4)ds, \frac{9}{4}\le t<4\end{cases}$$

then

$$I_{\frac{1}{2}}f_2(t)=\begin{cases}2t, \quad 0\le t<\frac{1}{4}\\ -1+4\sqrt{t}-2t, \quad \frac{1}{4}\le t<\frac{9}{4}\\ 2t-8\sqrt{t}+8, \quad \frac{9}{4}\le t<4\end{cases} \tag{3.6}$$

we have :

$$I_{\frac{1}{2}}f_2(4)=2\times 4-8\times\sqrt{4}+8=0.$$

The condition 4.1 is satisfied, then the function $I_{\frac{1}{2}}f_2$ is $\frac{1}{2}$-periodic function with period 4.

Theorem 3.6. *Let f, F, α, G_α be as above. If f is continuously $\mathcal{N}_F^\alpha$-differentiable $\mathcal{N}_F^\alpha$ p-periodic function, and g is as in Definition 3.1, then*

(1) $\mathcal{N}_F^\alpha f(t)=g'(G_\alpha(t))$.
(2) $\mathcal{N}_F^\alpha f$ *is* $\mathcal{N}_F^\alpha$ *p-periodic.*

Proof. By Definition 3.1, g satisfies

$$f(t)=g(G_\alpha(t))=g(G_\alpha(t)+G_\alpha(p)).$$

For the first statement, if $t>0$, Theorem 2.2 implies

$$\mathcal{N}_F^\alpha f(t)=\frac{1}{F(t,\alpha)}f'(t)=g'(G_\alpha(t))=g_1(G_\alpha(t)), \tag{3.7}$$

where $g_1(t)=g'(t)=\mathcal{N}_F^\alpha f((G_\alpha^{-1}(t))$. Moreover, since $\mathcal{N}_F^\alpha f(t)$ is continuous, it follows that $g\in\mathcal{C}^1((0,+\infty))$. On the other hand, if $t=0$, we have $\mathcal{N}_F^\alpha f(0)=lim_{t\to 0^+}\mathcal{N}_F^\alpha f(t)$ exists, and

$$\lim_{t\to 0^+}g'(t)=\lim_{t\to 0^+}\mathcal{N}_F^\alpha(f)(G_\alpha^{-1(t)})=\mathcal{N}_F^\alpha(f)(0)=g'(0).$$

So, for all $t\ge 0$, we have $\mathcal{N}_F^\alpha f(t)=g'(G_\alpha(t)), \quad t\in[0,\infty)$.
For the second assertion, we know that if f is $\mathcal{N}_F^\alpha$-periodic, then $g(G_\alpha(t)+G_\alpha(p))=g(G_\alpha(t))$ for all $t\in[0,\infty)$. Differentiating with respect to t implies $F(t,\alpha)g'(G_\alpha(t)+G_\alpha(p))=F(t,\alpha)g'(G_\alpha(t))$, and hence $g'(G_\alpha(t)+G_\alpha(p))=g'(G_\alpha(t))$. Thus $\mathcal{N}_F^\alpha f$ is $\mathcal{N}_F^\alpha$ p-periodic. □

Theorem 3.7. *Let f, F, α, G_α and g be as above. If f is n times continuously $\mathcal{N}_F^\alpha$-differentiable $\mathcal{N}_F^\alpha$ p-periodic function, then for all $j\in\{0,...,n\}$ and $t\in[0,\infty)$,*

(1) $\mathcal{N}_F^{j\alpha}f(t)=g^{(j)}(G_\alpha(t))$.

(2) $\mathcal{N}_F^{j\alpha} f(t)$ *is* $\mathcal{N}_F^{\alpha}$ *p-periodic.*

Here we adopt the standard convention that $\mathcal{N}_F^{(0)} f(t) = f(t)$.

Proof. We proceed by induction on $j \in \{0, ..., n\}$. For $j = 0$, Definition 3.1 implies that

$$f(t) = g(G_\alpha(t)) = g(G_\alpha(t) + G_\alpha(p))$$

Thus (1) and (2) are satisfied. The case $j = 1$ was proven in Theorem 3.6. Assume that, for all $j \in \{2, ..., n\}$, $\mathcal{N}_F^{(j-1)\alpha} f(t) = g^{(j-1)}(G_\alpha(t))$ with $g \in \mathcal{C}^{j-1}([0, \infty))$, and that $\mathcal{N}_F^{(j-1)\alpha} f(t)$ is $\mathcal{N}_F^{\alpha}$ p-periodic. If $t > 0$, we have

$$\mathcal{N}_F^{(j\alpha)} f(t) = \mathcal{N}_F^{\alpha}(\mathcal{N}_F^{(j-1)\alpha} f)(t) = \mathcal{N}_F^{\alpha}(g^{(j-1)}(G_\alpha(t)) = g^{(j)}(G_\alpha(t)).$$

When $t = 0$, we have $\mathcal{N}_F^{(j\alpha)} f(0) = \lim_{t \to 0^+} \mathcal{N}_F^{(j\alpha)} f(t)$ exists. Then

$$\lim_{t \to 0^+} g^{(j)}(t) = \lim_{t \to 0^+} \mathcal{N}_F^{(j\alpha)}(f(G_\alpha^{-1}(t)) = \mathcal{N}_F^{(j\alpha)} f(0) = g^{(j)}(0).$$

So, we have shown that

$$\mathcal{N}_F^{(j\alpha)} f(t) = g^{(j)}(G_\alpha(t)).$$

In particular, it holds when $j = n$. This completes the induction steps for the first assertion. For the second assertion, following a similar approach, we have $g^{(j-1)}(G_\alpha(t)) = g^{(j-1)}(G_\alpha(t) + G_\alpha(p))$, and then $F(t, \alpha) g^{(j)}(G_\alpha(t)) = F(t, \alpha) g^{(j)}(G_\alpha(t) + G_\alpha(p))$, which implies that $\mathcal{N}_F^{j\alpha} f(t)$ is $\mathcal{N}_F^{\alpha}$ p-periodic. □

4. $\mathcal{N}_F^{\alpha}$-Fourier transform

In this section, we define the notion of $\mathcal{N}_F^{\alpha}$-Fourier transform. We still have a nonnegative function F, with primitive G_α, as before. In what follows, all integrals are assumed to exist.

Definition 4.1. Let $f : [0, \infty) \to \mathbb{R}$ be $\mathcal{N}_F^{\alpha}$ p-periodic, and let $0 < \alpha \leq 1$. If $k \in \mathbb{Z}$, we define the k-th $\mathcal{N}_F^{\alpha}$-Fourier coefficient of f by

$$\mathcal{F}_F^{\alpha}(f(t))(k) := \widehat{f}_F^{\alpha}(k) = \frac{1}{G_\alpha(p)} \int_0^p e^{-ik \frac{2\pi}{G_\alpha(p)} G_\alpha(t)} f(t) F(t, \alpha) dt.$$

Example 4.2. Consider the function f_2 defined in Example 3.5. We have

$$\mathcal{F}_{t^{-\frac{1}{2}}}^{\alpha}(f_2(t))(k) = \frac{1}{4} \left(\int_0^1 t e^{-i \frac{k\pi}{2} t} dt + \int_1^3 (3 - t) e^{-i \frac{k\pi}{2} t} dt + \int_3^4 (t - 4) e^{-i \frac{k\pi}{2} t} dt \right).$$

Calculating, we obtain

$$\mathcal{F}_{t^{-\frac{1}{2}}}^{\alpha}(f_2(t))(k) = \begin{cases} 0, \text{ if } k = 2l, l \in \mathbb{Z} \\ 2(-1)^{-\frac{2l+1}{2}}, \text{ if } k = 2l + 1, l \in \mathbb{Z}. \end{cases}.$$

Remark 4.3. For $k = 0$, $\mathcal{F}_F^{\alpha}(f(t))(0) = \frac{1}{G_\alpha(p)} \mathcal{I}_F^{\alpha}(f)(p)$.

In what follows, we list some properties of the $\mathcal{N}_F^{\alpha}$-Fourier transform.

Theorem 4.4. *Let $f:[0,\infty)\to\mathbb{R}$ be $\mathcal{N}_F^\alpha$ p-periodic, and let $0<\alpha\leq 1$. If $k\in\mathbb{Z}$, then*

$$\mathcal{F}_F^\alpha\{f(t)\}(k)=\mathcal{F}\{f(G_\alpha^{-1}(t))\}(k).$$

Proof. By Remark 3.2, and the change of variable $u=G_\alpha(t)$ in Definition 4.1, we obtain

$$\begin{aligned}\mathcal{F}_F^\alpha(f(t))(k)&=\frac{1}{G_\alpha(p)}\int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}G_\alpha(t)}f(t)F(t,\alpha)dt\\&=\frac{1}{G_\alpha(p)}\int_0^{G_\alpha(p)} e^{-ik\frac{2\pi}{G_\alpha(p)}u}f(G_\alpha^{-1}(u))du\\&=\mathcal{F}\{f(G_\alpha^{-1}(t))\}(k).\end{aligned}$$

This completes the proof. □

Theorem 4.5. *Let $f:[0,\infty)\to\mathbb{R}$ be $\mathcal{N}_F^\alpha$ p-periodic function, such that $\mathcal{I}_F^\alpha(f)(p)=0$. Then $\mathcal{I}_F^\alpha(f)(t)$ is $\mathcal{N}_F^\alpha$ p-periodic, and, for $k\neq 0$,*

$$\mathcal{F}_F^\alpha(\mathcal{I}_F^\alpha f(t))(k)=\frac{G_\alpha(p)}{2ik\pi}\mathcal{F}_\alpha(f(t))(k).$$

Proof. By Theorem 3.4, $\mathcal{I}_F^\alpha f(t)$ is $\mathcal{N}_F^\alpha$ p-periodic. For $k\neq 0$, we have

$$\begin{aligned}\mathcal{F}_\alpha(\mathcal{I}_F^\alpha f(t))(k)&=\frac{1}{G_\alpha(p)}\int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}G_\alpha(t)}\mathcal{I}_F^\alpha f(t)F(t,\alpha)dt\\&=\frac{1}{G_\alpha(p)}\int_0^{G_\alpha(p)} e^{-ik\frac{2\pi}{G_\alpha(p)}u}\mathcal{I}_F^\alpha f(G_\alpha^{-1}(u))du.\end{aligned}$$

By (3.2), we can write

$$\mathcal{I}_F^\alpha f(t)=\int_0^{G_\alpha(t)} f(G_\alpha^{-1}(u))du,$$

which, upon integrating by parts, gives

$$\mathcal{F}_\alpha(\mathcal{I}_F^\alpha f(t))(k)=-\frac{1}{2ik\pi}\mathcal{I}_F^\alpha f(p)+\frac{G_\alpha(p)}{2ik\pi}\left(\frac{1}{G_\alpha(p)}\int_0^{G_p(\alpha)} e^{-i\frac{2k\pi}{G_\alpha(p)}t}f(G_\alpha^{-1}(t)dt\right).$$

Therefore,

$$\begin{aligned}\mathcal{F}_\alpha(\mathcal{I}_F^\alpha f(t))(k)&=\frac{G_\alpha(p)}{2ik\pi}\left(\frac{1}{G_\alpha(p)}\int_0^{G_p(\alpha)} e^{-i\frac{2k\pi}{G_\alpha(p)}t}f(G_\alpha^{-1}(t)dt\right)\\&=\frac{G_\alpha(p)}{2ik\pi}\mathcal{F}_\alpha(f(t))(k),\end{aligned}$$

where we have used the assumption that $\mathcal{I}_F^\alpha(f)(p)=0$ to obtain the last equality. This completes the proof.

□

Theorem 4.6. *Let f be $\mathcal{N}_F^\alpha$ p-periodic continuously $\mathcal{N}_F^\alpha$-differentiable function. Then $\mathcal{N}_F^\alpha(f)$ is $\mathcal{N}_F^\alpha$ p-periodic, and*

$$\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k)=\frac{2k\pi i}{G_\alpha(p)}\mathcal{F}_F^\alpha(f(t))(k).$$

Proof. By Theorem 3.6, $\mathcal{N}_F^\alpha f(t) = g'(G_\alpha(t))$ and $\mathcal{N}_F^\alpha(f)$ is $\mathcal{N}_F^\alpha$ p-periodic. For $k \in \mathbb{Z}$, we have

$$\begin{aligned}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) &= \mathcal{F}(\mathcal{N}_F^\alpha f(G_\alpha^{-1}(t)))(k)\\ &= \frac{1}{G_\alpha(p)}\int_0^{G_\alpha(p)} e^{-i\frac{2k\pi}{G_\alpha(p)}t}\mathcal{N}_F^\alpha((f(G_\alpha^{-1}(t))dt\\ &= \frac{1}{G_\alpha(p)}\int_0^{G_\alpha(p)} e^{-i\frac{2k\pi}{G_\alpha(p)}t}g'(t)dt.\end{aligned}$$

Integrating by parts yields

$$\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) = \frac{2k\pi i}{G_\alpha(p)}\mathcal{F}_F^\alpha(f(t))(k),$$

which completes the proof. □

Theorem 4.7. *Let f be $\mathcal{N}_F^\alpha$ p-periodic, and n times continuously $\mathcal{N}_F^\alpha$-differentiable. If $j \in \{0, ..., n\}$, then $\mathcal{N}_F^{(j\alpha)} f$ is $\mathcal{N}_F^\alpha$ p-periodic, and*

$$\mathcal{F}_F^\alpha(\mathcal{N}_F^{(j\alpha)} f(t))(k) = \left(\frac{2k\pi i}{G_\alpha(p)}\right)^j \mathcal{F}_F^\alpha(f(t))(k),$$

with the convention that $\mathcal{N}_F^{(0)} f(t) = f(t)$.

Proof. By Theorem 3.7, we have for all $j \in \{0, ..., n\}$, $\mathcal{N}_F^{(j\alpha)} f(t) = g^{(j)}(G_\alpha(t))$, and $\mathcal{N}_F^{(j\alpha)} f$ is $\mathcal{N}_F^\alpha$ p-periodic. We proceed by induction on $j \in \{0, ..., n\}$. Indeed, the property is clearly true for $j = 0$ and $j = 1$. Suppose that, for a certain j,

$$\mathcal{F}_F^\alpha(\mathcal{N}_F^{((j-1)\alpha)} f(t))(k) = \left(i\frac{2k\pi}{G_\alpha(p)}\right)^{j-1} \mathcal{F}_F^\alpha(f(t))(k).$$

We will show that

$$\mathcal{F}_F^\alpha(\mathcal{N}_F^{(j\alpha)} f(t))(k) = \left(\frac{2k\pi i}{G_\alpha(p)}\right)^j \mathcal{F}_F^\alpha(f(t))(k).$$

Indeed, by the induction hypothesis and Theorem 4.6, we have

$$\begin{aligned}\mathcal{F}_F^\alpha(\mathcal{N}_F^{(j\alpha)} f(t))(k) &= \mathcal{F}_F^\alpha(\mathcal{N}_F^{(\alpha)}(\mathcal{N}_F^{((j-1)\alpha)} f(t)))(k)\\ &= i\frac{2k\pi}{G_\alpha(p)}\mathcal{F}_F^\alpha\mathcal{N}_F^{((j-1)\alpha)} f(t))(k)\\ &= \left(\frac{2k\pi i}{G_\alpha(p)}\right)^j \mathcal{F}_F^\alpha(f(t))(k).\end{aligned}$$

This completes the proof. □

Theorem 4.8. *Let $F_1(\cdot, \alpha), F_2(\cdot, \alpha)$ be positive functions on $[0, \infty)$, and let $H^\alpha(t)$ and $K^\alpha(t)$ be their primitives, respectively. Assume that $H_\alpha(p) = K_\alpha(p)$, and that $f : [0, \infty) \to \mathbb{R}$ is $\mathcal{N}_{F_1}^\alpha$ and $\mathcal{N}_{F_2}^\alpha$ $p-$periodic. Then*

$$\mathcal{F}_{F_1+F_2}^\alpha(f(t))(k) = \mathcal{F}_{F_1}^\alpha\left(e^{-ik\frac{2\pi}{G_\alpha(p)}G_2^\alpha(t)} f(t)\right)(k) + \mathcal{F}_{F_2}^\alpha\left(e^{-ik\frac{2\pi}{G_\alpha(p)}G_1^\alpha(t)} f(t)\right)(k).$$

Proof. Let $F = F_1 + F_2$, and let G be its primitive. It can be seen that f is $\mathcal{N}_F^\alpha$ $p-$periodic, and that $G_\alpha(p) = H_\alpha(p) = K_\alpha(p)$. By Definition 4.1, we have

$$\begin{aligned}\mathcal{F}^\alpha_{F_1+F_2}(f(t))(k) &= \frac{1}{G_\alpha(p)}\int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}(H_\alpha(t)+K_\alpha(t))} f(t)(F_1(t,\alpha)+F_2(t,\alpha))dt \\ &= \frac{1}{G_\alpha(p)}\int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}H_\alpha(t)}\left(e^{-ik\frac{2\pi}{G_\alpha(p)}K_\alpha(t)}f(t)\right)F_1(t,\alpha)dt \\ &\quad + \frac{1}{G_\alpha(p)}\int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}K_\alpha(t)}\left(e^{-ik\frac{2\pi}{G_\alpha(p)}G_1^\alpha(t)}f(t)\right)F_2(t,\alpha)dt \\ &= \mathcal{F}^\alpha_{F_1}\left(e^{-ik\frac{2\pi}{G_\alpha(p)}G_2^\alpha(t)}f(t)\right)(k) + \mathcal{F}^\alpha_{F_2}\left(e^{-ik\frac{2\pi}{G_\alpha(p)}G_1^\alpha(t)}f(t)\right)(k).\end{aligned}$$

This completes the proof. □

By induction, it can be seen that Theorem 4.8 is extendable to n functions, as follows. Since the proof is similar to that of Theorem 4.8, we leave it to the interested reader. In this theorem, we use the notation G_i^α to denote $(G_i)_\alpha$, where G_i is the primitive of $F_i(\cdot,\alpha) > 0$.

Theorem 4.9. *For $i \in \{1,2,...,n\}$, let $F_i(\cdot,\alpha) > 0$, and let G_i^α be its primitive. Assume that $G_i^\alpha(p) = G_j^\alpha(p), \forall i,j \in \{1,2,...,n\}$. Also let $f : [0,\infty) \to \mathbb{R}$ be $\mathcal{N}_{F_i}^\alpha$ $p-$periodic, for all i. Then*

$$\mathcal{F}^\alpha_{(\sum_{i=1}^n F_i)}(f(t))(k) = \sum_{i=1}^n \mathcal{F}^\alpha_{F_i}(e^{-ik\frac{2\pi}{G_\alpha(p)}[\sum_{j=1,j\neq i}^n G_j^\alpha(t)]}f(t))(k),$$

where $G_\alpha(p)$ is the common value $G_i^\alpha(p)$.

In some applications, cosine and sine $\mathcal{N}_F^\alpha$-Fourier transforms might be needed. These are defined similar to the corresponding ones for the traditional Fourier transform. We still adopt the same notations from the above discussion.

Definition 4.10. Let $f : [0,\infty) \to \mathbb{R}$ be a given continuous $\mathcal{N}_F^\alpha$ p-periodic function. We define 1. The cosine $\mathcal{N}_F^\alpha$-Fourier coefficients of f as

$${}^c\mathcal{F}_F^\alpha(f(t))(k) = \frac{2}{G_\alpha(p)}\int_0^p \cos\left(\frac{2k\pi}{G_\alpha(p)}G_\alpha(t)\right) f(t)F(t,\alpha)dt.$$

2. The sine $\mathcal{N}_F^\alpha$-Fourier coefficients of f as

$${}^s\mathcal{F}_F^\alpha(f(t))(k) = \frac{2}{G_\alpha(p)}\int_0^p \sin\left(\frac{2k\pi}{G_\alpha(p)}G_\alpha(t)\right) f(t)F(t,\alpha)dt.$$

The following properties of ${}^c\mathcal{F}_F^\alpha$ and ${}^s\mathcal{F}_F^\alpha$ follow from the above definition, upon integration by parts.

Proposition 4.11. *Let $f : [0,\infty) \to \mathbb{R}$ be a given continuous $\mathcal{N}_F^\alpha$ p-periodic function. Then*

(1)

$${}^c\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) = \frac{2k\pi}{G_\alpha(p)}\ {}^s\mathcal{F}_F^\alpha f(t)(k).$$

(2)
$$ {}^{s}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) = -\frac{2k\pi}{G_\alpha(p)}\ {}^{c}\mathcal{F}_F^\alpha f(t)(k). $$
(3)
$$ {}^{c}\mathcal{F}_F^\alpha(\mathcal{N}_F^{2\alpha} f(t))(k) = -\left(\frac{2k\pi}{G_\alpha(p)}\right)^2\ {}^{c}\mathcal{F}_F^\alpha f(t)(k). $$
(4)
$$ {}^{s}\mathcal{F}_F^\alpha(\mathcal{N}_F^{2\alpha} f(t))(k) = -\left(\frac{2k\pi}{G_\alpha(p)}\right)^2\ {}^{s}\mathcal{F}_F^\alpha f(t)(k). $$

Proof. The first and second assertions follow immediately, upon a single integration by parts. For the third assertion, notice that
$$ \begin{aligned} {}^{c}\mathcal{F}_F^\alpha(\mathcal{N}_F^{2\alpha} f(t))(k) &= {}^{c}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) \\ &= \frac{2k\pi}{G_\alpha(p)}\ {}^{s}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) = -\left(\frac{2k\pi}{G_\alpha(p)}\right)^2\ {}^{c}\mathcal{F}_F^\alpha f(t)(k). \end{aligned} $$
Similarly, the fourth conclusion follows from the observations
$$ \begin{aligned} {}^{s}\mathcal{F}_F^\alpha(\mathcal{N}_F^{2\alpha} f(t))(k) &= {}^{s}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) \\ &= -\frac{2k\pi}{G_\alpha(p)}\ {}^{c}\mathcal{F}_F^\alpha(\mathcal{N}_F^\alpha f(t))(k) = -\left(\frac{2k\pi}{G_\alpha(p)}\right)^2\ {}^{s}\mathcal{F}_F^\alpha f(t)(k). \end{aligned} $$
This completes the proof. □

Theorem 4.12. *Let $f : [0,\infty) \to \mathbb{R}$ be $\mathcal{N}_F^\alpha$ p-periodic. Let $F(\cdot,\alpha) > 0$ and $G_\alpha(\cdot)$ be its primitive. If*
$$ \mathcal{I}_F^\alpha f(p) = \int_0^{G_\alpha(t)} f(G_\alpha^{-1})(u)du = 0, \tag{4.1} $$
then, for $k \neq 0$,

(1)
$$ {}^{c}\mathcal{F}_F^\alpha(\mathcal{I}_F^\alpha f(t))(k) = -\frac{G_\alpha(p)}{2k\pi}\ {}^{s}\mathcal{F}_F^\alpha(F(t,\alpha)f(t))(k). $$
(2)
$$ {}^{s}\mathcal{F}_F^\alpha(\mathcal{I}_F^\alpha f(t))(k) = \frac{G_\alpha(p)}{2k\pi}\ {}^{c}\mathcal{F}_F^\alpha(F(t,\alpha)f(t))(k). $$

Proof. We have $\mathcal{I}_F^\alpha f(p) = 0$. Then, for $k \neq 0$,
$$ \begin{aligned} {}^{c}\mathcal{F}_F^\alpha(\mathcal{I}_F^\alpha f(t))(k) &= \frac{2}{G_\alpha(p)}\int_0^p \cos\left(\frac{2k\pi}{G_\alpha(p)}G_\alpha(t)\right)\mathcal{I}_F^\alpha f(t)F(t,\alpha)dt \\ &= \frac{2}{G_\alpha(p)}\frac{G_\alpha(p)}{2k\pi}\left[\sin\left(\frac{2k\pi}{G_\alpha(p)}G_\alpha(t)\right)\mathcal{I}_F^\alpha f(t)\right]_0^p \\ &\quad - \frac{2}{G_\alpha(p)}\frac{G_\alpha(p)}{2k\pi}\int_0^p \sin\left(\frac{2k\pi}{G_\alpha(p)}G_\alpha(t)\right)\mathcal{I}_F^\alpha f(t)F(t,\alpha)dt \\ &= -\frac{G_\alpha(p)}{2k\pi}\ {}^{s}\mathcal{F}_F^\alpha(F(t,\alpha)f(t))(k). \end{aligned} $$

Furthermore, for $k \neq 0$,

$$\begin{aligned}
{}^s\mathcal{F}_F^\alpha(\mathcal{I}_F^\alpha f(t))(k) &= \frac{2}{G_\alpha(p)} \int_0^p \sin\left(\frac{2k\pi}{G_\alpha(p)} G_\alpha(t)\right) \mathcal{I}_F^\alpha f(t) F(t,\alpha) dt \\
&= -\frac{2}{G_\alpha(p)} \frac{G_\alpha(p)}{2k\pi} [\cos\left(\frac{2k\pi}{G_\alpha(p)} G_\alpha(t)\right) \mathcal{I}_F^\alpha f(t)]_0^p \\
&\quad + \frac{2}{G_\alpha(p)} \frac{G_\alpha(p)}{2k\pi} \int_0^p \cos\left(\frac{2k\pi}{G_\alpha(p)} G_\alpha(t)\right) \mathcal{I}_F^\alpha f(t) F(t,\alpha) dt \\
&= \frac{G_\alpha(p)}{2k\pi}\, {}^c\mathcal{F}_F^\alpha(F(t,\alpha) f(t))(k).
\end{aligned}$$

□

In the next section we will apply the previous results to some $\mathcal{N}_F^\alpha$-periodic solution of $\mathcal{N}_F^\alpha$-differential equation.

5. $\mathcal{N}_F^\alpha$-PERIODIC SOLUTION OF $\mathcal{N}_F^\alpha$-DIFFERENTIAL EQUATION

Let $F(t,\alpha)$ be a continuous function such that $F(t,\alpha) > 0$ for all $t > 0$, and let $G_\alpha(t)$ be its primitive function, verifying $G_\alpha(0) = 0$ and $\lim_{t\to\infty} G_\alpha(t) = \infty$.

5.1. **$\mathcal{N}_F^\alpha$-periodic solution in Banach space.** Let X be a complex Banach space and $p > 0$. If $f : [0,\infty) \to X$ is $\mathcal{N}_F^\alpha$ p-periodic, such that

$$||f||_q := \left(\int_0^p ||f(t)||^q dt\right)^{\frac{1}{q}} < \infty,$$

we write $f \in L_F^q(0,p;X)$, for $1 \leq q < +\infty$. So, $L_F^q(0,p;X))$ consists of all such f.

We list some definitions and properties from [2], to be able to proceed. The notation $\mathcal{L}(X,Y)$ is used for the set of all bounded linear operators from a Banach space X to another Banach space Y.

Definition 5.1. Let $q \geq 1$, and let X, Y be two Banach spaces. A family $\mathbf{T} \subset \mathcal{L}(X,Y)$ is called R_q-bounded if there exists $c_q \geq 0$ such that

$$\left\|\sum_{j=1}^n r_j \otimes T_j x_j\right\|_{L^q(0,1;X)} \leq c_q \left\|\sum_{j=1}^n r_j \otimes x_j\right\|_{L^q(0,1;X)} \tag{5.1}$$

for all $T_1,\dots,T_n \in \mathbf{T}, x_1,\dots,x_n \in X$ and $n \in \mathbb{N}$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (5.1) holds.

Definition 5.2. A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^q(\mathbb{R};\ X)$ for all $1 < q < \infty$.

Definition 5.3. For $F(\cdot,\alpha) > 0$, $1 \leq q < \infty$ and $0 < \alpha \leq 1$, a sequence $\{M_k^\alpha\}_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is said to be an (L_F^q, L_F^q)-multiplier (or L_F^q-multiplier) if for each $f \in L_F^q(0,p;X)$, there exists $u \in L_F^q(0,p;Y)$ such that $\widehat{u}_F^\alpha(k) = M_k^\alpha \widehat{f}_F^\alpha(k)$ for all $k \in \mathbb{Z}$.

Proposition 5.4. Let X be a Banach space and $\{M_k^\alpha\}_{k\in\mathbb{Z}}$ be an L_F^q-multiplier, where $1 \le q < \infty$ and $0 < \alpha \le 1$. Then the set $\{M_k^\alpha\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. Direct application of Definition 5.3, Theorem 4.4 and [Proposition 2.7] in [2]. □

Theorem 5.5. Let X, Y be UMD spaces and $\{M_k^\alpha\}_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. If the sets $\{M_k^\alpha\}_{k\in\mathbb{Z}}$ and $\left\{k(M_{k+1}^\alpha - M_k^\alpha)\right\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k^\alpha\}_{k\in\mathbb{Z}}$ is an L_F^q-multiplier for $q > 1$.

Proof. Direct application of [2, Theorem 2.8], Theorem 4.4 and Definition 5.3. □

Lemma 5.6. [1]. *Let $f, g \in L_F^q(0,q;X)$ be such that $\widehat{f}_F^\alpha(k) \in D(A)$ and $A\widehat{f}_F^\alpha(k) = \widehat{g}_F^\alpha(k)$ for all $k \in \mathbb{Z}$. Then*

$$f(t) \in D(A) \ \ \text{and} \ \ Af(t) = g(t) \ \ \text{for all} \quad t \in [0,q].$$

Where $A : D(A) \subset X \to X$ is a closed linear operator.

We still adopt the same notations from the previous section, where we have $F > 0$ and G_α as its primitive.

Proposition 5.7. [1] *Let $f \in L_F^q(0,p;X)$. Then*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{\delta_F^{\alpha,k} t} \widehat{f}_F^\alpha(k)$$

with convergence in $L_F^q(0,p;Y)$, where $\delta_F^{\alpha,k} = \frac{2ik\pi}{G_\alpha(p)}$.

Now consider the equation

$$\mathcal{N}_F^\alpha x(t) = Ax(t) + \mathcal{I}_F^\alpha x(t) + f(t) \ \text{for} \ t \in \mathbb{R}. \tag{5.2}$$

Put $\delta_F^{\alpha,k} = \frac{2ik\pi}{G_\alpha(p)}$, and define

$$\Delta_F^{\alpha,k} = (\delta_F^{\alpha,k} I - A - \frac{1}{\delta_F^{\alpha,k}})) \ \text{ and } \ \sigma_{\mathbb{Z}}(\Delta_F^{\alpha,k}) = \left\{ k \in \mathbb{Z}^* : \Delta_F^{\alpha,k} \ \text{is not bijective} \right\}.$$

The $\mathcal{N}_F^\alpha$-periodic vector-valued space is defined by

$$H_F^{\alpha,q}(0,p;X) = \left\{ u \in L_F^q(0,p,X) : \exists v \in L_F^q(0,p,X), \widehat{v}_F^\alpha(k) = \delta_F^{\alpha,k} \widehat{u}_F^\alpha(k), \ k \ne 0 \right\}.$$

Lemma 5.8. *Let $1 \le p < \infty, 0 < \alpha \le 1$ and $u, w \in L_F^q(0,p,X)$. The following are equivalent:*

(i) *$\mathcal{I}_F^\alpha(w)(p) = 0$, and there exists $x \in X$ such that*

$$u(t) = x + \mathcal{I}_F^\alpha(w)(t), \ t \in [0,p].$$

(ii) *$\widehat{w}_F^\alpha(k) = \delta_F^{\alpha,k} \ \widehat{u}_F^\alpha(k)$ for all $k \in \mathbb{Z}$.*

Proof. (i) *ightarrow* (ii). Let $k \neq 0$. We have

$$\begin{aligned}\widehat{u}_F^\alpha(k) &= \frac{1}{G_\alpha(p)} \int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}G_\alpha(t)} u(t)F(t,\alpha)dt \\ &= \frac{1}{G_\alpha(p)} \int_0^p e^{-ik\frac{2\pi}{G_\alpha(p)}G_\alpha(t)} \mathcal{I}_F^\alpha(w)(t)F(t,\alpha)dt.\end{aligned}$$

we also have $\mathcal{I}_F^\alpha(w)(G_\alpha^{-1}(t)) = \int_0^t w(G_\alpha^{-1}(s))ds$. Integrating by parts and using (4.1), we obtain,

$$\widehat{u}_F^\alpha(k) = \frac{1}{\delta_F^{\alpha,k}}\ \widehat{w}_F^\alpha(k).$$

This proves (ii). Now for (ii) *ightarrow* (i). As above, one has $\widehat{(\mathcal{I}_F^\alpha(w)(t))}_F^\alpha(k) = \frac{1}{\delta_F^{\alpha,k}}\widehat{(w(t))}_F^\alpha(k) = \widehat{(u(t))}_F^\alpha(k)$ for $k \neq 0$. Thus $u - \mathcal{I}_F^\alpha(w)$ is a constant function, and the proof is complete. $\square$

Definition 5.9. For $1 \leq q < \infty$, we say that a sequence $\{M_k^\alpha\}_{k\in\mathbb{Z}} \subset (X,Y)$ is an $(L_F^q, H_F^{\alpha,q})$-multiplier, if for each $f \in L_F^q(0,p;X)$, there exists $u \in H_F^{\alpha,q}(0,p;Y)$ such that

$$\widehat{u}_F^\alpha(k) = M_k^\alpha \widehat{f}_F^\alpha(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Lemma 5.10. *Let $1 \leq p < \infty, 0 < \alpha \leq 1$ and $(M_k^\alpha)_{k\in\mathbb{Z}} \subset (X)$ $((X)$ is the set of all bounded linear operators from X to $X)$. Then the following assertions are equivalent:*

(i) *$(M_k^\alpha)_{k\in\mathbb{Z}}$ is an $(L_F^q, H_F^{\alpha,q})$-multiplier.*
(ii) *$(\delta_F^{\alpha,k} M_k^\alpha)_{k\in\mathbb{Z}}$ is an (L_F^q, L_F^q)-multiplier.*

Proof. For (i) *ightarrow* (ii), let $f \in L_F^q(0,p;X)$. By assumption, there exists $g \in H_F^{\alpha,q}(0,p;X)$ such that $\widehat{g}_F^\alpha(k) = M_k^\alpha \widehat{f}_F^\alpha(k)$, by Definition of $H_F^{\alpha,q}(0,p;X)$ there exists $v \in L_F^q(0,p;X)$ such that $\widehat{v}_F^\alpha(k) = \delta_F^{\alpha,k} M_k^\alpha \widehat{f}_F^\alpha(k)$. Then $(\delta_F^{\alpha,k} M_k^\alpha)_{k\in\mathbb{Z}}$ is an (L_F^q, L_F^q)-multiplier.
For (ii) *ightarrow* (i), let $f \in L_F^q(0,p;X)$. By assumption, there exists $v \in L_F^q(0,p;X)$ such that $\widehat{v}_F^\alpha(k) = \delta_F^{\alpha,k} M_k^\alpha \widehat{f}_F^\alpha(k)$ for all $k \in \mathbb{Z}$. So, if $k \neq 0$, we have

$$\widehat{(\mathcal{I}_F^\alpha(v)(t))}_F^\alpha(k) = \frac{1}{\delta_F^{\alpha,k}}\widehat{(v(t))}_F^\alpha(k) = M_k^\alpha \widehat{f}_F^\alpha(k).$$

Let $u(t) = u(0) + \mathcal{I}_F^\alpha(v)(t)$. Lemma 5.8 implies $u \in H_F^{\alpha,q}(0,p;X)$. Moreover, $\widehat{(u(t))}_F^\alpha(k) = M_k^\alpha \widehat{f}_F^\alpha(k)$ for all $k \in \mathbb{Z}$. Then $(M_k^\alpha)_{k\in\mathbb{Z}}$ is an $(L_F^q, H_F^{\alpha,q})$-multiplier. $\square$

To proceed with the solution discussion of (5.2), we recall the notion of a strong solution; keeping in mind the adopted notions related to F, α, G_α.

Definition 5.11. Let $f \in L_F^q(0,p;X)$. A function $x \in H_F^{\alpha,p}(0,p;X)$ is said to be an $\mathcal{N}_F^\alpha$-periodic strong solution of Eq.(5.2) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (5.2) holds almost everywhere.

Proposition 5.12. *Let $1 < q < \infty$ and $0 < \alpha \leq 1$. Also let A be a closed linear operator defined on an UMD space X. Suppose that and $\sigma_{\mathbb{Z}}(\Delta_F^{\alpha,k}) = \phi$. Then the following assertions are equivalent*

(i): $\left(\delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\ )^{-1}\right)_{k\in\mathbb{Z}}$ *is an L^q_α-multiplier.*

(ii): $\left(\delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\ )^{-1}\right)_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) follows as a consequence of Proposition 5.4. For (ii) $\Rightarrow$ (i), define $M_k^\alpha = \delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\ )^{-1}$. By Theorem (5.5) it is sufficient to prove that the set $\left\{k(M_{k+1}^\alpha - M_k^\alpha)\right\}_{k\in\mathbb{Z}}$ is R-bounded. We use $k(\delta_F^{\alpha,k+1} - \delta_F^{\alpha,k}) = \delta_F^{\alpha,k}$ and $M_k^\alpha(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\ ) = \delta_F^{\alpha,k}$, to find

$$
\begin{aligned}
& k\left[M_{k+1}^\alpha - M_k^\alpha\right] \\
&= k[\delta_F^{\alpha,k+1}(\delta_F^{\alpha,k+1} - A - \frac{1}{\delta_F^{\alpha,k+1}}\ )^{-1} - \delta_F^{\alpha,k}(\delta_F^{\alpha,k}I - A - \frac{1}{\delta_F^{\alpha,k}}\ )^{-1}] \\
&= (\delta_F^{\alpha,k+1} - A - \frac{1}{\delta_F^{\alpha,k+1}}\ )^{-1}[-A\delta_F^{\alpha,k} + k(\frac{\delta_F^{\alpha,k}}{\delta_F^{\alpha,k+1}} - \frac{\delta_F^{\alpha,k+1}}{\delta_F^{\alpha,k}})](\delta_F^{\alpha,k}I - A - \frac{1}{\delta_F^{\alpha,k}}\ )^{-1} \\
&= -\frac{1}{\delta_F^{\alpha,k+1}}M_{k+1}^\alpha A M_k^\alpha - M_{k+1}^\alpha[\frac{\delta_F^{\alpha,k} + \delta_F^{\alpha,k+1}}{(\delta_F^{\alpha,k+1})^2\delta_F^{\alpha,k}}]M_k^\alpha \\
&= [\frac{\delta_F^{\alpha,k}}{\delta_F^{\alpha,k+1}}]M_{k+1}^\alpha - [\frac{(\delta_F^{\alpha,k})^2 - 1}{\delta_F^{\alpha,k+1}\delta_F^{\alpha,k}}]M_{k+1}^\alpha M_k^\alpha - M_{k+1}^\alpha[\frac{\delta_F^{\alpha,k} + \delta_F^{\alpha,k+1}}{(\delta_F^{\alpha,k+1})^2\delta_F^{\alpha,k}}]M_k^\alpha.
\end{aligned}
$$

We have that $\{\frac{\delta_F^{\alpha,k}}{\delta_F^{\alpha,k+1}}, k \neq 0\}, \{\frac{(\delta_F^{\alpha,k})^2-1}{\delta_F^{\alpha,k+1}\delta_F^{\alpha,k}}, k \in \mathbb{Z}^*\}$ and $\{\frac{\delta_F^{\alpha,k}+\delta_F^{\alpha,k+1}}{(\delta_F^{\alpha,k+1})^2\delta_F^{\alpha,k}}, k \in \mathbb{Z}^*\}$ are bounded. Since products and sums of R-bounded sequences is R-bounded [1, Remark 2.2], the proof is complete. ☐

Lemma 5.13. *Let $1 \leq q < \infty$ and $0 < \alpha \leq 1$. Suppose that $\sigma_{\mathbb{Z}}(\Delta_F^{\alpha,k}) = \phi$ and that for every $f \in L_F^q(0, p; X)$ there exists an $\mathcal{N}_F^\alpha$-periodic strong solution x of Eq. (5.2). Then x is the unique $\mathcal{N}_F^\alpha$-periodic strong solution.*

Proof. Suppose that x_1 and x_2 two $\mathcal{N}_F^\alpha$-periodic strong solution of Eq. (5.2). Then $x = x_1 - x_2$ is an $\mathcal{N}_F^\alpha$-periodic strong solution of Eq. (5.2) corresponding to $f = 0$. Taking $\mathcal{N}_F^\alpha$-Fourier transform in (5.2), we obtain that

$$(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})\widehat{x}_F^\alpha(k) = 0,\ k \neq 0.$$

It follows that $\widehat{x}_F^\alpha(k) = 0$ for every $k \neq 0$, and therefore $x = 0$, and the proof is complete. ☐

Theorem 5.14. *Let $0 < \alpha \leq 1$ and X be a Banach space. Suppose that for every $f \in L_F^q(0, p; X)$ there exists a unique $\mathcal{N}_F^\alpha$-periodic strong solution of Eq. (5.2) for $1 \leq q < \infty$. Then*

(1) *for every* $k \neq 0$, *the operator* $\Delta_F^{\alpha,k} = (\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})$ *has a bounded inverse.*

(2) $\left\{\delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})^{-1}\right\}_{k\in\mathbb{Z}}$ *is R-bounded.*

The following Lemma is needed to be able to prove Theorem 5.14.

Lemma 5.15. *If, for the aforementioned parameters,* $(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})(x) = 0$ *for all* $k \neq 0$, *then* $u(t) = e^{\delta_F^{\alpha,k} G_\alpha(t)}x$ *is an* $\mathcal{N}_F^{\alpha}$*-periodic strong solution of the following equation*

$$\mathcal{N}_F^{\alpha}u(t) = Au(t) + \mathcal{I}_F^{\alpha}u(t).$$

Proof. The fact that $(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})(x) = 0$ implies $\delta_F^{\alpha,k}x = Ax + \frac{1}{\delta_F^{\alpha,k}}x$. We have $u(t) = e^{\delta_F^{\alpha,k} G_\alpha(t)}x$ and, by Theorem 2.2,

$$\begin{aligned}
\mathcal{N}_F^{\alpha}u(t) &= e^{\delta_F^{\alpha,k} G_\alpha(t)}\delta_F^{\alpha,k}x \\
&= e^{\delta_F^{\alpha,k} G_\alpha(t)}\left(Ax + \frac{1}{\delta_F^{\alpha,k}}x\right) \\
&= Ae^{\delta_F^{\alpha,k} G_\alpha(t)}x + \frac{1}{\delta_F^{\alpha,k}}e^{\delta_F^{\alpha,k} G_\alpha(t)}x \\
&= Au(t) + \mathcal{I}_F^{\alpha}u(t).
\end{aligned}$$

Proof of Theorem 5.14: (1) Let $k \neq 0$ and $y \in X$. Then for $f(t) = e^{\delta_F^{\alpha,k} G_\alpha(t)}y$, there exists $x \in H_F^{\alpha,p}(0,q;X)$ such that

$$\mathcal{N}_F^{\alpha}x(t) = Ax(t) + \mathcal{I}_F^{\alpha}x(t) + f(t).$$

Taking $\mathcal{N}_F^{\alpha}$-Fourier transform, we have

$$(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})\widehat{x}_F^{\alpha}(k) = \widehat{f}_F^{\alpha}(k) = y.$$

Consequently, $(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})$ is surjective. If $\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)(u) = 0$, then by Lemma 5.15, $x(t) = e^{\delta_F^{\alpha,k} G_\alpha(t)}u$ is a $\mathcal{N}_F^{\alpha}$-periodic strong solution of Eq.(5.2) corresponding to the function $f(t) = 0$. Hence $x(t) = 0$ and $u = 0$, implying that $(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})$ is injective. This completes the proof of the first assertion.
For (2), let $f \in L_F^q(0,p;X)$. By assumptions, there exists a unique $x \in H_F^{\alpha,q}(0,q;X)$ such that the Eq. (5.2) is valid. Taking $\mathcal{N}_F^{\alpha}$-Fourier transform, we deduce that

$$\widehat{x}_F^{\alpha}(k) = \left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\widehat{f}_F^{\alpha}(k) \quad \text{for all} \quad k \neq 0.$$

Hence

$$\delta_F^{\alpha,k}\widehat{x}_F^{\alpha}(k) = \delta_F^{\alpha,k}\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\widehat{f}_F^{\alpha}(k) \quad \text{for all} \quad k \neq 0.$$

Since $x \in H_F^{\alpha,q}(0,p;X)$, there exists $v \in L_F^q(0,p;X)$ such that

$$\widehat{v}_F^{\alpha}(k) = \delta_F^{\alpha,k}\widehat{x}_F^{\alpha}(k) = \delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})^{-1}\widehat{f}_F^{\alpha}(k), k \neq 0.$$

Then $\left\{\delta_F^{\alpha,k}\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\right\}_{k\in\mathbb{Z}}$ is an L_F^q-multiplier and by Proposition 5.4, and $\left\{\delta_F^{\alpha,k}\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded. □

5.2. **$\mathcal{N}_F^{\alpha}$-periodic solutions in UMD space.** Our main result in this section is to establish that the converse of Theorem 5.14 is true, provided X is an UMD space.

Theorem 5.16. *Let X be an UMD space and $A : D(A) \subset X \to X$ be a closed linear operator. Then the following assertions are equivalent for $1 < q < \infty$.*

(1) *For every $f \in L_F^q(0,p;X)$, there exists a unique $\mathcal{N}_F^{\alpha}$-periodic strong solution of Eq. (5.2).*
(2) *$\sigma_{\mathbb{Z}}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}) = \phi$ and $\left\{\delta_F^{\alpha,k}(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}})^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. :
Theorem 5.14 implies the second assertion, assuming the first. For the other implication, let $f \in L_F^q(0,p;X)$. By Lemma 5.10, the family $\left\{\delta_F^{\alpha,k}\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\right\}_{k\in\mathbb{Z}}$ is an L_α^q-multiplier if and only if the family $\left\{\left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\right\}_{k\in\mathbb{Z}}$ is an L_F^q-multiplier that maps $L_F^p(0,p;X)$ into $H^{\alpha,p}(0,p;X)$. Namely, there exists $x \in H_F^{\alpha,p}(0,p;X)$ such that

$$\widehat{x}_F^{\alpha}(k) = \left(\delta_F^{\alpha,k} - A - \frac{1}{\delta_F^{\alpha,k}}\right)^{-1}\widehat{f}_F^{\alpha}(k). \tag{5.3}$$

In particular, $x \in L_F^q(0,p;X)$ and $\exists v \in L_F^q(0,p;X)$ such that $\widehat{v}_F^{\alpha}(k) = \delta_F^{\alpha,k}\widehat{x}_F^{\alpha}(k)$, and

$$\widehat{(\mathcal{N}_F^{\alpha}x(t))}_F^{\alpha}(k) := \widehat{v}_F^{\alpha}(k) = \delta_F^{\alpha,k}\widehat{x}_F^{\alpha}(k). \tag{5.4}$$

By Theorem 5.7, we have

$$x(t) = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{t\delta_F^{\alpha,k}}\widehat{x}_F^{\alpha}(k)$$

Using (5.3) and (5.4), we obtain

$$\widehat{(\mathcal{N}_F^{\alpha}x(t))}_F^{\alpha}(k) = \delta_F^{\alpha,k}\widehat{x}_F^{\alpha}(k) = A\widehat{x}_F^{\alpha} + \widehat{(\mathcal{I}_F^{\alpha}x(t))}_F^{\alpha}(k) + \hat{f}_F^{\alpha}(k), k \neq 0.$$

Since A is closed, then $x(t) \in D(A)$ by Lemma 5.6. Then, from the uniqueness theorem of Fourier coefficients, it follows that (5.2) is valid. This completes the proof. □

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