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A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER DEFINED BY THE SYMMETRIC Q-DERIVATIVE AND SUBORDINATION

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Abstract. In this paper, we introduce a new subclass of bi-univalent functions of complex order, using the symmetric q-derivative with subordination principles. We obtain upper bounds for the coefficients $|c_2|$, $|c_3|$ and estimate an upper bound for the Fekete-Szegö problem within this new subclass $\widetilde{\mathfrak{M}}^q_\Sigma(t, \varrho, \zeta)$.

1. Introduction

Let Δ be the class of analytic functions $\mathfrak{T}$ in $\psi = \{\mathcal{Z} \in \mathbb{C} : |\mathcal{Z}| < 1\}$, such that $\mathfrak{T}'(0) - 1 = \mathfrak{T}(0) = 0$. So, each function $\mathfrak{T} \in \Delta$ has the following form of Taylor series expansion:

$$\mathfrak{T}(\mathcal{Z}) = \mathcal{Z} + \sum_{\imath=2}^{\infty} c_\imath \mathcal{Z}^\imath, \quad \mathcal{Z} \in \psi. \tag{1.1}$$

Let $\mathbb{S}$ be a subset of Δ containing functions of the form (1.1) that are also univalent within ψ.

For two analytic functions $\mathfrak{T}$ and $\mathcal{L}$ in ψ, the function $\mathfrak{T}$ is considered subordinate to $\mathcal{L}$ and denoted by $\mathfrak{T}(\mathcal{Z}) \prec \mathcal{L}(\mathcal{Z})$ for all $\mathcal{Z} \in \psi$, if there exists an analytic function $\varpi(\mathcal{Z})$ in ψ and satisfies $\varpi(0) = 0$ and $|\varpi(\mathcal{Z})| < 1$ for all $\mathcal{Z} \in \psi$, such that $\mathfrak{T}(\mathcal{Z}) = \mathcal{L}(\varpi(\mathcal{Z}))$. This means that each function $\mathfrak{T} \in \mathbb{S}$ has an inverse $\mathfrak{T}^{-1}$ that meets the following requirements

$$\mathfrak{T}^{-1}(\mathfrak{T}(\mathcal{Z})) = \mathcal{Z}, \quad \mathcal{Z} \in \psi,$$

and

$$\mathfrak{T}(\mathfrak{T}^{-1}(\varpi)) = \varpi, \quad |\varpi| < r_0(\mathfrak{T}); \quad r_0(\mathfrak{T}) \geq \frac{1}{4}.$$

The expression for the inverse function $\mathfrak{T}^{-1}$ is:

$$\mathfrak{T}^{-1}(\varpi) = \varpi - c_2\varpi^2 + (2c_2^2 - c_3)\varpi^3 - (5c_2^3 - 5c_2c_3 + c_4)\varpi^4 + \cdots .$$

We refer to [7] . A function $\mathfrak{T}$ within the class Δ is considered bi-univalent in ψ if $\mathfrak{T}$ and $\mathfrak{T}^{-1}$ are univalent in ψ. The class of bi-univalent functions in ψ

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is symbolized by Σ (see [9]). Lewin [19] investigated the class Σ and showed that obtaining a bound of $|c_2| \leq 1.51$. Later, Brannan and Clunie [4] proposed that $|c_2| \leq \sqrt{2}$. Netanyahu [24] subsequently showed that the maximum value of $|c_2| \leq \frac{4}{3}$. Brannan and Taha [5] presented subclasses in Σ that are comparable to the starlike functions $S^*(\beta)$ and convex functions $K(\beta)$ of order β ($0 \leq \beta < 1$). They also introduced the classes of bi-starlike functions $S^*_\Sigma(\beta)$ and bi-convex functions $K_\Sigma(\beta)$. They provided non-sharp estimates for the initial coefficients of these classes. Bounds for different subclasses have recently been studied by a number of writers (see [2, 9, 13, 15, 20, 23, 27, 28]). However, there is limited knowledge regarding the bounds on the general coefficient $|c_\imath|$ for $\imath \geq 4$. Only a few studies have addressed the general coefficient bounds $|c_\imath|$ (see [8, 10, 16]). The problem of estimating the coefficient $|c_\imath|$ for $\imath \in \mathbb{N} - \{1, 2, 3\}$ remains an open problem.

Chebyshev polynomials play a crucial role in numerical analysis, with four distinct types. Most studies focus on the first two kinds, denoted as $T_\imath(t)$ and $U_\imath(t)$, due to their wide range of applications. For further details, see [6, 21].

The first and second kinds of Chebyshev polynomials are familiar in mathematics. For $t \in (-1, 1)$, they are defined as:

$$T_\imath(t) = \cos \imath\theta \text{ and } U_\imath(t) = \frac{\sin(\imath + 1)\theta}{\sin\theta},$$

where $\imath$ represents the degree of the polynomial, and $t = \cos\theta$.

A subfield of mathematics known as q-calculus, which focuses on the study of functions and operators in the setting of a parameter q (where q is usually a real or complex number), introduced the q-derivative, a generalization of the classical derivative. In the q-deformed space, it offers a method for differentiating functions.

Fractional q-calculus plays a key role in studying various subclasses of analytic functions. For instance, it extends the theory of univalent functions through q-calculus. Additionally, fractional q-integrals and q-derivative operators are widely used to introduce and analyze new subclasses of analytic functions (see [1, 12, 14, 25]). Further, Purohit and Raina [26] investigated how fractional q-calculus operators can be applied to define new classes of analytic functions. Afterwards, Mohammed and Darus [22] examined the geometric properties of these q-operators within certain analytic function subclasses.

We begin by presenting the fundamental definitions and concepts of q-calculus for $0 < q < 1$, see [11]. For complex-valued functions $\mathfrak{T}(\mathcal{Z})$, the definitions of fractional q-calculus operators are reviewed as follows:

$$D_q\mathfrak{T}(\mathcal{Z}) = \begin{cases} \frac{\mathfrak{T}(\mathcal{Z}) - \mathfrak{T}(q\mathcal{Z})}{(1-q)\mathcal{Z}}, & \text{if } \mathcal{Z} \neq 0, \\ \mathfrak{T}'(\mathcal{Z}), & \text{if } \mathcal{Z} = 0, \end{cases} \tag{1.2}$$

for $q \in (0, 1)$. It is clear that

$$\lim_{q \to 1^-} D_q\mathfrak{T}(\mathcal{Z}) = \mathfrak{T}'(\mathcal{Z})$$

for $\mathfrak{T}$ given by (1.1). The formula for the q-derivative of the function $\mathcal{Z}^{\imath}$ is as follows:

$$D_q \mathcal{Z}^{\imath} = [\imath]_q \mathcal{Z}^{\imath-1}, \quad \imath \in \mathbb{N},$$

where

$$[\imath]_q = \sum_{k=1}^{\imath} q^{k-1}, \quad [0]_q = 0$$

is called the basic number $\imath$. It follows that:

$$[\imath]_q = \frac{1-q^{\imath}}{1-q}, \quad \lim_{q\to 1^-} [\imath]_q = \imath \quad \text{for } q \in (0,1).$$

And on a product of functions, the Leibniz rule for powers of the q-derivative is:

$$D_q^{(\imath)}(\mathfrak{T}\mathcal{L})(\mathcal{Z}) = \sum_{k=0}^{\imath} \binom{\imath}{k}_q D_q^{(k)} \mathfrak{T}(q^{\imath-k}\mathcal{Z}) D_q^{(\imath-k)} \mathcal{L}(q^k \mathcal{Z}),$$

where $\binom{\wp}{k}_q$ is the q-binomial coefficients given by

$$\binom{\imath}{k}_q = \frac{[\imath]_q!}{[k]_q![\imath-k]_q!}, \quad \imath, k \in \mathbb{N}.$$

For $k = 0, 1, 2, \cdots$, such that

$$[\imath]_q! = \prod_{k=1}^{n} [k]_q, \quad [0]_q! = 1,$$

where $[\imath]_q!$ is the q-factorial function. Note that

$$\lim_{q\to 1^-} [\imath]_q! = \imath!$$

is the factorial function, and

$$\lim_{q\to 1^-} \binom{\imath}{k}_q = \binom{\imath}{k}$$

are binomial coefficients. For more details about some properties of the difference operator D_q see [3] . It follows that

$$D_q^{(2)}\mathfrak{T}(\mathcal{Z}) = D_q\left(D_q\mathfrak{T}(\mathcal{Z})\right), \quad \text{and} \quad D_q^{(\imath)}\mathfrak{T}(\mathcal{Z}) = D_q\left(D_q^{(\imath-1)}\mathfrak{T}(\mathcal{Z})\right).$$

Definition 1.1. [18]. The symmetric q-derivative $\widetilde{D}_q\mathfrak{T}$ of a function $\mathfrak{T}$ given by (1.1) has the following definition:

$$\widetilde{D}_q\mathfrak{T}(\mathcal{Z}) = \begin{cases} \frac{\mathfrak{T}(q\mathcal{Z})-\mathfrak{T}(q^{-1}\mathcal{Z})}{(q-q^{-1})\mathcal{Z}}, & \text{for } \mathcal{Z} \neq 0, \\ \mathfrak{T}'(0), & \text{for } \mathcal{Z} = 0. \end{cases}$$

From (1.3), we have that

$$\widetilde{D}_q \mathcal{Z}^{\imath} = \widetilde{[\imath]}_q \mathcal{Z}^{\imath-1}.$$

And a power series expansion of $\widetilde{D}_q\mathfrak{T}$ is given by

$$\widetilde{D}_q\mathfrak{T}(\mathcal{Z}) = 1 + \sum_{\imath=2}^{\infty} \widetilde{[\imath]}_q c_{\imath} \mathcal{Z}^{\imath-1}$$

where $\mathfrak{T}$ has the form (1.1) and $\widetilde{[\imath]}_q = \frac{q^{\imath}-q^{-\imath}}{q-q^{-1}}$. Verifying the properties

$$\widetilde{D}_q\left[\mathfrak{T}(\mathcal{Z}) + \mathcal{L}(\mathcal{Z})\right] = \widetilde{D}_q\mathfrak{T}(\mathcal{Z}) + \widetilde{D}_q\mathcal{L}(\mathcal{Z}),$$

and

$$\widetilde{D}_q\left[\mathfrak{T}(\mathcal{Z})\mathcal{L}(\mathcal{Z})\right] = \mathfrak{T}(q^{-1}\mathcal{Z})\widetilde{D}_q\mathcal{L}(\mathcal{Z}) + \mathcal{L}(q\mathcal{Z})\widetilde{D}_q\mathfrak{T}(\mathcal{Z}).$$

Finally, we obtain the relation given below:

$$\widetilde{D}_q\mathfrak{T}(\mathcal{Z}) = D_{q^2}\mathfrak{T}(q^{-1}\mathcal{Z}),$$

and

$$\widetilde{D}_q\mathcal{L}(\varpi) = \frac{\mathcal{L}(q\varpi) - \mathcal{L}(q^{-1}\varpi)}{(q-q^{-1})\varpi} = 1 - \widetilde{[2]}_q c_2\varpi + \widetilde{[3]}_q(2c_2^2 - c_3)\varpi^2 + \cdots .$$

We note that if $t = \cos\ltimes$ where $\ltimes \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(\mathcal{Z}, t) = \frac{1}{1 - 2t\mathcal{Z} + \mathcal{Z}^2} = 1 + \sum_{\imath=1}^{\infty} \frac{\sin((\imath+1)\ltimes)}{\sin\ltimes}\mathcal{Z}^{\imath}, \quad \mathcal{Z} \in \psi$$

Thus,

$$H(\mathcal{Z}, t) = 1 + 2\cos\ltimes\,\mathcal{Z} + (3\cos^2\ltimes - \sin^2\ltimes)\mathcal{Z}^2 + \cdots, \quad \mathcal{Z} \in \psi$$

Next, we write

$$H(\mathcal{Z}, t) = 1 + U_1(t)\mathcal{Z} + U_2(t)\mathcal{Z}^2 + \cdots, \quad \mathcal{Z} \in \psi,\ t \in (-1, 1)$$

where $U_{\imath-1}(t) = \frac{\sin(\imath\arccos t)}{\sqrt{1-t^2}}$ (for $\imath \in \mathbb{N}$) are the Chebyshev polynomials of the second kind. Additionally, it is acknowledged that

$$U_{\imath}(t) = 2tU_{\imath-1}(t) - U_{\imath-2}(t)$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad . \tag{1.3}$$

The Chebyshev polynomials $T_{\imath}(t)$, for $t \in (-1, 1)$, of the first kind have the following generating function:

$$\sum_{\imath=0}^{\infty} T_{\imath}(t)\mathcal{Z}^{\imath} = \frac{1 - t\mathcal{Z}}{1 - 2t\mathcal{Z} + \mathcal{Z}^2}, \quad \mathcal{Z} \in \psi$$

Definition 1.2. A function $\mathfrak{T} \in \Sigma$ given by (1.1) is be in the class $\widetilde{\mathfrak{M}}^q_{\Sigma}(t, \varrho, \zeta)$, if satisfied:

$$(\widetilde{\mathfrak{D}}_q\mathfrak{T}(\mathcal{Z}))^{\varrho+i\zeta} \prec H(\mathcal{Z}, t) := \frac{1}{1 - 2t\mathcal{Z} + \mathcal{Z}^2}, \quad \frac{1}{2} < t < 1,\ \varrho, \zeta \in \mathbb{R}, \mathcal{Z} \in \psi \tag{1.4}$$

and

$$(\widetilde{\mathfrak{D}}_q\mathcal{L}(\varpi))^{\varrho+i\zeta} \prec H(\varpi, t) := \frac{1}{1 - 2t\varpi + \varpi^2}, \quad \frac{1}{2} < t < 1,\ \varrho, \zeta \in \mathbb{R}, \varpi \in \psi, \text{ and } \mathcal{L} = \mathfrak{F}^{-1}$$

Our paper focuses on utilizing Chebyshev polynomial to derive the initial coefficients of certain subclass of complex order defined through the symmetric q-derivative. Fekete-Szegö inequalities are also established for the class $\widetilde{\mathfrak{M}}^q_{\Sigma}(t, \varrho, \zeta)$.

2. Estimating Coefficients for the Class $\widetilde{\mathfrak{M}}_{\Sigma}^{q}(t,\varrho,\zeta)$

We need the next fundamental lemma to prove our main theorems.

Lemma 2.1. [13]. *If the function p belonging to $\mathcal{P}$ ($\mathcal{P}$ be the class of analytic functions with a positive real part, including the functions $p:\psi\to\mathbb{C}$ that satisfy $Re(p(\mathcal{Z}))>0$, $p(0)=1$) is given by*

$$p(\mathcal{Z})=1+l_1\mathcal{Z}+l_2\mathcal{Z}^2+l_3\mathcal{Z}^3+\cdots,$$

then $|l_\imath|\leq 2$, for $\imath\in\mathbb{N}=\{1,2,\cdots\}$.

Next theorem, focus on estimating the coefficients for function in the class $\widetilde{\mathfrak{M}}_{\Sigma}^{q}(t,\varrho,\zeta)$.

Theorem 2.2. *Suppose $\mathfrak{T}$ defined by (1.1) is in the class $\widetilde{\mathfrak{M}}_{\Sigma}^{q}(t,\varrho,\zeta)$, where $q\in(0,1)$, $\varrho,\zeta\in\mathbb{R}$ Then:*

$$|c_2|\leq\frac{2t\sqrt{2t}}{\sqrt{\left|4(\varrho+i\zeta)t^2\left(\widetilde{[3]}_q+\widetilde{[2]}_q^2(\varrho-1+i\zeta)/2-(\varrho+i\zeta)\widetilde{[2]}_q^2\right)+(2t+1)\left((\varrho+i\zeta)^2\widetilde{[2]}_q^2\right)\right|}}$$

and

$$|c_3|\leq\frac{2t}{\widetilde{[3]}_q|\varrho+i\zeta|}+\frac{4t^2}{\widetilde{[2]}_q^2|\varrho+i\zeta|^2}.$$

Proof. Since $\mathfrak{T}\in\widetilde{\mathfrak{M}}_{\Sigma}^{q}(t,\varrho,\zeta)$, then, there exist two functions h and v, which are analytic in ψ, satisfying $h(0)=v(0)=0$, $|h(\mathcal{Z})|<1$, and $|v(\varpi)|<1$ for all $\mathcal{Z},\varpi\in\psi$, such that:

$$(\widetilde{\mathfrak{D}}_q\mathfrak{T}(\mathcal{Z}))^{\varrho+i\zeta}=H(h(\mathcal{Z}),t) \tag{2.1}$$

$$(\widetilde{\mathfrak{D}}_q\mathcal{L}(\varpi))^{\varrho+i\zeta}=H(v(\varpi),t) \tag{2.2}$$

Next, let the functions p and d be defined as:

$$p(\mathcal{Z})=\frac{1+h(\mathcal{Z})}{1-h(\mathcal{Z})}=1+p_1\mathcal{Z}+p_2\mathcal{Z}^2+\cdots$$

and

$$d(\varpi)=\frac{1+v(\varpi)}{1-v(\varpi)}=1+d_1\varpi+d_2\varpi^2+\cdots$$

So,

$$h(\mathcal{Z})=\frac{p(\mathcal{Z})-1}{p(\mathcal{Z})+1}=\frac{1}{2}p_1\mathcal{Z}+\frac{1}{2}(p_2-\frac{1}{2}p_1^2)\mathcal{Z}^2+\cdots \tag{2.3}$$

and

$$v(\varpi)=\frac{d(\varpi)-1}{d(\varpi)+1}=\frac{1}{2}d_1\varpi+\frac{1}{2}(d_2-\frac{1}{2}d_1^2)\varpi^2+\cdots \tag{2.4}$$

By combining equations (2.1) (2.2), (2.3), and(2.4) :

$$(\widetilde{\mathfrak{D}}_q\mathfrak{T}(\mathcal{Z}))^{\varrho+i\zeta}=1+\frac{1}{2}U_1(t)p_1\mathcal{Z}+\left(\frac{1}{4}U_2(t)p_1^2+\frac{1}{2}U_1(t)\left(p_2-\frac{1}{2}p_1^2\right)\right)\mathcal{Z}^2+\cdots \tag{2.5}$$

and

$$(\widetilde{\mathfrak{D}}_q\mathcal{L}(\varpi))^{\varrho+i\zeta} = 1+\frac{1}{2}U_1(t)d_1\varpi+\left(\frac{1}{4}U_2(t)d_1^2+\frac{1}{2}U_1(t)\left(d_2-\frac{1}{2}d_1^2\right)\right)\varpi^2+\cdots \tag{2.6}$$

Based on (2.5) and (2.6), it is obtained that:

$$(\varrho+i\zeta)\widetilde{[2]}_q c_2 = \frac{U_1(t)}{2}p_1, \tag{2.7}$$

$$(\varrho+i\zeta)\widetilde{[3]}_q c_3+\frac{(\varrho+i\zeta)(\varrho-1+i\zeta)}{2}\left(\widetilde{[2]}_q\right)^2 c_2^2 = \frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2, \tag{2.8}$$

$$-(\varrho+i\zeta)\widetilde{[2]}_q c_2 = \frac{U_1(t)}{2}d_1, \tag{2.9}$$

and

$$(\varrho+i\zeta)\widetilde{[3]}_q(2c_2^2-c_3)+\frac{(\varrho+i\zeta)(\varrho-1+i\zeta)}{2}\left(\widetilde{[2]}_q\right)^2 c_2^2 = \frac{U_1(t)}{2}\left(d_2-\frac{d_1^2}{2}\right)+\frac{U_2(t)}{4}d_1^2. \tag{2.10}$$

From equations (2.7) and (2.9), it is derived that:

$$p_1 = -d_1 \tag{2.11}$$

and

$$2(\varrho+i\zeta)^2\left(\widetilde{[2]}_q\right)^2 c_2^2 = \frac{U_1^2(t)}{4}\left(p_1^2+d_1^2\right) \tag{2.12}$$

By adding (2.8) to (2.10), it is obtained that:

$$\left(2(\varrho+i\zeta)\widetilde{[3]}_q+(\varrho+i\zeta)(\varrho-1+i\zeta)\left(\widetilde{[2]}_q\right)^2\right)c_2^2 = \frac{U_1(t)}{2}(p_2+d_2)+\frac{U_2(t)-U_1(t)}{4}\left(p_1^2+d_1^2\right) \tag{2.13}$$

Substituting (2.12) into equation (2.13), it has been found that:

$$\left(2(\varrho+i\zeta)\widetilde{[3]}_q+(\varrho+i\zeta)(\varrho-1+i\zeta)\left(\widetilde{[2]}_q\right)^2-\frac{U_2(t)-U_1(t)}{U_1^2(t)}\left(2(\varrho+i\zeta)^2\left(\widetilde{[2]}_q\right)^2\right)\right)c_2^2$$

$$= \frac{U_1(t)}{2}\left(p_2+d_2\right). \tag{2.14}$$

Using Lemma 2.1 and equation (1.3) foe equation (2.14), we get:

$$|c_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|4(\varrho+i\zeta)t^2\left(\widetilde{[3]}_q+\widetilde{[2]}_q^2(\varrho-1+i\zeta)/2-(\varrho+i\zeta)\widetilde{[2]}_q^2\right)+(2t+1)\left((\varrho+i\zeta)^2\widetilde{[2]}_q^2\right)\right|}}$$

From subtracting (2.10) from (2.8) is:

$$(\varrho+i\zeta)\widetilde{[3]}_q\left(2c_3-2c_2^2\right) = \frac{U_1(t)}{2}\left(p_2-d_2\right)+\frac{U_2(t)}{4}\left(p_1^2-d_1^2\right) \tag{2.15}$$

In view of (2.11) and (2.12), it is obtained that (2.15):

$$c_3 = \frac{U_1(t)}{4(\varrho+i\zeta)\widetilde{[3]}_q}\left(p_2-d_2\right)+c_2^2$$

$$= \frac{U_1(t)}{4(\varrho+i\zeta)\left(\widetilde{[3]}_q\right)}(p_2-d_2)+\frac{U_1^2(t)\left(p_1^2+d_1^2\right)}{8(\varrho+i\zeta)^2\left(\widetilde{[2]}_q\right)^2} \tag{2.16}$$

As indicated by Lemma 2.1 and equation (1.3) , we have

$$|c_3| \leq \frac{2t}{\widetilde{[3]}_q|\varrho+i\zeta|}+\frac{4t^2}{\widetilde{[2]}_q^2|\varrho+i\zeta|^2}.$$

□

By substituting $\zeta=0$ in the previous Theorem, yields the following outcome:

Corollary 2.3. *Suppose $\mathfrak{T}$ defined by (1.1) is in the class $\widetilde{\mathfrak{M}}_\Sigma^q(t,\varrho,0)$, where $q\in(0,1), \varrho\in\mathbb{R}$ Then:*

$$|c_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|4\varrho t^2\left(\widetilde{[3]}_q+(\varrho-1)/2\left(\widetilde{[2]}_q\right)^2-\varrho\left(\widetilde{[2]}_q\right)^2\right)+(2t+1)\left((\varrho)^2\left(\widetilde{[2]}_q\right)^2\right)\right|}}$$

and

$$|c_3| \leq \frac{2t}{\varrho\left(\widetilde{[3]}_q\right)}+\frac{4t^2}{\left(\varrho\widetilde{[2]}_q\right)^2}.$$

Remark 2.4. By substituting $\varrho=1$ in the previous corollary, the results obtained align with those presented in [3].

3. Fekete-Szegö for the Class $\widetilde{\mathfrak{M}}_\Sigma^q(t,\varrho,\zeta)$

In this section, the focus is on the Fekete-Szegö inequalities for the function in the class $\widetilde{\mathfrak{M}}_\Sigma^q(t,\varrho,\zeta)$.

Theorem 3.1. *Suppose $\mathfrak{T}$ as defined by equation (1.1), belongs to the class $\widetilde{\mathfrak{M}}_\Sigma^q(t,\varrho,\zeta)$, where $\eta,\varrho,\zeta\in\mathbb{R}$, then*

$$|c_3-\eta c_2^2| \leq \begin{cases} \frac{2t}{(\varrho+i\zeta)(\widetilde{[3]}_q)} & \textit{for } |1-\eta| \leq \frac{\left((4(\widetilde{[3]}_q)+2(\varrho-1+i\zeta)\left(\widetilde{[2]}_q\right)^2-4(\varrho+i\zeta)\left(\widetilde{[2]}_q\right)^2\right)t^2+(2t+1)(\varrho+i\zeta)^2\left(\widetilde{[2]}_q\right)^2}{4(\widetilde{[3]}_q)t^2}, \\ 8t|\Upsilon(\eta)| & \textit{for } |1-\eta| \geq \frac{\left((4(\widetilde{[3]}_q)+2(\varrho-1+i\zeta)\left(\widetilde{[2]}_q\right)^2-4(\varrho+i\zeta)\left(\widetilde{[2]}_q\right)^2\right)t^2+(2t+1)(\varrho+i\zeta)^2\left(\widetilde{[2]}_q\right)^2}{4(\widetilde{[3]}_q)t^2}. \end{cases}$$

Proof. From equations (2.14) and (2.16), it is derived that:

$$c_3-\eta c_2^2=(1-\eta)c_2^2+\frac{U_1(t)}{4(\varrho+i\zeta)\widetilde{[3]}_q}(p_2-d_2)$$

Also, using equation (2.14) , we get

$$c_3-\eta c_2^2=\frac{(1-\eta)\left(U_1^3(t)(p_2+d_2)\right)}{2(\varrho+i\zeta)U_1^2(t)\left(2\widetilde{[3]}_q+(\varrho-1+i\zeta)\widetilde{[2]}_q^2\right)-(U_2(t)-U_1(t))\left(4(\varrho+i\zeta)^2\widetilde{[2]}_q^2\right)}$$

$$+\frac{U_1(t)}{4(\varrho+i\zeta)(\widetilde{[3]_q})}(p_2-d_2).$$

Simplify to:

$$c_3-\eta c_2^2=U_1(t)\left[\left(\Upsilon(\eta)+\frac{1}{4(\varrho+i\zeta)(\widetilde{[3]_q})}\right)p_2+\left(\Upsilon(\eta)-\frac{1}{4(\varrho+i\zeta)(\widetilde{[3]_q})}\right)d_2\right],$$

where

$$\Upsilon(\eta)=\frac{(1-\eta)U_1^2(t)}{2(\varrho+i\zeta)U_1^2(t)\left(2\widetilde{[3]_q}+(\varrho-1+i\zeta)\widetilde{[2]_q}^2\right)-(U_2(t)-U_1(t))\left(4(\varrho+i\zeta)^2\widetilde{[2]_q}^2\right)}.$$

As indicated by Lemma 2.1 and equation (1.3) , we have

$$|c_3-\eta c_2^2|\leq\begin{cases}\frac{2t}{(\varrho+i\zeta)(\widetilde{[3]_q})} & \text{for } |1-\eta|\leq\frac{\left((4(\widetilde{[3]_q})+2(\varrho-1+i\zeta)\left(\widetilde{[2]_q}\right)^2-4(\varrho+i\zeta)\left(\widetilde{[2]_q}\right)^2\right)t^2+(2t+1)(\varrho+i\zeta)^2\left(\widetilde{[2]_q}\right)^2}{4(\widetilde{[3]_q})t^2},\\ 8t|\Upsilon(\eta)| & \text{for } |1-\eta|\geq\frac{\left((4(\widetilde{[3]_q})+2(\varrho-1+i\zeta)\left(\widetilde{[2]_q}\right)^2-4(\varrho+i\zeta)\left(\widetilde{[2]_q}\right)^2\right)t^2+(2t+1)(\varrho+i\zeta)^2\left(\widetilde{[2]_q}\right)^2}{4(\widetilde{[3]_q})t^2}.\end{cases}$$

□

By substituting $\zeta=0$ in the previous Theorem, yields the following outcome:

Corollary 3.2. *Suppose $\mathfrak{T}$, as defined by equation (1.1), belongs to the class $\widetilde{\mathfrak{M}}_{\Sigma}^{q}(t,\varrho,0)$, where $\eta,\varrho\in\mathbb{R}$. Then*

$$|c_3-\eta c_2^2|\leq\begin{cases}\frac{2t}{\varrho\widetilde{[3]_q}} & \textit{for } |1-\eta|\leq\frac{\left((4(\widetilde{[3]_q})+2(\varrho-1)\left(\widetilde{[2]_q}\right)^2-4(\varrho)\left(\widetilde{[2]_q}\right)^2\right)t^2+(2t+1)(\varrho)^2\left(\widetilde{[2]_q}\right)^2}{4(\widetilde{[3]_q})t^2},\\ 8t|\Upsilon(\eta)| & \textit{for } |1-\eta|\geq\frac{\left((4(\widetilde{[3]_q})+2(\varrho-1)\left(\widetilde{[2]_q}\right)^2-4(\varrho)\left(\widetilde{[2]_q}\right)^2\right)t^2+(2t+1)(\varrho)^2\left(\widetilde{[2]_q}\right)^2}{4(\widetilde{[3]_q})t^2},\end{cases}$$

where

$$\Upsilon(\eta)=\frac{(1-\eta)U_1^2(t)}{2\varrho U_1^2(t)\left(2\widetilde{[3]_q}+(\varrho-1)\widetilde{[2]_q}^2\right)-(U_2(t)-U_1(t))\left(4(\varrho)^2\widetilde{[2]_q}^2\right)}$$

Remark 3.3. By substituting $\varrho=1$ in the previous corollary, the results obtained align with those presented in [3].

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