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THE DIOPHANTINE EQUATION $x^3 + y^3 = 2z^2$

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Abstract. In this paper. We are intersted to show the methode of the solving of the diophantine equation $x^3 + y^3 = 2z^2$ by using the arithmetic technicals and the diophantine equations $x^2 + 3y^2 = z^2$ (see [1]) and $x^2 - 3y^2 = 2z^2$.

1. Introduction

The problem of the diophantine equations was intersted by a several Mathematicians (see [1], [2], [7], [8], [9], [11], [4])

We know that while the equation $x^2 + y^2 = z^2$ has infinitely many solutions in integers, the equation $x^4 + y^4 = z^2$ has none (see [12]). What about the equation $x^3 + y^3 = 2z^2$?.

The obvious $4^3 + 4^3 = 2 \times 8^2$. So we search to find the solutions the above diophantine equation.

For that we study this problem by showing the following results:

Theorem 1.1.

Let $(x_0, y_0, z_0) \in \mathbb{Z}^3$ *such that* $x_0 \wedge y_0 = 1$, $x_0 \equiv y_0 \bmod 4$ *and* $x_0 \not\equiv y_0 \bmod 3$. *then the following conditions are equivalent*

(i) (x_0, y_0, z_0) *is the solution of* $E : x^3 + y^3 = 2z^2$

(ii) $x_0 = (c^2 - 3d^2)^2 + 4(c^2 + 3d^2)(cd)$, $y_0 = (c^2 - 3d^2)^2 - 4(c^2 + 3d^2)(cd)$
$\mid z_0 \mid = (c^2 - 3d^2)((c^2 + 3d^2)^2 + 3(2cd)^2)$.

or

$x_0 = (\frac{c^2-3d^2}{2})^2 + (c^2 + 3d^2)(cd)$, $y_0 = (\frac{c^2-3d^2}{2})^2 - (c^2 + 3d^2)(cd)$
$\mid z_0 \mid = (c^2 - 3d^2)((\frac{c^2+3d^2}{2})^2 + 3(cd)^2)$.

Theorem 1.2.

Let $(x_0, y_0, z_0) \in \mathbb{Z}^3$ *such that* $x_0 \wedge y_0 = 1$, $x_0 \geq y_0$, $x_0 \equiv y_0 \bmod 4$ *and* $x_0 \equiv y_0 \bmod 3$. *then the following conditions are equivalent*

(i) (x_0, y_0, z_0) *is the solution of* $E : x^3 + y^3 = 2z^2$

(ii) $x_0 = 3z_5^2 z_6^2 + \frac{z_5^4 - 3z_6^4}{2}$, $y_0 = 3z_5^2 z_6^2 - \frac{z_5^4 - 3z_6^4}{2}$,
$\mid z_0 \mid = 3z_5 z_6 (\frac{z_5^4 + 3z_6^4}{2})$

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Theorem 1.3.

Let $E : x^3 + y^3 = 2z^2$ diophantine equation and $(x_0, y_0, z_0) \in \mathbb{Z}^3$ such that $x_0 \wedge y_0 = 1$, $x_0 \not\equiv y_0 \bmod 4$ and $x_0 \equiv -y_0 \bmod 3$. then the following properties are equivalent

(i) (x_0, y_0, z_0) *is the solution of E*

(ii) $x_0 = 12z_6^2z_7^2 + 4z_6^4 - 3z_7^4$, $\quad y_0 = 12z_6^2z_7^2 - 4z_6^4 + 3z_7^4$, $\mid z \mid= 6z_7z_6(4z_6^4 + 3z_7^4)$

or

$x_0 = 12z_6^2z_7^2 + z_6^4 - 12z_7^4$, $y_0 = 12z_6^2z_7^2 - z_6^4 + 12z_7^4$, $z_0 = 6z_7z_6(z_6^4 + 12z_7^4)$

Theorem 1.4.

Let $x_0, y_0, z_0 \in \mathbb{Z}$, such that (x_0, y_0, z_0) is the solution of the following equation $E : x^2 - 3y^2 = 2z^2$, $x_0 \wedge y_0 = 1$, $x_0 \not\equiv y_0 \bmod 4$ and $x_0 \not\equiv -y_0 \bmod 3$ then $x_0y_0z_0 = 0$

2. THE DIOPHANTINE EQUATION $x^2 - 3y^2 = 2z^2$

In this section we show that $(0, 0, 0)$ is only solution of the diophantine equation $x^2 - 3y^2 = 2z^2$.

Theorem 2.1.

Let $x_0, y_0, z_0 \in \mathbb{Z}$, such that (x_0, y_0, z_0) is the solution of the following equation $E : x^2 - 3y^2 = 2z^2$ then $x_0y_0z_0 = 0$.

Proof.

Assume that (x, y, z) the soluttion of the $E : x^2 - 3y^2 = 2z^2$ with $x \wedge y = 1$. We have $x^2 - 3y^2 = 2z^2$ then x and y are odds

1^{st} **case** x is divisible by 3 i.e $(x = 3x_1)$

$$\begin{array}{ccc} & x^2 + 3y^2 & = & 2z^2 \\ & (3x_1)^2 + 3y^2 & = & 2z^2 \\ wich\ implies\ that & z & = & 3z_1 \\ i.e & 3x_1^2 + y^2 & = & 3z_1^2 \\ i.e & y & = & 3y_1 \end{array}$$

We deduce that 3 divide $x \wedge y = 1$ contradiction

2^{nd} **case** z is divisible by 3 i.e $(z = 3z_1)$

$$\begin{array}{cccc} & x^2 + 3y^2 & = & 2z^2 \\ & x^2 + 3y^2 & = & 2(3z_1)^2 \\ wich\ implies\ that & x & = & 3x_1 \\ i.e & 3x_1^2 + y^2 & = & 3z_1^2 \\ i.e & y & = & 3y_1 \end{array}$$

We deduce that 3 divide $x \wedge y = 1$ contradiction

3^{rd} case x and y are not divisible by 3

We can write $xz \wedge 3 = 1$ then $x^2 \equiv 1 \bmod 3$ and $z^2 \equiv 1 \bmod 3$ **(A1)**.

Using **(A1)** it follows $3y^2 = x^2 - 2z^2 \equiv -1 \bmod 3$ contradiction $\square$

3. CHARACTERIZATION OF SOLUTIONS OF THE DIOPHANTINE EQUATION $x^3 + y^3 = 2z^2$

To find the solution of the above diophantine equation, we will study four cases
1^{er} **case** ($x \equiv y \bmod 4$ and $x \not\equiv -y \bmod 3$)

Theorem 3.1.

Let $(x_0, y_0, z_0) \in \mathbb{Z}^3$ *such that* $x_0 \wedge y_0 = 1$, $x_0 \equiv y_0 \bmod 4$ *and* $x_0 \not\equiv y_0 \bmod 3$. *then the following conditions are equivalent*

(i) (x_0, y_0, z_0) *is the solution of* $E : x^3 + y^3 = 2z^2$

(ii) $x_0 = (c^2 - 3d^2)^2 + 4(c^2 + 3d^2)(cd)$, $y_0 = (c^2 - 3d^2)^2 - 4(c^2 + 3d^2)(cd)$
$\mid z_0 \mid = (c^2 - 3d^2)((c^2 + 3d^2)^2 + 3(2cd)^2)$.
or
$x_0 = (\frac{c^2-3d^2}{2})^2 + (c^2 + 3d^2)(cd)$, $y_0 = (\frac{c^2-3d^2}{2})^2 - (c^2 + 3d^2)(cd)$
$\mid z_0 \mid = (c^2 - 3d^2)((\frac{c^2+3d^2}{2})^2 + 3(cd)^2)$.

Proof.

(i) $\Longrightarrow$ **(ii)**

Let (x, y, z) is the solution of E such that $x = u + v$ and $y = u - v$ **(a1)**.
We have $x \equiv y \bmod 4$ then $x - y = 4k$ So $v = 2k$. We deduce that u is odd and v is even.

It is clear that $u \wedge v = 1$. We can see

$$\begin{array}{rcl} (u+v)^3 + (u-v)^3 & = & 2u(u^2 + 3v^2) \\ & = & 2z^2 \\ \textit{wich implies that} \quad u(u^2 + 3v^2) & = & z^2 \qquad (\mathbf{a2}) \end{array}$$

Since $x \not\equiv -y \bmod 3$ then 3 doesn't divide $x+y = 2u$ wich implies that $u \wedge 3 = 1$ then $u^2 + 3v^2 \wedge u = 1$.

By **(a2)** we can write $u = z_1^2$, $u^2 + 3v^2 = z_2^2$ and $z = z_1 z_2$ **(a3)**.
Since [1], we can write $u = a^2 - 3b^2$, $\mid v \mid = 2ab$ and $\mid z_2 \mid = a^2 + 3b^2$ **(a4)**

Since **(a3)** and **(a4)** we have $a^2 - 3b^2 = z_1^2$ then $z_1^2 + 3b^2 = a^2$ therefore
$\mid z_1 \mid = c^2 - 3d^2$, $\mid b \mid = 2cd$ and $\mid a \mid = c^2 + 3d^2$ **(a5)** or
$\mid z_1 \mid = \frac{c^2-3d^2}{2}$, $\mid b \mid = cd$ and $\mid a \mid = \frac{c^2+3d^2}{2}$ **(a5)**
Assume $x < y$ then $v < 0$

$$\begin{array}{rcl} x & = & u + v \\ & = & a^2 - 3b^2 + 2ab \\ & = & (c^2 + 3d^2)^2 - 3(2cd)^2 + 2(c^2 + 3d^2)(2cd) \\ & = & (c^2 - 3d^2)^2 + 4(c^2 + 3d^2)(cd) \\ y & = & u - v \\ & = & a^2 - 3b^2 - 2ab \\ & = & (c^2 + 3d^2)^2 - 3(2cd)^2 - 2(c^2 + 3d^2)(2cd) \\ & = & (c^2 - 3d^2)^2 - 4(c^2 + 3d^2)(cd) \\ \mid z \mid & = & z_1 z_2 \\ & = & (c^2 - 3d^2)(a^2 + 3b^2) \\ & = & (c^2 - 3d^2)((c^2 + 3d^2)^2 + 3(2cd)^2) \end{array}$$

or

$$\begin{array}{rcl} x & = & u+v \\ & = & a^2-3b^2+2ab \\ & = & (\frac{c^2+3d^2}{2})^2-3(cd)^2+2(\frac{c^2+3d^2}{2})(cd) \\ & = & (\frac{c^2-3d^2}{2})^2+(c^2+3d^2)(cd) \\ y & = & u-v \\ & = & a^2-3b^2-2ab \\ & = & (\frac{c^2+3d^2}{2})^2-3(cd)^2-2(\frac{c^2+3d^2}{2})(cd) \\ & = & (\frac{c^2-3d^2}{2})^2-(c^2+3d^2)(cd) \\ \mid z \mid & = & z_1z_2 \\ & = & (\frac{c^2-3d^2}{2})(a^2+3b^2) \\ & = & (c^2-3d^2)((\frac{c^2+3d^2}{2})^2+3(cd)^2) \end{array}$$

□

2^{nd} **case** ($x \equiv y \bmod 4$ and $x \equiv y \bmod 3$)

Theorem 3.2.
Let $(x_0, y_0, z_0) \in \mathbb{Z}^3$ *such that* $x_0 \wedge y_0 = 1$, $x_0 \geq y_0$, $x_0 \equiv y_0 \bmod 4$ *and* $x_0 \equiv y_0 \bmod 3$. *then the following conditions are equivalent*
(i) (x_0, y_0, z_0) *is the solution of* $E : x^3 + y^3 = 2z^2$
(ii) $x_0 = 3z_5^2z_6^2 + \frac{z_5^4-3z_6^4}{2}$, $y_0 = 3z_5^2z_6^2 - \frac{z_5^4-3z_6^4}{2}$, $\mid z_0 \mid = 3z_5z_6(\frac{z_5^4+3z_6^4}{2})$

Proof.
(i) $\Longrightarrow$ **(ii)**
Let (x, y, z) is the solution of E such that $x = u + v$ and $y = u - v$ **(a1)**.
We have $x \equiv -y \bmod 3$ and $x \equiv y \bmod 4$ then $x + y = 3k$ and $x - y = 4k$
Therfore $u = 3u_1$, $v = 2v_1$ **(b1)**.
It is clear that $u \wedge v = 1$, $u \wedge 2 = 1$ and $v \wedge 3 = 1$ **(a2)**.
We can see

$$\begin{array}{rrcl} & (u+v)^3+(u-v)^3 & = & 2u(u^2+3v^2) \\ & & = & 2z^2 \\ \textit{wich implies that} & u(9u_1^2+3v^2) & = & z^2 \qquad \textbf{(a3)} \end{array}$$

Then $\mid z \mid = 3z_1$ **(b2)**.
Then $u_1(3u_1^2 + v^2) = z_1^2$. Since **(a2)** and **(b1)** We have $u_1 \wedge 3u_1 + v = 1$ therefore $z_1 = z_3z_4$,$u_1 = z_3^2$ and $3u_1 + v = z_4^2$ **(a4)**. Since [1] we can write $u_1 = ab$, $v = \frac{a^2-3b^2}{2}$ and $z_4 = \frac{a^2+3b^2}{2}$ **(a5)** $\Longrightarrow z_3^2 = ab \Longrightarrow z_3 = z_5z_6$, $a = z_5^2$ and $b = z_6^2$ **(a6)**

$$\begin{array}{rcl} x & = & u+v \\ & = & 3u_1 + v \quad (by\ (b1)) \\ & = & 3ab + \frac{a^2-3b^2}{2} \quad (by\ (a5)) \\ & = & 3z_5^2z_6^2 + \frac{z_5^4-3z_6^4}{2} \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} y & = & u-v \\ & = & 3u_1 - v \quad (by\ (b1)) \\ & = & 3ab - \frac{a^2-3b^2}{2} \quad (by\ (a5)) \\ & = & 3z_5^2z_6^2 - \frac{z_5^4-3z_6^4}{2} \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} \mid z \mid & = & 3z_1 \quad (by\ (b2)) \\ & = & 3z_3z_4 \quad (by\ (a5),(a6)) \\ & = & 3z_5z_6(\frac{a^2+3b^2}{2}) \quad (by\ (a6)) \\ & = & 3z_5z_6(\frac{z_5^4+3z_6^4}{2}) \quad (by\ (a5)) \end{array}$$

□

3^{rd} **case** ($x \not\equiv y \bmod 4$ and $x \equiv -y \bmod 3$)

Theorem 3.3.

Let $E : x^3 + y^3 = 2z^2$ diophantine equation and $(x_0, y_0, z_0) \in \mathbb{Z}^3$ such that $x_0 \wedge y_0 = 1$, $x_0 \not\equiv y_0 \bmod 4$ and $x_0 \equiv -y_0 \bmod 3$. then the following properties are equivalent

(i) *(x_0, y_0, z_0) is the solution of E*

(ii) $x_0 = 12z_6^2z_7^2 + 4z_6^4 - 3z_7^4$, $\quad y_0 = 12z_6^2z_7^2 - 4z_6^4 + 3z_7^4$, $\mid z \mid= 6z_7z_6(4z_6^4 + 3z_7^4)$

or

$x_0 = 12z_6^2z_7^2 + z_6^4 - 12z_7^4$, $y_0 = 12z_6^2z_7^2 - z_6^4 + 12z_7^4$, $z_0 = 6z_7z_6(z_6^4 + 12z_7^4)$

Proof.

(i) $\Longrightarrow$ (ii)

Let (x, y, z) is the solution of E such that $x = u + v$ and $y = u - v$ **(a1)**. We have $x \equiv -y \bmod 3$ and $x \not\equiv y \bmod 4$ then $x + y = 3k$ and $x - y = 2v =\not\equiv 4$ So $u = 3u_1$, and v is odd then u is even **(b1)**.

It is clear that $u \wedge v = 1$, $v \wedge 2 = 1$ and $v \wedge 3 = 1$ **(a2)**. We can see

$$\begin{array}{rrcl} & (u+v)^3 + (u-v)^3 & = & 2u(u^2+3v^2) \\ & & = & 2z^2 \\ wich\ implies\ that & u(9u_1^2+3v^2) & = & z^2 \qquad (\mathbf{a3}) \end{array}$$

Then $\mid z \mid= 3z_1$ **(b2)** $\Longrightarrow u_1(3u_1^2 + v^2) = z_1^2$. Since **(a2)** and **(b1)** We have $u_1 \wedge 3u_1^2 + v^2 = 1$ therefore $z_1 = z_3z_4$,$u_1 = z_3^2$ and $3u_1^2 + v^2 = z_4^2$ **(a4)**. Since [1] we can write $u_1 = 2ab$, $v = a^2 - 3b^2$ and $z_4 = a^2 + 3b^2$ **(a5)** $\Longrightarrow z_3^2 = 2ab \Longrightarrow z_3 = 2z_5$ $2z_5^2 = ab \Longrightarrow z_5 = z_6z_7$, $a = 2z_6^2$ and $b = z_7^2$ or $z_5 = z_6z_7$, $a = z_6^2$ and $b = 2z_7^2$ **(a6)**

$$\begin{array}{rcl} x & = & u+v \\ & = & 3u_1+v \quad (by\ (b1)) \\ & = & 6ab+a^2-3b^2 \quad (by\ (a5)) \\ & = & 12z_6^2z_7^2+4z_6^4-3z_7^4 \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} y & = & u-v \\ & = & 3u_1-v \quad (by\ (b1)) \\ & = & 6ab-a^2-3b^2 \quad (by\ (a5)) \\ & = & 12z_6^2z_7^2-4z_6^4+3z_7^4 \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} \mid z \mid & = & 3z_1 \quad (by\ (b2)) \\ & = & 3z_3z_4 \quad (by\ (a5),(a6)) \\ & = & 6z_5(a^2+3b^2) \quad (by\ (a6)) \\ & = & 6z_7z_6(4z_6^4+3z_7^4) \quad (by\ (a5)) \end{array}$$

or

$$\begin{array}{rcl} x & = & u+v \\ & = & 3u_1+v \quad (by\ (b1)) \\ & = & 6ab+a^2-3b^2 \quad (by\ (a5)) \\ & = & 12z_6^2z_7^2+z_6^4-12z_7^4 \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} y & = & u-v \\ & = & 3u_1-v \quad (by\ (b1)) \\ & = & 6ab-a^2-3b^2 \quad (by\ (a5)) \\ & = & 12z_6^2z_7^2-z_6^4+12z_7^4 \quad (by\ (a5)) \end{array}$$

$$\begin{array}{rcl} \mid z \mid & = & 3z_1 \quad (by\ (b2)) \\ & = & 3z_3z_4 \quad (by\ (a5),(a6)) \\ & = & 6z_5(a^2+3b^2) \quad (by\ (a6)) \\ & = & 6z_7z_6(z_6^4+12z_7^4) \quad (by\ (a5)) \end{array}$$

□

4^{th} **case** ($x \not\equiv y \bmod 4$ and $x \not\equiv -y \bmod 3$)

Theorem 3.4.

Let $x_0, y_0, z_0 \in \mathbb{Z}$, such that (x_0, y_0, z_0) is the solution of the following equation $E : x^2 - 3y^2 = 2z^2$, $x_0 \wedge y_0 = 1$, $x_0 \not\equiv y_0 \bmod 4$ and $x_0 \not\equiv -y_0 \bmod 3$ then $x_0y_0z_0 = 0$

Proof.

Let (x, y, z) is the solution of E such that $x = u + v$ and $y = u - v$ **(a1)**. We have $x \not\equiv y \bmod 4$ then $x + y = 4k$ So $u = 2k$. We deduce that u is even and v is odd.

It is clear that $u \wedge v = 1$. We can see

$$\begin{array}{rcl} x^3+y^3=(u+v)^3+(u-v)^3 & = & 2u(u^2+3v^2) \\ & = & 2z^2 \end{array}$$

wich implies that $\qquad u(u^2+3v^2) = z^2 \qquad$ **(a2)**

Since $x \not\equiv -y \bmod 3$ then 3 doesn't divide $x + y = 2u \Longrightarrow u \wedge 3 = 1$ then $u^2 + 3v^2 \wedge u = 1$.

By **(a2)** we can write $u = z_1^2$ **(a3)**, $u^2 + 3v^2 = z_2^2$ and $z = z_1 z_2$. By [1] therefore $u = \frac{a^2-3b^2}{2}$, $\mid v \mid= ab$ and $\mid z_2 \mid= \frac{a^2+3b^2}{2}$ because v is odd **(a4)**

Since **(a3)** and **(a4)** we have $\frac{a^2-3b^2}{2} = z_1^2$ then $a^2 - 3b^2 = 2z_1^2$. Since theorem 2.1 (a, b, z_1) isn't solution of the diophantine equation Contradiction

□

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