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ON UNIQUENESS OF LOCAL ENTROPY SOLUTION OF A CONVECTION-DIFFUSION TYPE INTEGRO-DIFFERENTIAL EQUATION

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Abstract. We study the uniqueness of entropy solution for a class of triply nonlinear parabolic integro-differential equations of the form

$$\partial_t\big(k*(j(v)-j(v_0))\big)-\nabla\cdot\bigg(a(x,\nabla\varphi(v))+F(\varphi(v))\bigg)=f$$

in a bounded domain with homogeneous Dirichlet boundary conditions. The source term f belongs to L^1 and the memory term $k*(j(v)-j(v_0))$ introduces a nonlocal dependence. The functions $j(v)$ and $\varphi(v)$, assumed to be non-decreasing, further contribute to the nonlinear nature of the problem. To prove uniqueness, we apply the method of doubling variables leading to an energy estimate that ensures the desired result.

1. Introduction

In this paper, we consider the triply nonlinear problems under the general form

$$(EP)^{j,F,k}_{f,\varphi}(v_0)\begin{cases}\partial_t\big(k*(j(v)-j(v_0))\big)-\nabla\cdot a(x,\nabla\varphi(v)+\nabla\cdot F(\varphi(v))=f & \text{in } Q_T,\\ j(v)(0,\cdot)=j(v_0) & \text{in } \Omega,\\ v=0 & \text{on } \Sigma_T.\end{cases}$$

Our aim is to study uniqueness of entropy solutions to the problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$. In the problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$, Ω is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$, $T>0$ is a fixed time, $Q_T:=(0,T)\times\Omega$, $\Sigma_T:=(0,T)\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω, $p>1$ is a real number, $p'=\dfrac{p}{p-1}$.

The operator $Av=-\nabla\cdot a(\cdot,\nabla\varphi(v))$ is a Leray-Lions operator. The scalar kernel k is of type $\mathcal{PC}$, meaning that $k\in L^1_{loc}([0,\infty))$, non–negative, non-increasing and satisfies the existence of a function $l\in L^1_{loc}([0,\infty))$ such that the convolution $(k*l)(t)=1$ for all $t>0$ where $k*l$ denotes the convolution on the positive half–line with respect to the time variable:

$$(k*l)(t)=\int_0^t k(t-s)l(s)ds,\ t>0.$$

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Under these assumptions on k, our work covers the case of a time-fractional derivative of order $0 < \alpha < 1$, i.e $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t \in (0, \infty)$ where Γ denotes the Gamma function. Hence, the problem $(EP)^{b,F,k}_{f,\varphi}(v_0)$ can be used to model anomalous diffusion, see Metzler & Klafter, 2000, [12]. There exists another application of $(EP)^{b,F,k}_{f,\varphi}(v_0)$ as the transport of fluids in porous media with memory. In certain geothermal regions, mineral precipitation within the pores of the medium can reduce their size. This reduction in permeability introduces a history-dependent effect, which can be modeled using a fractional time derivative (see Caputo, 1999, [4]). In addition, $(EP)^{b,F,k}_{f,\varphi}(v_0)$ is applicable to the heat flow in materials with memory (see Ph. Clement, 1988, [5]).

In the setting of L^1-data, we cannot expect weak solutions. As a result, we work with local entropy solutions.

The $(EP)^{j,F,k}_{f,\varphi}(v_0)$ type problem has been studied by several authors. For example, Jakubowski, 2011, [8], considering functions $b = \varphi = \mathrm{id}_{\mathbb{R}}$ and $F \equiv 0$ in $(EP)^{j,F,k}_{f,\varphi}(v_0)$, established a result of existence of an entropy solution. In the case $\varphi = \mathrm{id}_{\mathbb{R}}$, $F \equiv 0$ and b strictly–increasing in $(EP)^{j,F,k}_{f,\varphi}(v_0)$, M. Scholtes, 2018, [13], showed a result of existence of an entropy solution and also Sapountzoglou, 2020, [15], established a result of existence of an entropy solution for b non–decreasing. If $\varphi = \mathrm{id}_{\mathbb{R}}$ and b non–decreasing, Soma & Bance, 2024, [16], established a result of existence of an entropy solution.

The authors cited above use the theory of abstract Volterra equations (see Gripenberg, 1985, [7]) to establish their various results. Note that, in the case $\varphi = id$, a result of an uniqueness entropy solution for L^1 data have been established in [18]. A many authors are interested in decay estimates for time-fractional equations (of the porous medium type), see, M. Bonfonte, 2024, [3], Schmitz & Wittbold, 2024, [14] and Soma & Bance, 2024, [17].

The paper is organized as follows. In Section 2, we prepare assumptions that are essential for our proofs. In Section 3, we will recall the definition of local entropy solution of problem $(EP)^{b,F,k}_{f,\varphi}(v_0)$, state our main result and its proof. Uniqueness is proved by a refinement of the technique of doubling the time variable t.

2. **Assumptions and notations**

To formulate the main results of this paper and recall the existence result from [1], we first introduce assumptions that will remain in effect throughout the paper.

Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$, $T > 0$,is fixed time. We set $Q_T := (0,T) \times \Omega$ and $\Sigma_T := (0,T) \times \partial\Omega$.

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \text{ is Caratheodory function,} \tag{H1}$$

i.e. $a(\cdot, \xi) : \Omega \to \mathbb{R}^N$ is measurable for all $\xi \in \mathbb{R}^N$, and $a(x, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is continuous vector field a.e. $x \in \Omega$. Moreover, we assume that a satisfies the classical Leray−Lions conditions, meaning that the operator a is monotone

$$\forall \xi, \zeta \in \mathbb{R}^N \text{ and a.e } x \in \Omega : (a(x,\xi) - a(x,\zeta) \cdot (\xi - \zeta) \geq 0, \tag{H2}$$

coercive

$$\exists p \in (1,\infty),\ \exists \gamma > 0,\ \forall \xi \in \mathbb{R}^N \text{ and a.e } x \in \Omega : a(x,\xi)\cdot \xi \geq \gamma |\xi|^p, \tag{H3}$$

and satisfies the growth condition

$$\exists c > 0,\ d \in L^{p'}(\Omega),\ \forall \xi \in \mathbb{R}^N \text{ and a.e } x \in \Omega : |a(x,\xi)| \leq c\left(d(x) + |\xi|^{p-1}\right) \tag{H4}$$

where $p' = \frac{p}{p-1}$.

Next, we suppose that

$$k : \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ is of type } \mathcal{PC}, \tag{H5}$$

meaning that $k \in L^1_{loc}(\mathbb{R}^+)$, non–increasing, non–negative and such that there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ verifying $(k * l)(t) = 1$ for every $t > 0$.

$$\text{The functions } j \text{ and } \varphi \text{ are continuous and non-decreasing on } \mathbb{R} \text{ and satisfy the condition } j(0) = \varphi(0) = 0. \tag{H6}$$

$$F \text{ is locally Lipschitz-continuous function defined on } \mathbb{R} \text{ and takes values in } \mathbb{R}^N, \tag{H7}$$

$$v_0 : \Omega \longrightarrow \overline{R} \text{ is a measurable function satisfying } u_0 := j(v_0) \text{ belongs to } L^1(\Omega)\ . \tag{H8}$$

$$f : Q_T \to \mathbb{R} \text{ belongs to } L^1(Q_T). \tag{H9}$$

We will also use some notations and functions later on, which are presented as follows.

If $A \subset \Omega$ is a Lebesgue measurable set, its Lebesgue measure will be denoted by $|A|$ and its characteristic function by χ_A. For all real number $K > 0$, we denote by $T_K : \mathbb{R} \longrightarrow \mathbb{R}$ the truncation function at the level K defined by

$$T_K(r) = \min\left(\max(r,-K)\,,\ K\right).$$

More precisely, for $1 \leq p < \infty$, the functional space $\mathcal{T}_0^{1,p}(\Omega)$ can be defined by

$$\mathcal{T}_0^{1,p}(\Omega) := \left\{ v : \Omega \to \mathbb{R}\ :\ v \text{ is a measurable and } T_K(v) \in W_0^{1,p}(\Omega) \text{ for every } K > 0 \right\}.$$

For further details on the class of functions $v : \Omega \to \mathbb{R}$ whose truncations $T_K(v)$ belong to a certain Sobolev space for every $K > 0$ see for instance [2, 6].

We will frequently use the notation $T_{K,L} := T_L - T_K$ for $L > K$.

For $r \in \mathbb{R}$, the function $r \mapsto \text{sign}_0^+(r)$ is defined by

$$\text{sign}_0^+(r) = \begin{cases} 1, & \text{if } r > 0, \\ 0, & \text{if } r \leq 0. \end{cases}$$

Throughout this paper, for simplicity, given a function $u : Q_T \to \mathbb{R}$ and a positive real number K, we use the shorthand notation

$$\{|u| \leq (<,>,\geq,=)K\}$$

to denote the set

$$\{(t,x) \in Q_T : |u(t,x)| \leq (<,>,\geq,=)K\}.$$

Additionally, we define

$$\mathcal{P} := \left\{ S \in \mathcal{C}^1(\mathbb{R})\ :\ \forall t \in (0,\infty),\ S'(t) \geq 0,\ \text{supp}(S') \text{ is compact}, S(0) = 0 \right\}.$$

3. Uniqueness of local entropy solution

In this section, we recall the definition of an entropy solution for the problem $(EP)^{b,F,k}_{f,\varphi}(v_0)$ which is in [1], state our main result and prove it.

Definition 3.1. A measurable function $v : Q_T \longrightarrow \overline{\mathbb{R}}$ is called a local entropy solution of (1) if

(P1) $j(v) \in L^1(Q_T)$ and $T_K(w) \in L^p(0,T; W^{1,p}_0(\Omega))$ for any $K > 0$ with $w := \varphi(v)$,

and for any functions $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$, $S \in \mathcal{P}$, for all $m \in \mathbb{N}$, we have

(P2)

$$\begin{aligned}
&- \int_{Q_T} \left(\int_0^t k_{1,m}(t-\tau) \int_{v_0(x)}^{v(\tau,x)} S(\varphi(\sigma) - \phi(x)) dj(\sigma) \right) d\tau \; \xi_t(t) dx dt \\
&\quad + \int_{Q_T} k_{2,j}(0^+)\big(j(v(t,x)) - j(v_0(x))\big) S(\varphi(v(t,x)) - \phi(x)) \xi(t) dx dt \\
&\quad + \int_{Q_T} \left(\int_0^t \left(j(v(t-\tau,x)) - j(v_0(x))\right) dk_{2,m}(\tau) \right) S(\varphi(v(t,x)) - \phi(x)) \xi(t) dx dt \\
&\quad + \int_{Q_T} \big(a(x, \nabla w(t,x)) + F(w(t,x))\big) \cdot \nabla S(\varphi(v(t,x)) - \phi(x)) \xi(t) dx dt \\
&\leq \int_{Q_T} f(t,x) S(\varphi(v(t,x)) - \phi(x)) \xi(t) dx dt
\end{aligned}$$

where $k_{1,m} := (k-m)^+$ and $k_{2,m} := k - k_{1,m}$.

Our main result is as follows:

Theorem 3.2. *Under assumptions (H1)−(H9), suppose that for $i = 1, 2$, the initial function $v_{0,i} : \Omega \longrightarrow \overline{\mathbb{R}}$ is measurable and satisfies $j(v_{0,i}) \in L^1(\Omega)$ with $j(v_{0,1}) = j(v_{0,2})$ almost everywhere in Ω. Let v_i be an entropy solution of the problem $(EP)^{b,F,k}_{f,\varphi}(v_{0,i})$ with right-hand side f and initial data $v_{0,i}$, satisfying*

$$\lim_{t \to 0} \|j(v_1)(t,\cdot) - j(v_{0,1})\|_{L^1(\Omega)} = 0.$$

Then, $j(v_1) = j(v_2)$ a.e. in Q_T.

Remark 3.3. Observe that continuity at $t = 0$ is assumed only for one of the entropy solutions.

Theorem 3.2 can be proved in essentially the same way as Theorem 4 in [8]. The proof relies on the following a priori estimate.

Lemma 3.4. *Assume that (H1)−(H9) hold. Additional, let $v : Q_T \to \mathbb{R}$ be an entropy solution of $(EP)^{b,F,k}_{f,\varphi}(v_0)$.*
Then,

$$\int_{Q_T} \xi |\nabla T_{K,K+L}(w)|^p dx dt = \int_{Q_T \cap \{K < |w| < K+L\}} \xi |\nabla w|^p dx dt \longrightarrow 0$$

as $K \to \infty$ *for every* $L > 0$, $\xi \in \mathcal{D}^+([0,T))$.

Proof. In the Definition 3.1, for $K > L > 0$, take $\phi = 0$, $S = T_{K,K+L}$ and $k_{1,m} = k$ respectively. By multiplying inequality (P2) from Definition 3.1 by $\frac{1}{L}$, we obtain

$$
\begin{aligned}
&-\frac{1}{L}\int_{Q_T}\left(\int_0^t k(t-\tau)\int_{v_0(x)}^{v(\tau,x)} T_{K,K+L}(\varphi(\sigma))dj(\sigma)\right)d\tau\ \xi_t(t)dxdt \\
&\quad+\frac{1}{L}\int_{Q_T}\xi(t)a(x,\nabla\varphi(v(t,x))\cdot\nabla T_{K,K+L}(\varphi(v(t,x))dxdt \\
&\quad+\frac{1}{L}\int_{Q_T}\xi(t)F(\varphi(v(t,x))\cdot\nabla T_{K,K+L}(\varphi(v(t,x))dxdt \\
&\leq\frac{1}{L}\int_{Q_T} f(t,x)T_{K,K+L}(\varphi(v(t,x)))\xi(t)dxdt,
\end{aligned}
$$

for every $\xi \in \mathcal{D}^+([0,T))$.
Observe that by applying the Gauss-Green Theorem along with the boundary condition,

$$\int_\Omega F(\varphi(v(t))\cdot\nabla T_{K,K+L}(\varphi(v(t))dx = 0.$$

Next, proceeding along the same lines as in the proof of [Lemma 5, [8]] we conclude the proof of Lemma 3.4. □

Now we can establish the proof of Theorem 3.2.

Proof. The proof of Theorem 3.2 is baseed on Kruzhkov's doubling of variable technique (see [11]) that has been adapted by some authors (see e.g. [19]) to prove uniqueness results and comparison principles for elliptic-parabolic problems.
In our particular case, we proceed as in [8] p. 72-83. Let $(t,s) \in [0,T]\times[0,T]$. We write the t variable in the local entropy solution v_1 of $(EP)^{b,F,k}_{f,\varphi}(v_{0,1})$ and s variable in the local entropy solution v_2 of $(EP)^{b,F,k}_{f,\varphi}(v_{0,2})$. For $K > 0$, we chose, $\phi = T_K(w_2)(s)$ as a test function in the definition of entropy solution for v_1 and $\phi = T_K(w_1)(t)$ in the definition for v_2 with $S = T_L$ for $L > 0$. Setting $u_1 := j(v_1)$, $u_2 := j(v_2)$, $u_0 := j(v_{0,1}) = b(v_{0,2})$ and $Q_{2,T} := (0,T)\times(0,T)\times\Omega$ and summing up both inequalities, we find

$$I_1^{K,L} + I_2^{K,L} + I_3^{K,L} + I_4^{K,L} + I_5^{K,L} \leq I_6^{K,L} \tag{3.1}$$

for any $K, L > 0$ where

$$I_1^{K,L} := -\int_{Q_{2,T}} \xi_t(t,s)\big[\int_0^t k_{1,m}(t-\sigma)\int_{v_{01}}^{v_1(\sigma)} T_L(\varphi(r) - T_K(w_2)(s))dj(r)d\sigma\big]dxdtds$$

$$I_2^{K,L} := -\int_{Q_{2,T}} \xi_s(t,s)\big[\int_0^s k_{1,m}(s-\tau)\int_{v_{02}}^{v_2(\tau)} T_L(\varphi(r) - T_K(w_1)(t))dj(r)d\tau\big]dxdtds$$

$$I_3^{K,L} := \int_{Q_{2,T}} \xi(t,s)\big[\partial_t[k_{2,m} * (u_1 - u_0)](t)\big]T_L[w_1(t) - T_K(w_2)(s)]dxdtds + \int_{Q_{2,T}} \big[\xi(t,s)\partial_s[k_{2,m} * (u_2 - u_0)](s)\big]T_L[w_2(s) - T_K(w_1)(t)]dxdtds$$

$$I_4^{K,L} = \int_{Q_{2,T}} \xi(t,s)\big[a(x,\nabla w_1(t))\cdot\nabla T_L[w_1(t) - T_K(w_2)(s)] + a(x,\nabla w_2(s))\cdot\nabla T_L[w_2(t) - T_K(w_1)(t)]\big]dxdtds$$

$$I_5^{K,L} := \int_{Q_{2,T}} \xi(t,s)\big[F(w_1(t))\cdot\nabla T_L[w_1(t) - T_K(w_2)(s)] + F(w_2(t))\cdot\nabla T_L[w_2(t) - T_K(w_1)(t)]\big]dxdtds$$

$$I_6^{K,L} := \int_{Q_{2,T}} \xi(t,s)\big[f(t)T_L[w_1(t) - T_K(w_2)(s)] + f(s)T_L[w_2(s) - T_K(w_1)(t)]\big]dxdtds$$

for any $\xi \in \mathcal{D}^+((0,T)\times(0,T))$.
We want to pass to the limit when $K \to \infty$ in (3.1).
We first consider the term $I_5^{K,L}$. We will show that $\lim_{K\to\infty} \inf I_5^{K,L} \geq 0$.
First, we write that

$$I_5^{K,L} = I_{5,1}^{K,L} + I_{5,2}^{K,L} + I_{5,3}^{K,L} + I_{5,4}^{K,L} + I_{5,5}^{K,L}$$

where

$$I_{5,1}^{K,L} = \int_{Q_2\cap\{|w_1(t)|,|w_2(s)|<K\}} \xi(t,s)\left[F\left(w_1(t)\right) - F\left(w_2(s)\right)\right]\cdot\nabla T_L\left[w_1(t) - w_2(s)\right]$$

$$I_{5,2}^{K,L} = \int_{Q_2\cap\{|w_2(s)|\geq K,|w_1(t)-T_K(w_2(s))|<L\}} \xi(t,s)F\left(w_1(t)\right)\cdot\nabla w_1(t)$$

$$I_{5,3}^{K,L} = \int_{Q_2\cap\{|w_1(t)|\geq K,|w_2(s)-T_K(w_1(t))|<L\}} \xi(t,s)F\left(w_2(s)\right)\cdot\nabla w_2(s)$$

$$I_{5,4}^{K,L} = \int_{Q_2\cap\{|w_2(s)|<K,|w_1(t)|\geq K\}} \xi(t,s)F\left(w_1(t)\right)\cdot\nabla T_L\left[w_1(t) - w_2(s)\right]$$

$$I_{5,5}^{K,L} = \int_{Q_2\cap\{|w_1(t)|<K,|w_2(s)|\geq K\}} \xi(t,s)F\left(w_2(s)\right)\cdot\nabla T_L\left[w_2(s) - w_1(t)\right].$$

Since F is locally Lipschitz continuous, we have

$$I_{5,1}^{K,L} = 0, \quad I_{5,2}^{K,L} = 0 \text{ and } I_{5,3}^{K,L} = 0.$$

Let now define $t_0 := \inf\{|t-T| \,:\, (t,s) \in \operatorname{Supp}\xi\}$, $\quad s_0 := \inf\{|s-T| \,:\, (t,s) \in \operatorname{Supp}\xi\}$ and $\widetilde{T} = T - \min(t_0, s_0)$.
By the definition of t_0, s_0 and $\widetilde{T}$, we can find $\rho \in \mathcal{D}(0,T)$ such that $0 \leq \rho \leq 1$ and $\rho(t) = 1$ for all $t \in [0, \widetilde{T}]$. As we have chosen ρ such that $\rho^{\frac{1}{p'}}(t)\rho^{\frac{1}{p}}(s) = 1$ for all $(t,s) \in \operatorname{Supp}\xi$, the term $I_{5,4}^{K,L}$ can be estimated for each $K > L$, we have,

$$\begin{aligned} I_{5,4}^{K,L} = &\int_{Q_2 \cap \{|w_2(s)|<K, |w_1(t)| \geq K, |w_1(t)-w_2(s)|<L\}} \xi(t,s) F(w_1(t)) \cdot \nabla w_1(t) \\ &- \int_{Q_2 \cap \{|w_2(s)|<K, |w_1(t)| \geq K, |w_1(t)-w_2(s)|<L\}} \xi(t,s) F(w_1(t)) \cdot \nabla w_2(s). \end{aligned}$$

Taking ρ such that $\operatorname{Supp}(\rho) \subset [-K-L, K+L]$, by Gauss-Green theorem, we have

$$\int_{Q_2 \cap \{|w_2(s)|<K, |w_1(t)| \geq K, |w_1(t)-w_2(s)|<L\}} \xi(t,s) F(w_1(t)) \cdot \nabla w_1(t) = 0.$$

Thus,

$$\begin{aligned} I_{5,4}^{K,L} &= -\int_{Q_2 \cap \{|w_2(s)|<K, |w_1(t)| \geq K, |w_1(t)-w_2(s)|<L\}} \xi(t,s) F(w_1(t)) \cdot \nabla w_2(s) \\ &\geq -\|\xi\|_\infty \int_{Q_2 \cap \{K \leq |w_1(t)| < K+L\} \cap \{K-L < |w_2(s)| < K\}} \rho^{\frac{1}{p'}}(t)\rho^{\frac{1}{p}}(s) |F(w_1(t))| \, |\nabla w_2(s)| \\ &\geq -T\|\xi\|_\infty \Big(\int_{Q_T \cap \{K \leq |w_1(t)| < K+L\}} \rho |F(w_1(t))|^{p'} \Big)^{\frac{1}{p'}} \\ &\quad \times \Big(\int_{Q_T \cap \{K-L < |w_2(s)| < K\}} \rho |\nabla w_2(s)|^p \Big)^{\frac{1}{p}} \\ &\geq -T\|\xi\|_\infty \Big(\int_{Q_T \cap \{K \leq |w_1(t)| < K+L\}} \rho |F(T_{K+L}(w_1)(t))|^{p'} \Big)^{\frac{1}{p'}} \\ &\quad \times \Big(\int_{Q_T \cap \{K-L < |w_2(s)| < K\}} \rho |\nabla w_2(s)|^p \Big)^{\frac{1}{p}} \\ &\geq -T\|\xi\|_\infty \chi_{\{K \leq |w_1(t)|\}} \|\rho^{\frac{1}{p'}} F(T_{K+L}(w_1))\|_{(L^{p'}(Q_T))^N} \times \|\rho^{\frac{1}{p}} \nabla T_{K-L,K}(w_2)\|_{(L^p(Q_T))^N}. \end{aligned}$$

Since, $\|\rho^{\frac{1}{p'}} F(T_{K+L}(w_1))\|_{(L^{p'}(Q_T))^N} < \infty$, then by Lemma 3.4 we deduce that

$$\lim_{K\to\infty} \inf I_{5,4}^{K,L} \geq 0.$$

Using analogous arguments estimate with the roles of $w_1(t,x)$ and $w_2(s,x)$ interchanged, we show also that $\lim\limits_{K\to\infty} \inf I_{5,5}^{K,L} \geq 0$. This implies that

$$\lim_{K\to\infty} \inf I_5^{K,L} \geq 0.$$

With the same arguments and the growth bound assumption (H4), we conclude also

$$\lim_{K\to\infty} \inf I_4^{K,L} \geq 0.$$

Now, we can pass to the limit with $K \to \infty$ in (3.1).
Since as $K \to \infty$, $T_L[w_1(t) - T_K(w_2)(s)] \longrightarrow T_L[w_1(t) - w_2(s)]$ and $T_L[w_2(s) - T_K(w_1)(t) \longrightarrow T_L[w_2(s) - w_1(t)]$ a.e. in $Q_{2,T}$, then by Lebesgue Dominated Convergence Theorem, we get

$$I_6^{K,L} \xrightarrow{K\to\infty} \int_{Q_{2,T}} \xi(s,t)\big(f(t) - f(s)\big)T_L[w_1(t) - w_2(s)]dxdtds := \tilde{I_6^L}.$$

In additional, since $k_{2,m}(0^+) < \infty$, we get that

$$\partial_t[k_{2,m} * (u_1 - u_0)] \text{ and } \partial_s[k_{2,m} * (u_2 - u_0)] \text{ belong to } L^1(Q_{2,T}).$$

Therefore, using again Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} I_3^{K,L} \xrightarrow{K\to\infty} \int_{Q_{2,T}} \xi(t,s)\Big[k_{2,m}(0^+)(u_1(t) - u_2(s)) + \int_0^t (u_1(t-\sigma) - u_0)dk_{2,m}(\sigma) \\ - \int_0^s (u_2(s-\tau) - u_0)dk_{2,m}(\tau)\Big]T_L[w_1(t) - w_2(s)]dxdtds := \tilde{I_3^L}. \end{aligned}$$

We have also,

$$\left|\int_{v_{01}}^{v_1(\sigma)} T_L(\varphi(r) - T_K(w_2)(s))dj(r)\right| \leq L(|j(v_1)| + |j(v_{0,1})|) \in L^1(Q_{2,T})$$

and

$$\int_{v_{0,1}}^{v_1(\sigma)} T_L(\varphi(r) - T_K(w_2)(s))dj(r) \xrightarrow{K\to\infty} \int_{v_{0,1}}^{v_1(\sigma)} T_L(\varphi(r) - w_2(s))dj(r) \text{ a.e. in } Q_{2,T}.$$

Then, Lebesgue Dominated Convergence Theorem implies that

$$I_1^{K,L} \xrightarrow{K\to\infty} -\int_{Q_{2,T}} \xi_t(t,s)\big[\int_0^t k_{1,m}(t-\sigma)\int_{v_{01}}^{v_1(\sigma)} T_L(\varphi(r) - w_2(s))dj(r)d\sigma\big]dxdtds := \tilde{I}_1^L.$$

Using the arguments as above, we also obtain

$$I_2^{K,L} \xrightarrow{K\to\infty} -\int_{Q_{2,T}} \xi_s(t,s)\big[\int_0^s k_{1,m}(s-\tau)\int_{v_{0,2}}^{v_2(\tau)} T_L(\varphi(r) - w_1(t))dj(r)d\tau\big]dxdtds := \tilde{I}_2^L.$$

Thus, passing to the limit with $K \to \infty$ in (3.1) and dividing and dividing the resulting inequality by $L > 0$, we obtain

$$\frac{1}{L}\tilde{I}_1^L + \frac{1}{L}\tilde{I}_2^L + \frac{1}{L}\tilde{I}_3^L \leq \frac{1}{L}\tilde{I}_6^L. \tag{3.2}$$

Remark that, we can be estimate $\frac{1}{L}\tilde{I}_6^L$ as follows

$$|\frac{1}{L}\tilde{I}_6^L| \leq \int_0^T\int_0^T \xi(s,t)\|f(t) - f(s)\|_{L^1(\Omega)}dtds := J_6.$$

Since, $\frac{1}{L}T_L[w_1(t)-w_2(s)] \longrightarrow \operatorname{sign}_0(w_1(t)-w_2(s))$ a.e. in $Q_{2,T}$ as $L\to 0^+$ and $w_i=\varphi(v_i)$, i=1,2 with φ is strictly increasing, then $sign_0(w_1(t)-w_2(s)) = \operatorname{sign}_0(v_1(t)-v_2(s))$ a.e. in $Q_{2,T}$. We deduce that

$$\frac{1}{L}\tilde{I}_3^L \xrightarrow{L\to 0^+} \int_{Q_{2,T}} \xi(t,s)\Big[k_{2,m}(0^+)(u_1(t)-u_2(s)) + \int_0^t (u_1(t-\sigma)-u_0)dk_{2,m}(\sigma) - \int_0^s (u_2(s-\tau)-u_0)dk_{2,m}(\tau)\Big] sign_0(v_1(t)-v_2(s))dxdtds := J_3.$$

Now, we define $e:\mathbb{R}^2\to\mathbb{R}$ by

$$e(t,s) := \begin{cases} \|u_1(t)-u_2(s)\|_{L^1(\Omega)} & \text{if } t>0,\ s>0;\\ \|u_0-u_2(s)\|_{L^1(\Omega)} & \text{if } t=0,\ s\geq 0;\\ \|u_1(t)-u_0\|_{L^1(\Omega)} & \text{if } t\geq 0,\ s=0;\\ 0 & \text{if } t<0 \text{ or } s<0.\end{cases}$$

Remark that $\big(u_1(t)-u_2(s)\big)\operatorname{sign}_0\big(v_1(t)-v_2(s)\big) = |u_1(t)-u_2(s)|$ because j is non-decreasing (we recall that $u_i := j(v_i)$ for $i=1,2$). So, the fact that $dk_{2,m}$ is a negative measure on $(0,\infty)$, we get that

$$\begin{aligned} J_3 \geq & \int_0^T\int_0^T \xi(t,s)\Big(k_{2,m}(0^+)e(t,s) + \int_0^{\min(t,s)} e(t-\sigma,s-\sigma)dk_{2,m}(\sigma)\Big)dtds \\ & + \int_0^T\int_0^T \xi(t,s)\Big(\chi_{\{s<t\}}\int_s^t e(t-\sigma,0)dk_{2,m}(\sigma) \\ & + \chi_{\{t<s\}}\int_t^s e(0,s-\tau)dk_{2,m}(\tau)\Big)dtds := J_{3,1}+J_{3,2}. \end{aligned}$$

In noticing that, $\operatorname{sign}_0\big(\varphi(r)-w_2(s)\big) = \operatorname{sign}_0\big(r-v_2(s)\big)$ then as $L\to 0^+$

$$\int_{v_{01}}^{v_1(\sigma)} \frac{1}{L}T_L(\varphi(r)-w_2(s))dj(r) \longrightarrow \int_{v_{01}}^{v_1(\sigma)} \operatorname{sign}_0\big(r-v_2(s)\big)dj(r) = |u_1(t)-u_2(s)|-|u_2(s)-u_0|$$

a.e. in $Q_{2,T}$ and

$$\int_{v_{01}}^{v_1(\sigma)} \frac{1}{L}T_L(\varphi(r)-w_2(s))dj(r) \leq |u_1(t)-u_2(s)| + |u_2(s)-u_0| \in L^1(Q_{2,T}).$$

So, it follows that

$$\begin{aligned} \frac{1}{L}\tilde{I}_1^L \xrightarrow{L\to 0^+} & \int_0^T\int_0^T \xi_t(t,s)\int_0^t k_{1,m}(t-\sigma)e(t,\sigma)d\sigma dtds \\ & - \int_0^T\int_0^T \xi(t,s)k_{1,m}(t)e(0,s)dtds := J_{1,1}+J_{1,2}. \end{aligned}$$

By the same arguments as above, we also get

$$\begin{aligned} \frac{1}{L}\tilde{I}_2^L \xrightarrow{L\to 0^+} & \int_0^T\int_0^T \xi_s(t,s)\int_0^s k_{1,m}(s-\tau)e(\tau,s)d\tau dtds \\ & - \int_0^T\int_0^T \xi(t,s)k_{1,m}(s)e(t,0)dtds := J_{2,1}+J_{2,2}. \end{aligned}$$

So, $L \to 0^+$ in (3.2) gives

$$J_{1,1} + J_{1,2} + J_{2,1} + J_{2,2} + J_{3,1} + J_{3,2} \leq J_6. \tag{3.3}$$

Now, let $(\rho_q)_{q>0}$ be a sequence mollifiers in $\mathbb{R}$ and satisfying $\rho_q \geq 0$, supp $\rho_q \subset [0,q]$ and $\rho_q \longrightarrow \delta_0$ in $\mathcal{D}'(\mathbb{R})$ as $q \to 0^+$. Setting $\xi_q(t,s) := \rho_q(t-s)\phi(t)$ for $\phi \in \mathcal{D}([0,T))$ with $\phi \geq 0$, it follows that $\xi_q \in \mathcal{D}([0,T) \times [0,T))$ with $\xi_q \geq 0$ for $q > 0$ small enough. We can replace ξ by ξ_q in (3.3) and pass to the limit with $m \to \infty$ and then $q \to 0^+$. First, remark that $\lim\limits_{m\to\infty}(J_{1,2} + J_{2,2}) = 0$ since $k_{1,m} := (k-m)^+ \longrightarrow 0$ as $m \to \infty$. Next, using the transformation $(\eta,\nu) \mapsto (t,s) = (\eta + \frac{\nu}{2}, \eta - \frac{\nu}{2})$, we get that

$$\begin{aligned} J_{3,1} = &\int_{-2T}^{2T} \int_{|\frac{\nu}{2}|}^{T-|\frac{\nu}{2}|} \xi_q(\eta + \frac{\nu}{2}, \eta - \frac{\nu}{2}) \Big(k_{2,m}(0^+) e\left(\eta + \frac{\nu}{2}, \eta - \frac{\nu}{2}\right) \\ &+ \int_0^\eta e\left(\eta + \frac{\nu}{2} - \sigma, \eta - \frac{\nu}{2} - \sigma\right) dk_{2,m}(\sigma) \Big) d\eta d\nu \\ = &- \int_{-2T}^{2T} \int_{|\frac{\nu}{2}|}^{T-|\frac{\nu}{2}|} \rho_q(\nu)\phi'\left(\eta + \frac{\nu}{2}\right) \int_0^\eta k_{2,m}(\eta - \nu) e\left(\sigma + \frac{\nu}{2}, \sigma - \frac{\nu}{2}\right) d\sigma d\eta d\nu. \end{aligned}$$

Here, we used partial integration and the fact that the boundary terms vanish. Note that $k_{2,m} := k - k_{1,m} \longrightarrow k$ as $m \to \infty$, we deduce that

$$\lim_{q\to 0^+} \lim_{m\to\infty} J_{3,1} = -\int_0^T \phi'(s) \int_0^t k(t-\sigma) e(\sigma,\sigma) d\sigma dt.$$

For estimate $J_{3,2}$, we suppose that u_1 is continuous at $t = 0$, so $\lim\limits_{t\to 0}\|u_1(t) - u_0\|_{L^1(\Omega)} = 0$. Since, we also have $\xi_q(t,s) = \rho_q(t-s)\phi(t) = 0$ for $t < s$, we obtain that

$$0 \geq J_{3,2} \geq -\left(\sup_{\sigma\in[0,q]} e(\sigma,0)\right) \int_0^T \int_s^T \xi_q(t,s)(k_{2,m}(s) - k_{2,m}(t)) dt ds \longrightarrow 0,$$

as first $m \to \infty$ and then $q \to 0^+$. Last, remark that $\lim\limits_{q\to 0^+} J_6 = 0$. Thus, combining the above results, we finally obtain that

$$-\int_0^T \phi'(s) \int_0^t k(t-\sigma) e(\sigma,\sigma) d\sigma dt \leq 0 \tag{3.4}$$

for every $\phi \in \mathcal{D}([0,T))$ with $\phi \geq 0$. By [8] p. 133, there exists a unique completely positive Radon measure α on $[0,T)$ defined by

$$\int_0^t \alpha[0,t-s] k(s) ds = t, \quad t \in (0,T).$$

As, $\dfrac{d}{dt}\left(\displaystyle\int_0^t k(t-\sigma)e(\sigma,\sigma)d\sigma\right) \leq 0$, in the sense of distributions, then the convolution of this inequality with the completely positive measure α gives

$$e(t,t) \leq 0,$$

in the sense of distributions thus for a.e. $t \in (0,T)$. Hence, we obtain that

$$e(t,t) = 0.$$

So, for a.e. $t \in (0,T)$

$$\|u_1(t) - u_2(t)\|_{L^1(\Omega)} = 0.$$

Hence, for a.e. $t \in (0,T)$

$$j(v_1) = j(v_2) \text{ in } L^1(Q_T.$$

□

Conclusion

In this work, we have studied the uniqueness of entropy solutions for the problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$, characterized by the presence of a memory term, nonlinear operators, and homogeneous Dirichlet boundary conditions. Assuming that the functions j and φ are non-decreasing, we proved uniqueness using the method of doubling variables, an effective approach for obtaining a priori estimates and establishing uniqueness results in the framework of entropy solutions.

This work extends the results of [1, 13, 16]. A promising direction for future research would be to extend this analysis to cases where the assumptions on j and φ are less restrictive or to investigate the existence and regularity of solutions under more general conditions.

References

[1] M. Bance and S. Soma. *Local entropy solution of a convection-diffusion type integro-differential equation.* Journal of Mathematics Research, p1–14 (2025). https://doi.org/10.5539/jmr.v16n6p1

[2] Ph. Bénilan. *On the p-laplacian in L^1.* In Conference at the Analysis Seminar, (1987).

[3] M. Bonforte, M. Gualdani and Peio Ibarrondo. *Time-fractional porous medium type equations. sharp time decay and regularization.* arXiv preprint arXiv:2401.03234, 2024.

[4] M. Caputo. *Diffusion of fluids in porous media with memory.* Geothermics, pages 113–130, (1999). https://doi.org/10.1016/S0375-6505(98)00047-9

[5] Ph. Clement and E. Mitidieri. *Qualitative properties of solutions of volterra equations in banach spaces.* Israel J. Math., 64:1–24, (1988). https://doi.org/10.1007/BF02767365

[6] T. Gallouet R. Gariepy M. Pierre J.L. Varquez Ph. Benilan, L. Boccardo. *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations.* Ann. Scuola. Norm. Sup. Pisa, 22(2):241–273, (1991).

[7] G. Gripenberg. *Volterra integro-differential equations with accretive nonlinearity.* J. Differential Equations, 60:57–79, (1985). https://doi.org/10.1016/0022-0396(85)90120-2

[8] V.G. Jakubowski. Nonlinear elliptic-parabolic integro-differential equations with L^1- data: existence, uniqueness, asymptotics. PhD thesis, University Duisburg-Essen, (2011).

[9] I. Konate and S. Ouaro *Multivalued anisotropic problem with Robin boundary condition involving diffuse Radon measure data and variable exponents.* Gulf Journal of Mathematics, 6, (3) : 2018 https://doi.org/10.56947/gjom.v6i3.138

[10] I. Konate and S. Ouaro *Nonlinear multivalued problems with variable exponent and diffuse measure data in anisotropic space.* Gulf Journal of Mathematics, 6, (2) : 2018 https://doi.org/10.56947/gjom.v6i2.130

[11] S. Kruzhkov. *First-order quasilinear equations in several independent variables.* Math. USSR Sb., pages 217–243., (1970).

[12] R. Metzler and J. Klafter. *The randon walks guide to anomalous diffusion: a fractional dynamics approach.* Phys. Rep., 339, (2000).

[13] N. Sapountzoglou. *Entropy solutions to doubly nonlinear integro-differential equations.* Nonlinear Analysis, 192:31 pp, (2020). https://doi.org/10.1016/j.na.2019.111656

[14] K. Schmitz and P. Wittbold. *Entropy solutions for time-fractional porous medium type equations.* Differential and Integral Equations, 37(5/6):309–322, 2024.

[15] M. Scholtes and P. Wittbold. *Existence of entropy solutions to a doubly nonlinear integro-differential equations.* Differ. Integral Equ., 31(5-6):465–496, (2018). https://doi.org/10.57262/die/1516676439

[16] S. Soma and M. Bance. *On existence of entropy solution for a doubly nonlinear differential equation with L^1-data.* Journal of Applied Mathematics and Physics, 11(12):4092–4127, (2023). https://doi.org/10.4236/jamp.2023.1112261

[17] S. Soma and M. Bance. *Entropy solution of nonlinear integro-differential equations with diffuse measure data.* Differential and Integral Equations, 27(1):1–27, 2024. https://doi.org/10.4236/jamp.2023.1112261

[18] S. Soma and M. Bance. *On uniqueness of entropy solution for a doubly nonlinear differential equation with L^1-data.* Submitted, (2024).

[19] A. Zimmermann. Renormalized solutions for nonlinear partial differential equations with variable exponents and L^1-data. PhD thesis.

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