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ON k-POTENT ARMENDARIZ RINGS

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Abstract. In this paper, we introduce the concept of k-potent Armendariz rings, generalizing the notion of Armendariz rings by requiring that products of coefficients of annihilating polynomials be k-potent instead of zero. We investigate the behavior of k-potent Armendariz rings under various ring constructions, such as direct product, skew polynomial extension, trivial extension and matrix rings. Furthermore, we show that if R is a reversible ring which also satisfies k-potent Armendariz property, then R is Armendariz. Lastly, it has been shown that if R is a k-potent Armendariz ring, then R is an abelian ring and McCoy ring.

1. Introduction and preliminaries

In this paper, unless explicitly stated otherwise, we assume a ring R with a multiplicative identity element. We adopt the following notations throughout: $Z(R)$ represents the center of R, $T(R,R)$ denotes the trivial extension of R, $N(R)$ is the set of all nilpotent elements in R, $Q(R)$ signifies the quasi-center of R, and $J(R)$ stands for the Jacobson radical of R. An element $t \in R$ is termed *k-potent* if there exists an integer $k \geq 2$ such that $t^k = t$. The set of all k-potent elements in R is denoted by $Po(R)$. Specifically, when $k = 2$, t is known as an *idempotent*, and when $k = 3$, it is called *tripotent*. We denote the polynomial ring over R by $R[x]$. Unless indicated otherwise, we assume that polynomials $f(x)$ and $g(x)$ in $R[x]$ are expressed in the forms $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$, respectively. The ring of all $n \times n$ matrices with entries from R is denoted by $M_n(R)$, while $T_n(R)$ represents the ring of all $n \times n$ upper triangular matrices over R. Recall that an additive map $d : R \to R$ is called a *derivation* if it satisfies the Leibniz rule: $d(ab) = d(a)b + a\,d(b)$ for all $a, b \in R$. For a given ring R, the *Ore extension*, denoted $R[x; \alpha, d]$, is constructed using an endomorphism α of R and an α-derivation d (meaning $d(ab) = d(a)b + \alpha(a)d(b)$). In this extension, multiplication is defined such that $xr = \alpha(r)\,x + d(r)$ for all $r \in R$. When $d = 0$, the Ore extension simplifies to the *skew polynomial ring* $R[x; \alpha]$.

The study of k-potent elements and matrices has gathered considerable attention in recent years due to their rich algebraic structures and significant applications in various fields, particularly in statistics. In linear algebra and matrix

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theory, k-potent matrices (matrices satisfying $A^k = A$ for some positive integer $k \geq 2$) play a crucial role in statistical models, especially in the analysis of quadratic forms and projection operators. For instance, idempotent matrices ($k = 2$) and tripotent matrices ($k = 3$) are extensively used in regression analysis, analysis of variance, and the study of linear models in statistics. Also, the k-potent elements are widely used in ring theory to study idempotent-based decompositions, matrix theory, and polynomial extensions. Their significance is evident in their connections to von Neumann regularity, unit-regularity, and clean elements, which help characterize rings with desirable algebraic properties. Several researchers have contributed to the understanding of k-potent matrices and their properties. Baksalary and Trenkler in [18] provided a systematic investigation of k-potent complex matrices, with a particular focus on tripotent matrices. Their work was motivated by potential applications in statistics, highlighting the role of tripotent matrices in problems concerning the distribution of quadratic forms and generalized projectors.

Inspired by these developments in matrix theory, it becomes natural to explore analogous concepts within the framework of ring theory. In [13], Mosić explored the multiple algebraic characterizations of k-potent elements within rings and substantially simplify the proofs of previously established characterizations.

Concurrently, significant research has been directed toward Armendariz rings. The concept of an Armendariz ring was initially established by Rege and Chhawchharia in [15]. A ring R is called an *Armendariz ring* if, for any polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in the polynomial ring $R[x]$, the equation $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all indices i and j. The term "Armendariz" is used because E.P. Armendariz demonstrated in [3] that reduced rings possess this property. Similarly, McCoy rings are defined based on results from [12]. A ring R is called a *right McCoy* if for any nonzero polynomial $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ in the polynomial ring $R[x]$ and any nonzero polynomial $g(x)$ in $R[x]$, there exists a nonzero element $c \in R$ such that $f(x) \cdot c = 0$ whenever $f(x)g(x) = 0$. The definition of a *left McCoy ring* is analogous. A ring that is both left and right McCoy is simply referred to as a *McCoy ring*.

Extensive research on Armendariz rings has been done, leading to numerous significant generalizations (see [1, 10, 11, 16, 17] for more details). For example, Liu and Zhao in [11], defined a ring R to be *weak Armendariz* if, whenever the product of two nonzero polynomials $f(x)g(x) = 0$, then $a_i b_j \in N(R)$ for all i and j. In another generalization, Agayev et al. presented the idea of *central Armendariz rings* in [1], defining a ring R to be central Armendariz if $f(x)g(x) = 0$ implies $a_i b_j \in Z(R)$ for all i and j.

Moreover, Sanaei et al. introduced *J-Armendariz rings* in [17]. A ring R is called J-Armendariz if, whenever $f(x)g(x) = 0$, each product $a_i b_j$ lies in the Jacobson radical $J(R)$ for all i and j. Liu further defined *quasi-central Armendariz rings* in [10] by stipulating that $f(x)g(x) = 0$ implies $Ra_i b_j = a_i b_j R$ for all i and j, indicating that each $a_i b_j$ is a quasi-central element of R. Additionally, Razaghi and Sahebi, in [16], introduced the notion of *idempotent Armendariz rings*, where a ring R is idempotent Armendariz if $f(x)g(x) = 0$ leads to each a_ib_j being an idempotent element of R.

In [4], Bakkari explored how the power Armendariz property transfers to various ring extensions. Patel, in [14], introduced the concept of the power serieswise Armendariz graph of a ring and studied certain properties such as diameter, clique number, and girth of the graph. Relationships between Armendariz rings and other classes of rings including semicommutative rings, Gaussian rings, and reversible rings have been investigated in works like [2, 6, 9].

In seeking to bridge the gap between k-potent elements and Armendariz rings, we introduce the notion of k-potent Armendariz rings by requiring that products of coefficients of annihilating polynomials be k-potent instead of zero. Specifically, every k-potent ring (ring where all elements are k-potent) is a k-potent Armendariz ring, but the converse is not necessarily true, see Example 2.6. We also examine the relationships between k-potent Armendariz rings and other rings, such as McCoy rings and Abelian rings. The main results of this paper include characterizations of k-potent Armendariz rings, investigations into their properties under various ring constructions (such as direct products, matrix rings, and trivial extensions), and examples illustrating their behavior. Finally, we give a relationship diagram connecting k-potent Armenariz rings and other rings.

2. k-potent Armendariz Rings

Definition 2.1. A ring R is called *k-potent Armendariz* if for any polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$, whenever $f(x)g(x) = 0$, the product a_ib_j is a *k-potent element* of R; that is, $(a_ib_j)^k = a_ib_j$ for all indices i, j, where $k \geq 2$ is a fixed integer.

Lemma 2.2. *If R is an Armendariz ring, then R is k-potent Armendariz.*

Proof. This follows directly from the definitions of Armendariz and k-potent Armendariz rings. □

Lemma 2.3. [3, Lemma 1] *Let R be a reduced ring. Then the product of two nonzero polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$, satisfy $f(x)g(x) = 0$ if and only if $a_ib_j = 0$ for all i, j.*

Lemma 2.4. [8, Theorem 2.3] *Let R be a ring. Then the trivial extension $T(R, R)$ of R satisfies Armendariz property if R is reduced and converse also holds.*

Definition 2.5. A ring is R is said to be a k-potent ring if each of its element is k-potent.

It is clear that all k-potent rings are k-potent Armendariz but the converse need not be true in general.

Example 2.6. Let $\mathbb{Z}$ be the ring of integers. Since $\mathbb{Z}$ is reduced, it is Armendariz by Lemma 2.3 and hence k-potent Armendariz by Lemma 2.2. Clearly, $\mathbb{Z}$ is not a k-potent ring.

Let R be a ring and M an R-bimodule. The *trivial extension* of R by M, denoted $T(R, M)$, is constructed as the direct sum $R \oplus M$ with component-wise addition. The multiplication operation is defined for pairs $(r,p),(s,q)\in T(R,M)$ by

$$(r,p)\cdot(s,q)=(rs,rq+ps),$$

where $r,s\in R$ and $p,q\in M$. This ring is isomorphic to the matrix ring consisting of all 2×2 matrices of the form

$$\begin{pmatrix} r & p \\ 0 & r\end{pmatrix},$$

where $r\in R$ and $p\in M$, under standard matrix addition and multiplication.

One may suspect that the trivial extension $T(R,R)$ over a k-potent Armendariz ring is k-potent Armendariz but the following example shows that if R is k-potent Armendariz, then it is not always true that $T(R,R)$ is also k-potent Armendariz.

Example 2.7. Suppose $\mathbb{H}=\{c_1+c_2i+c_3j+c_4k \mid c_1,c_2,c_3,c_4\in\mathbb{R}\}$ be the ring of Hamiltonian quaternions over the field of real numbers. Since $\mathbb{H}$ is reduced, $R=T(\mathbb{H},\mathbb{H})$ is Armendariz by Lemma 2.4 and hence k-potent Armendariz by Lemma 2.2. Let $S=T(R,R)$. We argue that S is not k-potent Armendariz by considering the polynomials $r(x)=T+Ux$ and $s(x)=V+Wx$ in $S[x]$, where

$$T=\begin{pmatrix}\begin{pmatrix}0 & j\\ 0 & 0\end{pmatrix} & \begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & j\\ 0 & 0\end{pmatrix}\end{pmatrix}, U=\begin{pmatrix}\begin{pmatrix}0 & j\\ 0 & 0\end{pmatrix} & \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & j\\ 0 & 0\end{pmatrix}\end{pmatrix}$$

$$V=\begin{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\end{pmatrix}, W=\begin{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & \begin{pmatrix}-k & 0\\ 0 & -k\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\end{pmatrix}.$$

Then $r(x)s(x)=0$. But $TW=\begin{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & k-i\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\end{pmatrix}$ is not a k-potent element. Hence, S is not k-potent Armendariz.

In [2] Anderson and Camillo proved the following result:

Theorem 2.8. [2, Theorem 5] *For any ring R and any natural number $n\geq 2$, the quotient ring $R[x]/(x^n)$ is Armendariz if and only if R is reduced.*

We prove the similar result for k-potent Armendariz ring, that is R is still reduced even if we take R to be k-potent Armendariz instead of Armendariz.

Theorem 2.9. *Let R be a ring. Then $R[x]/(x^n)$ is k-potent Armendariz if and only if R is reduced for any natural number $n\geq 2$.*

Proof. Suppose that R is a reduced ring. Then $R[x]/(x^n)$ is Armendariz by Theorem 2.8 and hence it is k-potent Armendariz by Lemma 2.2. Now, conversely suppose that $R[x]/(x^n)$ is k-potent Armendariz. Let $r\in R$ with $r^n=0$. Since r and $\bar{x}$ commute, $0=r^n-\bar{x}^nt^n=(r-\bar{x}t)(r^{n-1}+r^{n-2}\bar{x}t+\cdots+\bar{x}^{n-1}t^{n-1})$ where t is indeterminate. Since $R[x]/(x^n)$ is k-potent Armendariz, it gives $(r\bar{x}^{n-1})^k=r\bar{x}^{n-1}$ which implies $r=0$ and hence R is reduced. $\square$

Theorem 2.10. *Let $\{R_\alpha\}_{\alpha\in I}$ be a family of rings indexed by a set I. Then, the direct product ring $R = \prod_{\alpha\in I} R_\alpha$ is k-potent Armendariz if and only if each individual ring R_α is k-potent Armendariz.*

Proof. Assume that each ring R_α is k-potent Armendariz. Let $R = \prod_{\alpha\in I} R_\alpha$ be their direct product. Consider polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ such that $f(x)g(x) = 0$. Each coefficient a_i and b_j is a tuple $(a_{i,\alpha})_{\alpha\in I}$ and $(b_{j,\alpha})_{\alpha\in I}$, respectively. For each $\alpha \in I$, define polynomials in $R_\alpha[x]$:

$$f_\alpha(x) = \sum_{i=0}^{m} a_{i,\alpha} x^i, \quad g_\alpha(x) = \sum_{j=0}^{n} b_{j,\alpha} x^j.$$

Since $f(x)g(x) = 0$ in $R[x]$, it implies that

$$a_0 b_0 = 0 \tag{2.1}$$

$$a_0 b_1 + a_1 b_0 = 0 \tag{2.2}$$

$$\vdots$$

$$a_m b_n = 0. \tag{2.3}$$

From equation 2.1 we have

$$a_{0,1}b_{0,1} = a_{0,2}b_{0,2} = \cdots = a_{0,\alpha}b_{0,\alpha} = \cdots = 0.$$

From equation 2.2 we have

$$a_{0,1}b_{1,1} + a_{1,1}b_{0,1} = a_{0,2}b_{1,2} + a_{1,2}b_{0,2} = \cdots = a_{0,\alpha}b_{1,\alpha} + a_{1,\alpha}b_{0,\alpha} = \cdots = 0.$$

From equation 2.3 we have

$$a_{m,1}b_{n,1} = a_{m,2}b_{n,2} = \cdots = a_{m,\alpha}b_{n,\alpha} = \cdots = 0.$$

This implies that for every $\alpha \in I$:

$$f_\alpha(x)g_\alpha(x) = 0 \quad \text{in } R_\alpha[x].$$

Given that each R_α is k-potent Armendariz, it follows that for all i and j:

$$a_{i,\alpha}b_{j,\alpha} \in \mathrm{Po}(R_\alpha).$$

Hence, by the equation

$$\mathrm{Po}(\prod_{\alpha\in I} R_\alpha) = \prod_{\alpha\in I} \mathrm{Po}(R_\alpha),$$

it follows that $a_i b_j \in \mathrm{Po}(R)$. Therefore, R is k-potent Armendariz. Conversely, suppose that the direct product $R = \prod_{\alpha\in I} R_\alpha$ is k-potent Armendariz. Let α_0 be any fixed element in I. Consider polynomials $f_{\alpha_0}(x) = a_{0,\alpha_0} + a_{1,\alpha_0}x + \cdots + a_{m,\alpha_0}x^m$ and $g_{\alpha_0}(x) = b_{0,\alpha_0} + b_{1,\alpha_0}x + \cdots + b_{n,\alpha_0}x^n$ in $R_{\alpha_0}[x]$ such that $f_{\alpha_0}(x)g_{\alpha_0}(x) = 0$ in $R_{\alpha_0}[x]$. We will construct polynomials $P(x)$ and $Q(x)$ in $R[x]$ as $P(x) = \sum_{i=0}^{m} a_i x^i, Q(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ where $a_i = (0, \cdots, 0, a_{i,\alpha}, 0, \cdots)$ and $b_j = (0, \cdots, 0, b_{j,\alpha}, 0, \cdots)$ is in R. Now, we have $P(x)Q(x) = 0$, since $f_{\alpha_0}(x)g_{\alpha_0}(x) = 0$. Therefore, $a_i b_j$ is a k-potent element of R, since R is k-potent Armendariz. Hence, $a_{i,\alpha}b_{j,\alpha} \in Po(R_\alpha)$ and thus, each ring R_α is k-potent Armendariz. $\square$

In [7, Example 3], Kim and Lee established that the ring R_n is not Armendariz for $n \geq 4$. Extending this result, we demonstrate that R_n also fails to satisfy the k-potent Armendariz property for $n \geq 4$ by using the same example.

Example 2.11. Let A be a ring and let

$$R_4 = \left\{ \begin{pmatrix} b & b_{12} & b_{13} & b_{14} \\ 0 & b & b_{23} & b_{24} \\ 0 & 0 & b & b_{34} \\ 0 & 0 & 0 & b \end{pmatrix} \;\middle|\; b, b_{ij} \in A \right\}.$$

Let

$$p(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

and

$$q(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

be polynomials in $R_4[x]$. Then $p(x)q(x) = 0$, but

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin Po(R_4).$$

So R_4 is not k-potent Armendariz and the same is true for $n \geq 5$.

The notion of reversible rings was first established by Cohn in [5].

Definition 2.12. A ring R is said to be *reversible* if, for any elements $c, d \in R$, the condition $cd = 0$ implies $dc = 0$.

In [16], Razaghi and Sahebi showed that if R is both idempotent Armendariz and reversible, then R is Armendariz. We establish a similar result for k-potent Armendariz rings.

Theorem 2.13. *Let R be a reversible ring that is also k-potent Armendariz. Then R is Armendariz.*

Proof. Let

$$f(x) = \sum_{i=0}^{m} c_i x^i \quad \text{and} \quad g(x) = \sum_{j=0}^{n} d_j x^j$$

be polynomials in $R[x]$ with $f(x)g(x) = 0$. This forces all coefficients of the product to vanish, i.e., for every k with $0 \leq k \leq m + n$,

$$\sum_{i+j=k} c_i d_j = 0.$$

Explicitly:

$$c_0d_0 = 0 \quad (1)$$
$$c_0d_1 + c_1d_0 = 0 \quad (2)$$
$$c_0d_2 + c_1d_1 + c_2d_0 = 0 \quad (3)$$
$$\vdots$$

Since R is k-potent Armendariz, $c_id_j \in \mathrm{Po}(R)$. We show that $c_id_j = 0$.
By reversibility, (1) implies $d_0c_0 = 0$. Right-multiplying (2) by c_0 yields:

$$c_0d_1c_0 + c_1d_0c_0 = 0 \implies c_0d_1c_0 = 0 \implies c_0d_1 = (c_0d_1)^k = 0.$$

From (2), $c_1d_0 = 0$. Next, right-multiply (3) by c_0:

$$c_0d_2c_0 + c_1d_1c_0 + c_2d_0c_0 = 0 \implies c_0d_2c_0 = 0 \implies c_0d_2 = (c_0d_2)^k = 0.$$

Equation (3) reduces to $c_1d_1 + c_2d_0 = 0$. Right-multiplying by c_1:

$$c_1d_1c_1 + c_2d_0c_1 = 0 \implies c_1d_1c_1 = 0 \implies c_1d_1 = (c_1d_1)^k = 0.$$

Hence, $c_2d_0 = 0$. Iterating this argument inductively gives $c_id_j = 0$ for all i, j. Thus, R is Armendariz. □

Lemma 2.14. [12, Theorem 2] *Let R be a commutative ring. If an element $f(x)$ of $R[x]$ is a zero divisor in $R[x]$, then $f(x).c = 0$ for some nonzero element $c \in R$.*

Theorem 2.15. *If R is a k-potent Armendariz ring, then R is a McCoy ring.*

Proof. Consider a k-potent Armendariz ring R and let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ be polynomials in $R[x]$ such that $f(x)g(x) = 0$. By the k-potent Armendariz property, we have $(a_ib_j)^k = a_ib_j$ for all i, j. This yields $(a_ib_j)^k - a_ib_j = 0$, which can be rewritten as $a_i((b_ja_i)^{k-1}b_j - b_j) = 0$, establishing that R is right McCoy. Similarly, the equation $((a_ib_j)^{k-1}a_i - a_i)b_j = 0$ demonstrates that R is left McCoy. Therefore, R is a McCoy ring. □

The converse of the theorem above does not hold universally, as illustrated by the following example.

Example 2.16. Consider the ring T of 2×2 matrices over $\mathbb{Z}_4$ (integers modulo 4) of the form:

$$T = \left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \mid u, v \in \mathbb{Z}_4 \right\}$$

Since T is commutative, it is McCoy by Lemma 2.14. Consider the polynomials $t(x), w(x) \in T[x]$ defined as:

$$t(x) = w(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$$

While $t(x)w(x) = 0$, the product of their coefficients

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

is not k-potent for any k. Therefore, T is not a k-potent Armendariz ring.

The *Ore extension*, denoted as$R[x;\alpha,d]$ is a generalization of the polynomial ring $R[x]$ where the multiplication is defined by the rule

$$xr = \alpha(r)x + d(r) \quad \text{for all } r \in R.$$

When $d = 0$, this reduces to the *skew polynomial ring* $R[x;\alpha]$, where multiplication simplifies to $xr = \alpha(r)x$.

Notably, even if R is a k-potent Armendariz ring, the skew polynomial ring $R[x;\alpha]$ may fail to preserve the k-potent Armendariz property. This is demonstrated in the example below.

Example 2.17. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the ring of integers modulo 2. The operations in R are defined component-wise. Clearly R is a commutative ring which is also reduced. According to Lemma 2.3, every reduced ring is Armendariz. Thus, by Lemma 2.2, R inherits the k-potent Armendariz property.

Define an automorphism $\alpha : R \to R$ by swapping the components of each element, specifically, $\alpha((a,b)) = (b,a)$. We will demonstrate that the skew polynomial ring $R[x;\alpha]$ does not possess the k-potent Armendariz property.

Consider the polynomials in $R[x;\alpha][t]$ given by

$$p(t) = (1,0) + [(1,0)]x, \quad \text{and} \quad q(t) = (0,1) + [(1,0)x]t.$$

Multiplying these polynomials, we find that

$$p(t)q(t) = 0.$$

However, the element $(1,0)[(1,0)x]$ is not k-potent in $R[x;\alpha]$. This shows that $R[x;\alpha]$ is not a k-poptent Armendariz ring.

A ring R is defined as *abelian* if all of its idempotent elements are contained within its center, $Z(R)$. In [1, Proposition 2.1], Agayev et al. proved that if R is central Armendariz then R is abelian. We establish an analogous result for k-potent Armendariz rings.

Theorem 2.18. *A ring R is k-potent Armendariz if and only if it satisfies the following equivalent conditions:*

- *R is abelian (all idempotents are central), and for every idempotent $e \in R$, the subrings eR and $(1-e)R$ are k-potent Armendariz.*
- *There exists a central idempotent $e \in R$ such that eR and $(1-e)R$ are k-potent Armendariz.*

Proof. **($\Rightarrow$):** Assume R is k-potent Armendariz. To prove R is abelian, let $e \in R$ be an idempotent. For arbitrary $r \in R$, define:

$$h(x) = e - er(1-e)x \quad \text{and} \quad k(x) = (1-e) + er(1-e)x.$$

Then $h(x)k(x) = 0$. By the k-potent Armendariz property, $er(1-e)$ is k-potent, so $(er(1-e))^k = er(1-e)$. Since $(er(1-e))^2 = 0$, iterating yields $er(1-e) = 0$, implying $er = ere$. Similarly, define:

$$u(x) = (1-e) - (1-e)rex \quad \text{and} \quad v(x) = e + (1-e)rex.$$

As $u(x)v(x) = 0$, the k-potent Armendariz property forces $(1-e)re = 0$, so $ere = re$. Thus, e is central, proving R is abelian. Also, eR and $(1-e)R$ are k-potent Armendariz as subrings of k-potent Armendariz is k-potent Armendariz.

(⇐): Let e be a central idempotent in R with eR and $(1-e)R$ both k-potent Armendariz. Suppose $h(x) = \sum_{i=0}^{m} a_i x^i$ and $k(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $h(x)k(x) = 0$. Decompose:

$$h(x) = eh(x) + (1-e)h(x), \quad k(x) = ek(x) + (1-e)k(x).$$

In $eR[x]$ and $(1-e)R[x]$, we have $(ea_i)(eb_j) \in eR$ and $((1-e)a_i)((1-e)b_j) \in (1-e)R$, both k-potent. Hence

$$\begin{aligned}
(a_i b_j)^k &= (ea_i b_j + (1-e)a_i b_j)^k \\
&= (ea_i b_j)^k + e(1-e)(a_i b_j)^k + (1-e)e(a_i b_j)^k + ((1-e)a_i b_j)^k \\
&= (ea_i b_j + (1-e)a_i b_j) \\
&= a_i b_j.
\end{aligned}$$

Then, $a_i b_j$ is k-potent in R for all indices i and j. Therefore, R is k-potent Armendariz. □

Corollary 2.19. *$M_n(R)$ and $T_n(R)$ are not k-potent Armendariz.*

Proof. Since some idempotents of $M_n(R)$ and $T_n(R)$ are non-central, then these rings are not abelian. Therefore, these rings are not k-potent Armendariz by Theorem 2.18. □

Remark 2.20. Let $e \in R$ be any nontrivial idempotent. Then eRe is k-potent Armendariz whenever R is k-potent Armendariz; but $M_n(R)$ is not k-potent Armendariz. Since $M_n(R)$ is morita equivalent to R it means "k-potent Armendariz" is not a morita invariant property.

Consider a ring R and let $P(R)$ represent its prime radical. Using the argument in [7, Example 13], we show that R may fail to be k-potent Armendariz even when both $P(R)$ and $R/P(R)$ are k-potent Armendariz.

Example 2.21. Consider the ring of integers $\mathbb{Z}$ and suppose

$$R = \left\{ \begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \middle| \; u - v \equiv w \equiv 0 \pmod 2 \right\}.$$

Then $P(R) = \left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \middle| \; w \equiv 0 \pmod 2 \right\}$, and so $P(R)$ is k-potent Armendariz. Also, the only idempotents of R are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, R is an abelian ring.

$R/P(R)$ is k-potent Armendariz as it is reduced. Now, we claim that R is not k-potent Armendariz. Let

$$t(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \quad \text{and} \quad s(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x.$$

Clearly $t(x)s(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is not a k-potent element of R. Therefore, R is not a k-potent Armendariz ring.

3. Conclusion

The paper concludes by establishing key properties and relationships of k-potent Armendariz rings. It demonstrates that every k-potent Armendariz ring is abelian and a McCoy ring, but the converse does not always hold. The study also explores how these rings behave under various constructions, such as direct products, matrix rings, and trivial extensions, showing that some structures do not necessarily preserve the k-potent Armendariz property. Additionally, we show that if a ring is both reversible and k-potent Armendariz, then it is Armendariz. We conclude the paper by the following relation.

$$\begin{array}{ccccccc} \text{k-potent} & \Longrightarrow & \text{Reduced} & \Longrightarrow & \text{Armendariz} & \Longrightarrow & \textbf{k-potent Armendariz} \\ & & & & & & \Downarrow \\ & & & & & & \text{McCoy} \end{array}$$

Overall, the work extends the study of Armendariz rings and provides a deeper understanding of their algebraic structures and applications.

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