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RECENT PROGRESS IN DETERMINING p-CLASS FIELD TOWERS

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Abstract. For a fixed prime p, the p-class tower $\mathrm{F}_p^\infty K$ of a number field K is considered to be known if a pro-p presentation of the Galois group $G = \mathrm{Gal}(\mathrm{F}_p^\infty K|K)$ is given. In the last few years, it turned out that the Artin pattern $\mathrm{AP}(K) = (\tau(K), \varkappa(K))$ consisting of targets $\tau(K) = (\mathrm{Cl}_p L)$ and kernels $\varkappa(K) = (\ker J_{L|K})$ of class extensions $J_{L|K} : \mathrm{Cl}_p K \to \mathrm{Cl}_p L$ to unramified abelian subfields $L|K$ of the Hilbert p-class field $\mathrm{F}_p^1 K$ only suffices for determining the two-stage approximation $\mathfrak{M} = G/G''$ of G. Additional techniques had to be developed for identifying the group G itself: searching strategies in descendant trees of finite p-groups, iterated and multilayered IPADs of second order, and the cohomological concept of Shafarevich covers involving relation ranks. This enabled the discovery of three-stage towers of p-class fields over quadratic base fields $K = \mathbb{Q}(\sqrt{d})$ for $p \in \{2, 3, 5\}$. These non-metabelian towers reveal the new phenomenon of various tree topologies expressing the mutual location of the groups G and $\mathfrak{M}$.

1. Introduction

The reasons why our recent progress in determining p-class field towers [20, 43, 46] became possible during the past four years is due, firstly, to a few crucial theoretical results by Artin [1, 2] and Shafarevich [54], secondly, to actual implementations of group theoretic, resp. class field theoretic, algorithms by Newman [50] and O'Brien [52], resp. Fieker [22], and finally, to several striking phenomena discovered by ourselves [35, 38, 40, 41, 44, 45], partially inspired by Bartholdi, Boston, Bush, Hajir, Leedham-Green and Nover [17, 19, 9, 18, 51, 15]. In chronological order, these indispensable foundations can be summarized as follows.

1.1. Class extension and transfer.

Let p be a prime number and suppose that K is a number field with non-trivial p-class group $\mathrm{Cl}_p K := \mathrm{Syl}_p \mathrm{Cl}_K > 1$. Then

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K possesses unramified abelian extensions $L|K$ of relative degree a power of p, the biggest of them being the Hilbert p-class field $\mathrm{F}_p^1 K$ of K. For each of the extensions $L|K$, let $J_{L|K} : \mathrm{Cl}_p K \to \mathrm{Cl}_p L$ be the *class extension* homomorphism.

Artin used his reciprocity law of class field theory [1] for translating the arithmetical properties of $J_{L|K}$, the p-capitulation kernel $\ker J_{L|K}$ and the target p-class group $\mathrm{Cl}_p L$, into group theoretic properties of the *transfer* homomorphism $T_{\mathfrak{M},H} : \mathfrak{M} \to H/H'$ from the Galois group $\mathfrak{M} := \mathrm{Gal}(\mathrm{F}_p^2 K|K)$ of the second Hilbert p-class field $\mathrm{F}_p^2 K = \mathrm{F}_p^1 \mathrm{F}_p^1 K$ to the abelianization of the subgroup $H := \mathrm{Gal}(\mathrm{F}_p^2 K|L)$. The reciprocity map establishes an isomorphism between the targets, $H/H' \simeq \mathrm{Cl}_p L$, and an isomorphism between the domains, $\mathfrak{M}/\mathfrak{M}' \simeq \mathrm{Cl}_p K$, in particular, between the kernels $\ker \tilde{T}_{\mathfrak{M},H} \simeq \ker J_{L|K}$, of the *induced* transfer $\tilde{T}_{\mathfrak{M},H} : \mathfrak{M}/\mathfrak{M}' \to H/H'$ and $J_{L|K}$ [2]. In § 1.6, we introduce the Artin pattern AP of K, resp. $\mathfrak{M}$, as the collection of all targets and kernels of the homomorphisms $J_{L|K}$, resp. $\tilde{T}_{\mathfrak{M},H}$, where L varies over intermediate fields $K \le L \le \mathrm{F}_p^1 K$, resp. H varies over intermediate groups $\mathfrak{M}' \le H \le \mathfrak{M}$. The Artin pattern has turned out to be sufficient for identifying a finite batch of candidates for the *second p-class group* $\mathfrak{M}$ of K, frequently even a unique candidate.

1.2. **Relation rank of the p-tower group.** An invaluably precious aid in identifying the *p-tower group*, that is the Galois group $G := \mathrm{Gal}(\mathrm{F}_p^\infty K|K)$ of the maximal unramified pro-p extension $\mathrm{F}_p^\infty K$ of a number field K, has been elaborated by Shafarevich [54], who determined bounds $\varrho \le d_2 G \le \varrho + r + \theta$ for the *relation rank* $d_2 G := \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ of G in terms of the p-class rank ϱ of K, the torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1$ of a number field K with signature (r_1, r_2), and the invariant θ which takes the value 1, if K contains a primitive pth root of unity, and the value 0, otherwise.

The derived length of the group G is called the *length* $\ell_p K = \mathrm{dl}(G)$ of the p-class tower of K. The metabelianization G/G'' of the p-tower group G is isomorphic to the second p-class group $\mathfrak{M}$ of K in § 1.1, which can be viewed as a two-stage approximation of G.

1.3. **Cover and Shafarevich cover.** Let p be a prime and $\mathfrak{M}$ be a finite *metabelian* p-group.

Definition 1.1. By the *cover* of $\mathfrak{M}$ we understand the set of all (isomorphism classes of) finite p-groups whose second derived quotient is isomorphic to $\mathfrak{M}$,

$$\mathrm{cov}(\mathfrak{M}) := \{G \mid \mathrm{ord}(G) < \infty,\ G/G'' \simeq \mathfrak{M}\}.$$

By eliminating the finiteness condition, we obtain the *complete cover* of $\mathfrak{M}$,

$$\mathrm{cov}_c(\mathfrak{M}) := \{G \mid G/G'' \simeq \mathfrak{M}\}.$$

Remark 1.2. The unique *metabelian* element of $\mathrm{cov}(\mathfrak{M})$ is (the isomorphism class of) $\mathfrak{M}$ itself.

Theorem 1.3. *(Shafarevich* 1964*)*
Let p be a prime number and denote by ζ a primitive pth root of unity. Let K be a number field with signature (r_1, r_2) and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1$, and let S be a finite set of non-archimedean or real archimedean

places of K. Assume that no place in S divides p. Then the relation rank $d_2 G_S := \dim_{\mathbb{F}_p} H^2(G_S, \mathbb{F}_p)$ of the Galois group $G_S := \mathrm{Gal}(K_S|K)$ of the maximal pro-p extension K_S of K which is unramified outside of S is bounded from above by

$$d_2 G_S \le \begin{cases} d_1 G_S + r & \text{if } S \neq \emptyset \text{ or } \zeta \notin K, \\ d_1 G_S + r + 1 & \text{if } S = \emptyset \text{ and } \zeta \in K, \end{cases} \tag{1.1}$$

where $d_1 G_S := \dim_{\mathbb{F}_p} H^1(G_S, \mathbb{F}_p)$ denotes the generator rank of G_S.

Proof. The original statement in [54, Thm. 6, (18′)] contained a serious misprint which was corrected in [43, Thm. 5.5, p. 28]. □

Definition 1.4. Let p be a prime and K be a number field with p-class rank $\varrho := d_1 \mathrm{Cl}_p K$, torsionfree Dirichlet unit rank r, and second p-class group $\mathfrak{M} := \mathrm{G}_p^2 K$. By the *Shafarevich cover*, $\mathrm{cov}(\mathfrak{M}, K)$, of $\mathfrak{M}$ *with respect to* K we understand the subset of $\mathrm{cov}(\mathfrak{M})$ whose elements G satisfy the following condition for their relation rank $d_2 G$:

$$\varrho \le d_2 G \le \varrho + r + \theta, \text{ where } \theta := \begin{cases} 1 & \text{if } K \text{ contains the } p\text{th roots of unity}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Definition 1.5. A finite p-group or an infinite topological pro-p group G, with a prime number $p \ge 2$, is called a *σ-group*, if it possesses a *generator inverting* (GI-)automorphism $\sigma \in \mathrm{Aut}(G)$ which acts as the inversion mapping on the derived quotient G/G', that is,

$$\sigma(g)G' = g^{-1}G' \quad \text{for all } g \in G. \tag{1.3}$$

G is called a *Schur σ-group* if it is a σ-group with balanced presentation $d_2 G = d_1 G$.

1.4. **p-Group generation algorithm.** The descendant tree $\mathcal{T}R$ of a finite p-group R [40] can be constructed recursively by starting at the root R and successively determining immediate descendants by iterated executions of the *p-group generation algorithm* [29], which was designed by Newman [50], implemented for $p \in \{2, 3\}$ and $R = C_p \times C_p$ by Ascione and collaborators [3, 4], and implemented in full generality for GAP [25] and MAGMA [13, 14, 34] by O'Brien [52].

1.5. **Construction of unramified abelian p-extensions.** Routines for constructing all intermediate fields $K \le L \le \mathrm{F}_{(c)} K$ between a number field K and the ray class field $\mathrm{F}_{(c)} K$ modulo a given conductor c of K have been implemented in MAGMA [34] by Fieker [22]. Here, we shall use this class field package for finding unramified abelian p-extensions $L|K$ with conductor $c = 1$ only. These fields are located between K and its Hilbert p-class field $\mathrm{F}_p^1 K$.

1.6. **The Artin pattern.** Let p be a fixed prime and K be a number field with p-class group Cl_pK of order p^v, where $v \ge 0$ denotes a non-negative integer.

Definition 1.6. For each integer $0 \le n \le v$, the system $\mathrm{Lyr}_nK := \{K \le L \le \mathrm{F}_p^1K \mid [L : K] = p^n\}$ is called the *nth layer* of abelian unramified p-extensions of K.

Definition 1.7. For each intermediate field $K \le L \le \mathrm{F}_p^1K$, let $J_{L|K} : \mathrm{Cl}_pK \to \mathrm{Cl}_pL$ be the *class extension*, which can also be called the number theoretic *transfer* from K to L.

(1) Let $\tau(K) := [\tau_0K; \ldots ; \tau_vK]$ be the *multi-layered transfer target type* (TTT) of K, where $\tau_nK := (\mathrm{Cl}_pL)_{L\in\mathrm{Lyr}_nK}$ for each $0 \le n \le v$.
(2) Let $\varkappa(K) := [\varkappa_0G; \ldots ; \varkappa_vG]$ be the *multi-layered transfer kernel type* (TKT) or *multi-layered p-capitulation type* of K, where $\varkappa_nK := (\ker J_{L|K})_{L\in\mathrm{Lyr}_nK}$ for each $0 \le n \le v$.

Definition 1.8. The pair $\mathrm{AP}(K) := (\tau(K), \varkappa(K))$ is called the *abelian Artin pattern* of K.

Let p be a prime number and G be a pro-p group with finite abelianization G/G', more precisely, assume that the commutator subgroup G' is of index $(G : G') = p^v$ with an integer exponent $v \ge 0$.

Definition 1.9. For each integer $0 \le n \le v$, let $\mathrm{Lyr}_nG := \{G' \le H \le G \mid (G : H) = p^n\}$ be the *nth layer* of normal subgroups of G containing G'.

Definition 1.10. For any intermediate group $G' \le H \le G$, we denote by $T_{G,H} : G \to H/H'$ the *Artin transfer* homomorphism from G to H/H' [44, Dfn. 3.1], and by $\tilde{T}_{G,H} : G/G' \to H/H'$ the *induced transfer*.

(1) Let $\tau(G) := [\tau_0G; \ldots ; \tau_vG]$ be the *multi-layered transfer target type* (TTT) of G, where $\tau_nG := (H/H')_{H\in\mathrm{Lyr}_nG}$ for each $0 \le n \le v$.
(2) Let $\varkappa(G) := [\varkappa_0G; \ldots ; \varkappa_vG]$ be the *multi-layered transfer kernel type* (TKT) of G, where $\varkappa_nG := (\ker \tilde{T}_{G,H})_{H\in\mathrm{Lyr}_nG}$ for each $0 \le n \le v$.

Definition 1.11. The pair $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ is called the *abelian Artin pattern* of G.

Theorem 1.12. *Let p be a prime number. Assume that K is a number field, and let $\mathfrak{M} := \mathrm{G}_p^2K$ be the second p-class group of K. Then $\mathfrak{M}$ and K share a common abelian Artin pattern,*

$$\mathrm{AP}(\mathfrak{M}) = \mathrm{AP}(K), \quad \textit{that is} \quad \tau(\mathfrak{M}) = \tau(K) \textit{ and } \varkappa(\mathfrak{M}) = \varkappa(K), \tag{1.4}$$

in the sense of componentwise isomorphisms.

Proof. A sketch of the proof is indicated in [49], [36, § 2.3, pp. 476–478] and [37, Thm. 1.1, p. 402], but the precise proof has been given by Hasse in [28, § 27, pp. 164–175]. □

Theorem 1.13. *Let $\mathfrak{M}$ be a finite metabelian p-group. Then all elements of the complete cover of $\mathfrak{M}$ share a common abelian Artin pattern:*

$$\mathrm{AP}(G) = \mathrm{AP}(\mathfrak{M}), \quad \text{for all} \quad G \in \mathrm{cov}_c(\mathfrak{M}). \tag{1.5}$$

Proof. This is the Main Theorem of [44, Thm. 5.4, p. 86]. □

Definition 1.14. The first order approximation $\tau^{(1)}K := [\tau_0 K; \tau_1 K]$ of the TTT, resp. $\varkappa^{(1)}K := [\varkappa_0 K; \varkappa_1 K]$ of the TKT, is called the *index-p abelianization data* (IPAD), resp. *index-p obstruction data* (IPOD), of K.

Definition 1.15. The first order approximation $\tau^{(1)}G := [\tau_0 G; \tau_1 G]$ of the TTT, resp. $\varkappa^{(1)}G := [\varkappa_0 G; \varkappa_1 G]$ of the TKT, is called the *index-p abelianization data* (IPAD), resp. *index-p obstruction data* (IPOD), of G.

Remark 1.16. The IPOD and the TKT contain some standard information which can be omitted.

(1) Since the zeroth layer (top layer), $\mathrm{Lyr}_0 G = \{G\}$, consists of the group G alone, and $T_{G,G} : G \to G/G'$ is the natural projection onto the commutator quotient with kernel $\ker T_{G,G} = G'$, resp. $\ker \tilde{T}_{G,G} = G'/G' \simeq 1$, we usually omit the trivial top layer $\varkappa_0 G = \{1\}$ and identify the IPOD $\varkappa^{(1)}G$ with the first layer $\varkappa_1 G$ of the TKT.

(2) In the case of an elementary abelianization of rank two, $(G : G') = p^2$, we also identify the TKT $\varkappa(G)$ with its first layer $\varkappa_1 G$, since the second layer (bottom layer), $\mathrm{Lyr}_2 G = \{G'\}$, consists of the commutator subgroup G' alone, and the kernel of $T_{G,G'} : G \to G'/G''$ is always *total*, that is $\ker T_{G,G'} = G$, resp. $\ker \tilde{T}_{G,G'} = G/G'$, according to the *principal ideal theorem* [23]. Thus we omit the well-known bottom layer $\varkappa_2 G = \{G/G'\}$.

As mentioned in § 1.1, the TTT and TKT, frequently even the IPAD and IPOD, are sufficient for identifying the second p-class group $\mathfrak{M} = \mathrm{G}_p^2 K$ of a number field K. This was discovered by ourselves in [35, 38, 37], and independently by Boston and collaborators [17, 15].

For finding the p-class tower group $G = \mathrm{G}_p^\infty K$, however, we need the following non-abelian generalization, which requires computing extensions of relative degree p^2 instead of p over K, and was introduced by ourselves in [41, 43, 45] and by Bartholdi, Bush, and Nover in [19, 9, 18, 51].

Definition 1.17. $\tau^{(2)}K := [\tau_0 K; (\tau^{(1)}L)_{L \in \mathrm{Lyr}_1 K}]$ is called *iterated* IPAD *of second order* of K.

Definition 1.18. $\tau^{(2)}G := [\tau_0 G; (\tau^{(1)}H)_{H \in \mathrm{Lyr}_1 G}]$ is called *iterated* IPAD *of second order* of G.

Theorem 1.19. *Let p be a prime number. Assume that K is a number field with p-class tower group $G := \mathrm{G}_p^\infty K$. Then G, G/G''' and K share a common iterated IPAD of second order,*

$$\tau^{(2)}G = \tau^{(2)}G/G''' = \tau^{(2)}K, \tag{1.6}$$

in the sense of componentwise isomorphisms.

Proof. In Theorem 1.12, we proved that $\tau(\mathfrak{M}) = \tau(K)$, and from Theorem 1.13 we know that $\tau(G) = \tau(\mathfrak{M})$. Thus we have, in particular, $\tau^{(1)}G = [\tau_0 G; \tau_1 G] = [\tau_0 K; \tau_1 K] = \tau^{(1)}K$, where $\tau_0 G = G/G' \simeq \mathrm{Cl}_p K = \tau_0 K$ and $\tau_1 G = (\tau_0 H)_{H\in\mathrm{Lyr}_1 G} = (\tau_0 L)_{L\in\mathrm{Lyr}_1 K} = \tau_1 K$, i.e., $\tau_0 H = H/H' \simeq \mathrm{Cl}_p L = \tau_0 L$, for all $H \in \mathrm{Lyr}_1 G$ such that $H = \mathrm{Gal}(\mathrm{F}_p^\infty K|L)$ with $L \in \mathrm{Lyr}_1 K$.

It remains to show that $\tau_1 H = (\tau_0 U)_{U\in\mathrm{Lyr}_1 H} = (\tau_0 M)_{M\in\mathrm{Lyr}_1 L} = \tau_1 L$. Let $U \in \mathrm{Lyr}_1 H$, then $U = \mathrm{Gal}(\mathrm{F}_p^\infty K|M)$ for some $M \in \mathrm{Lyr}_1 L$, since $G''' \trianglelefteq H'' \trianglelefteq U' \trianglelefteq H' \trianglelefteq U \trianglelefteq H$ and $p = (H : U) = \#(H/U) = \#\mathrm{Gal}(M|L) = [M : L]$. Since U' is the smallest subgroup of U with abelian quotient U/U', we must have $U' = \mathrm{Gal}(\mathrm{F}_p^\infty K|\mathrm{F}_p^1 M)$ and thus $\tau_0 U = U/U' \simeq \mathrm{Gal}(\mathrm{F}_p^1 M|M) \simeq \mathrm{Cl}_p M = \tau_0 M$, as required. Observe that, in general, neither $U \trianglelefteq G$ nor $G'' \le U'$, and thus $G = \mathrm{G}_p^\infty K$ cannot be replaced by $\mathfrak{M} = \mathrm{G}_p^2 K$. We could, however, take $\mathrm{G}_p^3 K \simeq G/G'''$ instead of G. □

1.7. **Monotony on descendant trees.**

Definition 1.20. Let p be a prime and G, H and R be finite p-groups.

The *lower central series* (LCS) $(\gamma_n G)_{n\ge 1}$ of G is defined recursively by $\gamma_1 G := G$, and $\gamma_n G := [\gamma_{n-1} G, G]$ for $n \ge 2$.

We call G an *immediate descendant* (or *child*) of H, and H *the parent* of G, if $H \simeq G/\gamma_c G$ is isomorphic to the biggest non-trivial lower central quotient of G, that is, to the image of the natural projection $\pi : G \to G/\gamma_c G$ of G onto the quotient by the last non-trivial term $\gamma_c G > 1$ of the LCS of G, where $c := \mathrm{cl}(G)$ denotes the *nilpotency class* of G. In this case, we consider the projection π as a directed edge $G \to H$ from G to $H \simeq \pi G$, and we speak about the *parent operator*, $\pi : G \to \pi G = G/\gamma_c G \simeq H$.

We call G a *descendant* of H, and H an *ancestor* of G, if there exists a finite *path* of directed edges $(Q_j \to Q_{j+1})_{0\le j<\ell}$ such that $G = Q_0$, $Q_{j+1} = \pi Q_j$ for $0 \le j < \ell$, and $H = Q_\ell$, that is, $G = Q_0 \to Q_1 \to \ldots \to Q_{\ell-1} \to Q_\ell = H$, where $\ell \ge 0$ denotes the *path length*.

The *descendant tree* of R, denoted by $\mathcal{T}R$, is the rooted directed tree with root R having the isomorphism classes of all descendants of R as its *vertices* and all (child, parent)-pairs (G, H) among the descendants G, H of R as its *directed edges* $G \to \pi G \simeq H$. By means of formal iterations π^j of the parent operator π, each vertex of the descendant tree $\mathcal{T}R$ can be connected with R by a finite path of edges: $\mathcal{T}R = \{G \mid G = \pi^0 G \to \pi^1 G \to \pi^2 G \to \ldots \to \pi^\ell G = R \text{ for some } \ell \ge 0\}$.

Theorem 1.21. *Let $\mathcal{T}R$ be the descendant tree with root $R > 1$, a finite non-trivial p-group, and let $G \to \pi G$ be a directed edge of the tree. Then the abelian Artin pattern* $\mathrm{AP} = (\tau, \varkappa)$ *satisfies the following monotonicity relations (in componentwise sense)*

$$\begin{aligned} \tau(G) &\ge \tau(\pi G), \\ \varkappa(G) &\le \varkappa(\pi G), \end{aligned} \tag{1.7}$$

that is, the TTT τ is an isotonic mapping and the TKT $\varkappa$ is an antitonic mapping with respect to the partial order $G > \pi G$ induced by the directed edges $G \to \pi G$.

Proof. This is Theorem 3.1 in [46]. □

This result yields the crucial break-off condition for recursive executions of the p-group generation algorithm, when we want to find a finite p-group with assigned Artin pattern.

1.8. **State of research.** The state of research on p-class field towers in the year 2008 was summarized in a succinct form by McLeman. He literally pointed out the following problem on p. 200 and p. 205 of his paper [33].

Problem 1.22. For odd primes p, there are no known examples of *imaginary quadratic* number fields with p-class rank 2 and either an infinite p-class tower or a p-class tower of length bigger than 2. For $p = 3$, the longest known finite towers are of length 2.

McLeman's survey on the state of the art changed when Bush and ourselves found the first *imaginary quadratic* fields having 3-class field towers of exact length 3 [20] on 24 August 2012.

The reason why McLeman formulated this open problem for *imaginary quadratic* fields with $\varrho = 2$ is the well-known fact that $\varrho = 1$ implies an abelian single-stage tower, and, on the other hand, the strong criterion by Koch and Venkov [31] that $\varrho \geq 3$ enforces an infinite p-class tower for odd primes $p \geq 3$. The result in [31] sharpens the Golod-Shafarevich Theorem [26]. We shall come back to this criterion in § 8 on infinite 3-class towers.

However, until 2012, a much more extensive problem for finite p-class towers of any algebraic number field K was open.

Problem 1.23. No examples are known of number fields K having a 2-class tower of length $\ell_2 K \geq 4$ or a 3-class tower of length $\ell_3 K \geq 3$ or a p-class tower of length $\ell_p K \geq 2$ for $p \geq 5$.

Since the joint discovery with Bush [20], we unsuccessfully tried to extend the result from $p = 3$ to $p = 5$ for nearly 4 years, as documented in the historical introduction of [46], until a lucky coincidence of several unexpected facts enabled a significant break-through on 07 April 2016:

(1) Due to a bug in earlier MAGMA versions, the 5-capitulation type $\varkappa(K)$ of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = 3\,812\,377$ could not be computed until MAGMA V2.21-11 for machines running Mac OS was released on 04 April 2016 and revealed $\varkappa(K) = (100000)$.

(2) In the tree $\mathcal{T}R$ of 5-groups with coclass 1, where $R := \langle 5^2, 2\rangle$ in the notation of the SmallGroups Library [10, 11], the crucial bifurcation [40] at the 4th mainline vertex $\langle 5^5, 30\rangle$ was unknown up to now. It gives rise to the candidates $\langle 5^5, 30\rangle - \#2; n$, $n \in \{10, 22, 23, 34, 35\}$, for the 5-tower group $G = \mathrm{G}_5^\infty K$ of $K = \mathbb{Q}(\sqrt{3\,812\,377})$, in the notation of [24].

(3) Whereas the smallest non-metabelian p-tower groups $G = \mathrm{G}_p^\infty K$, for $p \geq 3$ an odd prime, of *imaginary* quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, are of order $|G| = p^8$ and coclass $\mathrm{cc}(G) = 3$, those of *real* quadratic fields, $d > 0$, have order only $|G| = p^7$, coclass $\mathrm{cc}(G) = 2$, and do not require

arithmetical computations of high complexity with extensions $M|K$ of relative degree $[M:K]=p^2$ for their justification. IPADs of first order are sufficient.

1.9. **Overview.** To avoid any misinterpretations of our notation, an essential remark must be made at the beginning: Throughout this article, we use the *logarithmic form* of type invariants of finite abelian p-groups A, that is, we abbreviate the cumbersome power form of type invariants $\Bigl(\overbrace{p^{e_1},\ldots,p^{e_1}}^{r_1\text{ times}},\ldots,\overbrace{p^{e_n},\ldots,p^{e_n}}^{r_n\text{ times}}\Bigr)$, with strictly decreasing $e_1>\ldots>e_n$, by writing $(e_1^{r_1},\ldots,e_n^{r_n})$ with formal exponents $r_i\ge 1$ denoting iteration. If $e_1<10$, which will always be the case in this paper, then we even omit the separating commas, thus saving a lot of space.

The layout of this survey article is the following. In § 2 we immediately celebrate our most recent sensational discovery of the long desired three-stage towers of 5-class fields. We continue with a recall of the meanwhile well-known three-stage towers of 3-class fields in § 3 and of 2-class fields in § 4. In § 5 we present the new phenomenon of *tree topologies* expressing the mutual location of second and third p-class groups on descendant trees of finite p-groups. An important remark has to be made in § 6 on the published form of the Shafarevich theorem on the relation rank of the p-tower group when the base field contains a primitive pth root of unity. Although the focus will mainly be on the novelty of three-stage towers, we use § 7 for communicating the first criteria for two-stage towers of p-class fields, $\ell_pK=2$, with $p\in\{5,7\}$, independently of the base field K. In § 8 we consider 3-class towers of three imaginary quadratic fields $K=\mathbb{Q}(\sqrt{d})$ with 3-class group Cl_3K of type (1^3), which have infinite length $\ell_3K=\infty$, according to [31]. We emphasize that we are far from having explicit pro-3 presentations of the 3-tower groups $G=\mathrm{G}_3^\infty K$ and we do not know the rate of growth for the orders of successive derived quotients $G/G^{(n)}\simeq\mathrm{G}_3^nK$, $n\ge 2$, of G. Even for the second 3-class groups $\mathfrak{M}=\mathrm{G}_3^2K$, we only have lower bounds for the orders.

2. Three-stage towers of 5-class fields

Experiment 2.1. As documented in § 3.2.7, p. 427, of [37], we used the class field package by Fieker [22] in the computational algebra system MAGMA [34] for constructing the unramified cyclic quintic extensions $L_i|K$, $1\le i\le 6$, of each of the 377 real quadratic fields $K=\mathbb{Q}(\sqrt{d})$ with discriminants $0<d\le 26\,695\,193$ and 5 class group of type (1^2). However, at that early stage, we only computed the first component $\tau(K)$ of the Artin pattern $\mathrm{AP}(K)=(\tau(K),\varkappa(K))$, since $\tau(K)$ uniquely determines $\varkappa(K)$ for the ground state of the TKTs a.2 and a.3, according to Theorem 3.8 and Table 3.3 in [37, § 3.2.5, pp. 423–424]. In this manner, we were able to classify $360=13+55+292$ cases with TKTs a.1, a.2, a.3, where the second 5-class group $\mathfrak{M}=\mathrm{G}_5^2K$ is of coclass $\mathrm{cc}(\mathfrak{M})=1$, listed in Table 3.4 of [37, § 3.2.7, p. 427], and to separate 14 cases with $\mathrm{cc}(\mathfrak{M})=2$, discussed in [37, § 3.5.3, p. 449]. There remained 3 cases of first excited states of the TKTs a.2 and a.3, where the TTT $\tau(K)=[21^3,(1^2)^5]$ is unable to distinguish between the TKTs. As mentioned at the end of § 1.8, the first MAGMA version

which admitted the computation of the TKTs for these difficult cases was V2.21-11, released on 04 April 2016. The result was TKT a.2, $\varkappa(K) = (1, 0^5)$, for $d \in \{3\,812\,377, 19\,621\,905\}$ and TKT a.3, $\varkappa(K) = (2, 0^5)$, for $d = 21\,281\,673$.

After this initial number theoretic experiment with computational techniques of § 1.5, a translation from arithmetic to group theory with Artin's reciprocity law, described in § 1.1, maps the Artin pattern $\mathrm{AP}(K) = (\tau(K), \varkappa(K))$ of the real quadratic fields K to the Artin pattern $\mathrm{AP}(\mathfrak{M}) = (\tau(\mathfrak{M}), \varkappa(\mathfrak{M}))$ of their second 5-class groups $\mathfrak{M} = \mathrm{G}_5^2 K$, which forms the input for the strategy of *pattern recognition via Artin transfers* by conducting a search for suitable finite 5-groups $\mathfrak{M}$ having the prescribed Artin pattern $\mathrm{AP}(\mathfrak{M})$. This is done by recusive iterations of the p-group generation algorithm in § 1.4 until a termination condition is satisfied, due to the monotony of Artin patterns on descendant trees in § 1.7.

The reason why we decided to take the Artin pattern $\mathrm{AP}(\mathfrak{M}) = (\tau(\mathfrak{M}), \varkappa(\mathfrak{M}))$, with $\tau_0\mathfrak{M} = (1^2)$, $\tau_1\mathfrak{M} = [21^3, (1^2)^5]$, $\varkappa_1\mathfrak{M} = (1, 0^5)$, of the first excited state of the TKT a.2 as the search pattern for seeking three-stage towers of 5-class fields was as follows. Firstly, since the derived subgroup $\mathfrak{M}'$ for the ground state of TKT a.2 and a.3 is of type (1^2), a result of Blackburn [12] ensures a two-stage tower with $\ell_5 K = 2$ for these 347 cases. Secondly, the first excited state of the TKT a.3 does not admit a unique candidate for the second 5-class groups $\mathfrak{M} = \mathrm{G}_5^2 K$, and finally, the 13 cases of 5-groups with TKT a.1 form a subgraph with considerable complexity of the coclass tree $\mathcal{T}^1 R$ with root $R := C_5 \times C_5$. Therefore, we arrived at the following group theoretic results.

Proposition 2.1. *Up to isomorphism, there exists a unique finite metabelian* 5*-group* $\mathfrak{M}$ *such that*

$$\tau_0\mathfrak{M} = (1^2), \quad \tau_1\mathfrak{M} = [21^3, (1^2)^5], \quad \text{and} \quad \varkappa_1\mathfrak{M} = (1, 0^5). \tag{2.1}$$

Theorem 2.2. *The unique metabelian* 5*-group* $\mathfrak{M}$ *in Proposition 2.1 is of order* $|\mathfrak{M}| = 5^6 = 15\,625$*, nilpotency class* $\mathrm{cl}(\mathfrak{M}) = 5$*, coclass* $\mathrm{cc}(\mathfrak{M}) = 1$*, and is isomorphic to* $\langle 5^6, 635\rangle$ *in the SmallGroups Library* [10, 11]*. It is a terminal vertex (leaf) on branch* $\mathcal{B}(5)$ *of the coclass tree* $\mathcal{T}^1 R$ *with abelian root* $R := \langle 5^2, 2\rangle$ *of type* (1^2)*. The group* $\mathfrak{M}$ *has relation rank* $d_2\mathfrak{M} = 4$ *and its derived subgroup* $\mathfrak{M}'$ *is abelian of type* (1^4)*.*

Proposition 2.3. *Up to isomorphism, there exist precisely five pairwise non-isomorphic finite non-metabelian* 5*-groups* $H_1, \ldots, H_5$ *whose second derived quotient* H_j/H_j'' *is isomorphic to the group* $\mathfrak{M}$ *of Theorem 2.2, for* $1 \le j \le 5$*.*

Theorem 2.4. *The five non-metabelian* 5*-groups* H_j *in Proposition 2.3 are of order* $|H_j| = 5^7 = 78\,125$*, nilpotency class* $\mathrm{cl}(H_j) = 5$*, coclass* $\mathrm{cc}(H_j) = 2$ *and derived length* $\mathrm{dl}(H_j) = 3$*, for* $1 \le j \le 5$*. They are isomorphic to* $\langle 5^7, n\rangle$ *with* $n \in \{361, 373, 374, 385, 386\}$ *in the SmallGroups Library* [11]*, located as terminal vertices (leaves) on the descendant tree* $\mathcal{T}R$*, but not on the coclass tree* $\mathcal{T}^1 R$*, of the abelian root* $R = \langle 5^2, 2\rangle$*; in fact, they are sporadic and do not belong to any coclass tree. Their second derived subgroup* H_j'' *is cyclic of order* 5*, and is contained in the centre* $\zeta_1 H_j$ *of type* (1^2)*, i.e., each* H_j *is centre by metabelian.*

The groups H_j have relation rank $d_2 H_j = 3$ and the abelianization H_j'/H_j'' of their derived subgroup H_j' is of type (1^4).

Proof. The detailed proof of Proposition 2.1, Theorem 2.2, Proposition 2.3, and Theorem 2.4 is conducted in [46, § 4]. A diagram of the pruned descendant tree $\mathcal{T}_* R$ with root $R = \langle 25, 2\rangle$, where the finite 5-groups $\mathfrak{M}$ and H_j are located, is shown in [46, § 7, Fig. 1]. Polycyclic power commutator presentations of the groups H_j are given in [46, § 7] and a diagram of their normal lattice, including the lower and upper central series, is drawn in [46, § 7, Fig. 2]. $\square$

Now we come to the number theoretic harvest of the group theoretic results by translating back to arithmetic in the manner of § 1.1. Here, we exceptionally use the power form of abelian type invariants, and we dispense with formal exponents denoting iteration.

Theorem 2.5. *Let K be a real quadratic field with 5-class group $\mathrm{Cl}_5 K$ of type $[5, 5]$ and denote by $L_1, \ldots, L_6$ its six unramified cyclic quintic extensions. If K possesses the 5-capitulation type*

$$\varkappa(K) \sim (1, 0, 0, 0, 0, 0), \text{ with fixed point } 1, \tag{2.2}$$

in the six extensions L_i, and if the 5-class groups $\mathrm{Cl}_5 L_i$ are given by

$$\tau(K) \sim ([25, 5, 5, 5], [5, 5], [5, 5], [5, 5], [5, 5], [5, 5]), \tag{2.3}$$

then the 5-class tower $K < \mathrm{F}_5^1 K < \mathrm{F}_5^2 K < \mathrm{F}_5^3 K = \mathrm{F}_5^\infty K$ of K has exact length $\ell_5 K = 3$.

Corollary 2.6. *A real quadratic field K which satisfies the assumptions in Theorem 2.5, in particular the Formulas (2.2) and (2.3), has the unique second 5-class group*

$$\mathfrak{M} = \mathrm{G}_5^2 K \simeq \langle 15625, 635\rangle \tag{2.4}$$

with order 5^6, class 5, coclass 1, derived length 2, and relation rank 4, and one of the following five candidates for the 5-class tower group

$$G = \mathrm{G}_5^\infty K = \mathrm{G}_5^3 K \simeq \langle 78125, n\rangle, \quad \text{where } n \in \{361, 373, 374, 385, 386\}, \tag{2.5}$$

with order 5^7, class 5, coclass 2, derived length 3, and relation rank 3.

Proof. For the proof of Theorem 2.5 and Corollary 2.6, the methods of §§ 1.2 and 1.3 come into the play. The cover of the metabelian 5-group $\mathfrak{M} = \mathrm{G}_5^2 K \simeq \langle 5^6, 635\rangle$ consists of the six elements $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}, H_1, \ldots, H_5\}$, but since the relation rank of the metabelian group is too big, the Shafarevich cover of $\mathfrak{M}$ with respect to any real quadratic number field K with $\varrho = 2$ reduces to $\mathrm{cov}(\mathfrak{M}, K) = \{H_1, \ldots, H_5\}$, as explained in the proofs of [46, Thm. 6.1] and [46, Thm. 6.3]. $\square$

Example 2.7. The minimal fundamental discriminant d of a real quadratic field $K = \mathbb{Q}(\sqrt{d})$ satisfying the conditions (2.2) and (2.3) is given by

$$d = 3\,812\,377 = 991 \cdot 3\,847. \tag{2.6}$$

The next occurrence is $d = 19\,621\,905 = 3 \cdot 5 \cdot 307 \cdot 4\,261$, which is currently the biggest known example, whereas $d = 21\,281\,673 = 3 \cdot 7 \cdot 53 \cdot 19\,121$ satisfies (2.3) but has a different $\varkappa(K) \sim (200000)$, without fixed point, and $d = 27\,186\,289 = 13 \cdot 677 \cdot 3\,089$, which is the last exceptional case in the range $0 < d < 3 \cdot 10^7$, has $\tau(K) \sim [1^5, (1^2)^5]$ and $\varkappa(K) \sim (200000)$.

3. Three-stage towers of 3-class fields

Since we have devoted the preceding § 2 to a detailed explanation of the general way from a number theoretic experiment, which prescribes a certain Artin pattern, over the group theoretic interpretation of data and the identification of suitable groups, to the final arithmetical statement of a criterion for three-stage towers, we can restrict ourselves to number theoretic end results, in the sequel.

The situation in § 2 gives rise to a *finite* cover with $\#\mathrm{cov}(\mathfrak{M}) = 6$ and a Shafarevich cover $\mathrm{cov}(\mathfrak{M}, K)$, with respect to a real quadratic field K, all of whose members H have the same derived length $\mathrm{dl}(H) = 3$, which we shall call a *homogeneous* Shafarevich cover. These homogeneity considerations will be the guiding principle for a subdivision of the following results.

3.1. **Finite homogeneous Shafarevich cover.** The finiteness of the cover $\mathrm{cov}(\mathfrak{M})$ of descendants $\mathfrak{M}$ either of $\langle 3^5, 6\rangle$ with types c.18, E.6, E.14 or of $\langle 3^5, 8\rangle$ with types c.21, E.8, E.9 has been proven up to a certain nilpotency class $c = \mathrm{cc}(\mathfrak{M})$ in [40] for section E, and in [43] for section c. In fact, the cardinality of the cover is expected to increase linearly with c, for sections E and c. We present an examplary result with TKT $\varkappa$ of type E.9, where a *complex* quadratic base field K compels a homogeneous Shafarevich cover $\mathrm{cov}(\mathfrak{M}, K)$ of its second 3-class group $\mathfrak{M} = \mathrm{G}_3^2 K$.

Theorem 3.1. *Let K be a complex quadratic field with 3-class group $\mathrm{Cl}_3 K$ of type (1^2) and denote by $L_1, \ldots, L_4$ its four unramified cyclic cubic extensions. If K possesses the 3-capitulation type*

$$\varkappa(K) \sim (2334), \text{ with two fixed points } 3, 4 \text{ and without a transposition,} \tag{3.1}$$

in the four extensions L_i, and if the 3-class groups $\mathrm{Cl}_3 L_i$ are given by

$$\tau(K) \sim (21, (j+1, j), 21, 21)\,, \text{ for some } 2 \le j \le 6, \tag{3.2}$$

then the 3-class tower $K < \mathrm{F}_3^1 K < \mathrm{F}_3^2 K < \mathrm{F}_3^3 K = \mathrm{F}_3^\infty K$ of K has exact length $\ell_3 K = 3$.

Corollary 3.2. *A complex quadratic field K which satisfies the assumptions in Theorem 3.1, in particular the Formulas (3.1) and (3.2) with $2 \le j \le 6$, has the second 3-class group*

$$\mathfrak{M} = \mathrm{G}_3^2 K \simeq \langle 3^6, 54\rangle(-\#1; 1 - \#1; 1)^{j-2} - \#1; n, \qquad \text{where } n \in \{4, 6\}, \tag{3.3}$$

with order 3^{2j+3}, class $2j+1$, coclass 2, derived length 2, and relation rank 3, and the 3-class tower group

$$G = \mathrm{G}_3^\infty K = \mathrm{G}_3^3 K \simeq \langle 3^6, 54\rangle(-\#2; 1-\#1; 1)^{j-2} - \#2; n, \ \textit{where } n \in \{4, 6\}, \tag{3.4}$$

with order 3^{3j+2}*, class* $2j+1$*, coclass* $j+1$*, derived length* 3*, and relation rank* 2*.*

Proof. The statements of Theorem 3.1 and Corollary 3.2 arise by specialization from the more general [42, Thm. 6.1 and Cor. 6.1, pp. 751–753]. Here, we restrict to the single TKT E.9 and to the smaller range $2 \le j \le 6$, which has been realized by explicit numerical results already, as shown in Example 3.3. □

Example 3.3. The fundamental discriminant d of complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$, with minimal absolute value $|d|$, satisfying the conditions (3.1) and (3.2), with increasing values of the parameter $2 \le j \le 6$, corresponding to excited states of increasing order of type E.9, are given by

$$d \in \{-9\,748, -297\,079, \ -1\,088\,808, \ -11\,091\,140, \ -99\,880\,548\}. \tag{3.5}$$

The associated second 3-class groups $\mathfrak{M} = \mathrm{G}_3^2 K$ form a periodic sequence of vertices on the coclass tree $\mathcal{T}^2\langle 3^5, 8\rangle$ drawn in the diagram [42, Fig. 4, p. 755]. The smallest example $d = -9\,748$ with $j = 2$, corresponding to the ground state of type E.9, was the object of our joint investigations with Bush in [20, Cor. 4.1.1, p. 775].

3.2. **Finite heterogeneous Shafarevich cover.** In contrast to the previous § 3.1, a *real* quadratic base field K with TKT $\varkappa$ of type E.9 is not able to enforce a homogeneous Shafarevich cover of its second 3-class group $\mathfrak{M} = \mathrm{G}_3^2 K$. In this situation, $\mathrm{cov}(\mathfrak{M}, K)$ contains elements H of derived lengths $2 \le \mathrm{dl}(H) \le 3$, and there arises the necessity to establish *criteria for distinguishing between two- and three-stage towers.* According to Theorem 1.13, the (simple) IPAD of first order, $\tau^{(1)}K = [\tau_0 K; \tau_1 K]$, which forms the first order approximation of the layered TTT $\tau(K) = [\tau_0 K; \ldots; \tau_v K]$, is unable to admit a decision, and we have to proceed to abelian type invariants of second order. The computation of the *iterated* IPAD *of second order,* $\tau^{(2)}K = [\tau_0 K; (\tau^{(1)}L)_{L \in \mathrm{Lyr}_1 K}]$, where $\tau^{(1)}L = [\tau_0 L; \tau_1 L] = [\tau_0 L; (\tau_0 M)_{M \in \mathrm{Lyr}_1 L}]$, for each $L \in \mathrm{Lyr}_1 K$, requires the construction of unramified abelian and non-abelian extensions $M|K$ of relative degree $[M : K] = 3^2$, that is, of absolute degree $[M : \mathbb{Q}] = 18$, whereas in § 3.1, cyclic extensions $L|K$ of absolute degree $[L : \mathbb{Q}] = 6$ were sufficient. Due to the complexity of the scenario, we now prefer a restriction to the ground state.

Theorem 3.4. *Let K be a real quadratic field with* 3*-class group* $\mathrm{Cl}_3 K$ *of type* (1^2) *and denote by* $L_1, \ldots, L_4$ *its four unramified cyclic cubic extensions. If* K *possesses the* 3*-capitulation type*

$$\varkappa(K) \sim (2334), \ \textit{with two fixed points } 3, 4 \textit{ and without a transposition,} \tag{3.6}$$

in the four extensions L_i*, and if the* 3*-class groups* $\mathrm{Cl}_3 L_i$ *are given by*

$$\tau(K) \sim (21, 32, 21, 21)\,, \tag{3.7}$$

then each L_i has four unramified cyclic cubic extensions $M_{i,1},\ldots,M_{i,4}$, which are also unramified but not necessarily abelian over K, and the 3-class groups $\mathrm{Cl}_3 M_{i,j}$ admit the following decision about the length $\ell_3 K$ of the 3-class tower of K:

$$\text{if } \tau(L_i) \sim \left(2^2 1, 21, 21, 21\right), \text{ for } i \in \{1,3,4\}, \tag{3.8}$$

then the 3-class tower $K < \mathrm{F}_3^1 K < \mathrm{F}_3^2 K = \mathrm{F}_3^\infty K$ of K has exact length $\ell_3 K = 2$,

$$\text{but if } \tau(L_i) \sim \left(2^2 1, 31, 31, 31\right), \text{ for } i \in \{1,3,4\}, \tag{3.9}$$

then the 3-class tower $K < \mathrm{F}_3^1 K < \mathrm{F}_3^2 K < \mathrm{F}_3^3 K = \mathrm{F}_3^\infty K$ of K has exact length $\ell_3 K = 3$.

Corollary 3.5. *A real quadratic field K which satisfies the assumptions in Theorem 3.1, in particular the Formulas (3.6) and (3.7), has the second 3-class group*

$$\mathfrak{M} = \mathrm{G}_3^2 K \simeq \langle 3^7, n \rangle, \qquad \text{where } n \in \{302, 306\}, \tag{3.10}$$

with order 2 187, class 5, coclass 2, derived length 2, and relation rank 3, and, if Formula (3.9) is satisfied, the 3-class tower group

$$G = \mathrm{G}_3^\infty K = \mathrm{G}_3^3 K \simeq \langle 3^6, 54 \rangle - \#2; n, \qquad \text{where } n \in \{2, 6\}, \tag{3.11}$$

with order 6 561, class 5, coclass 3, derived length 3, and relation rank 2, otherwise the 3-class tower group coincides with the second 3-class group.

Proof. The statements of Theorem 3.4 and Corollary 3.5 have been proved in [41, Thm. 6.3, pp. 298–299] and [45, Thm. 4.2]. We point out that the remaining component $\tau(L_2) \sim (2^2 1, 31^2, 31^2, 31^2)$ of the iterated IPAD of second order does not admit a decision, and that the common entry $\mathrm{Cl}_3 M_{i,1} \simeq (2^2 1)$ of all components $\tau(L_i)$ corresponds to the Hilbert 3-class field $M_{i,1} = \mathrm{F}_3^1 K$ of K, whereas all other extensions $M_{i,j}$ with $j > 1$ are non-abelian over K. $\square$

Example 3.6. The fundamental discriminants $0 < d < 10^7$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, satisfying the conditions (3.6), (3.7), and (3.9), resp. (3.8), are given by

$$d \in \{342\,664,\ 1\,452\,185,\ 1\,787\,945,\ 4\,861\,720,\ 5\,976\,988,\ 8\,079\,101,\ 9\,674\,841\}, \tag{3.12}$$

resp.

$$d \in \{4\,760\,877,\ 6\,652\,929,\ 7\,358\,937,\ 9\,129\,480\}. \tag{3.13}$$

Evidence of a similar behaviour of real quadratic fields K with the first excited state of one of the TKTs $\varkappa(K)$ in section E is provided in [47, Example 4.1].

3.3. **Infinite cover.** The infinitude of the cover, $\#\mathrm{cov}(\mathfrak{M}) = \infty$, has been proven by Bartholdi and Bush [9] for the sporadic 3-group $\mathfrak{M} = N := \langle 3^6, 45\rangle$, with $\tau = [1^3, 1^3, 21, 1^3]$ and $\varkappa = (4443)$ of type H.4*, and it is conjectured for $\mathfrak{M} = W := \langle 3^6, 57\rangle$, with $\tau = [(21)^4]$ and $\varkappa = (2143)$ of type G.19*.

In the former case, $\mathrm{cov}(N)$ contains an infinite sequence of Schur σ-groups. Consequently, even the Shafarevich cover $\mathrm{cov}(N, K)$ with respect to complex quadratic fields K of 3-class rank $\varrho = 2$ is infinite. Furthermore, it may be called *infinitely heterogeneous* in the sense of unbounded derived length [9]. This fact causes the considerable difficulty that *iterated* IPADs *of increasing order* are required for the distinction between the members of $\mathrm{cov}(N, K)$. Already for separating the leading two members, we have to compute extensions $M|K$ of absolute degree $[M : \mathbb{Q}] = 54$ in the second layer over K, as the following theorem shows.

Theorem 3.7. *Let K be a complex quadratic field with 3-class group Cl_3K of type (1^2) and denote by $L_1, \ldots, L_4$ its four unramified cyclic cubic extensions. If K possesses the 3-capitulation type*

$$\varkappa(K) \sim (4443),\ \textit{nearly constant, without fixed points, and containing a transposition,} \tag{3.14}$$

in the four extensions L_i, and if the 3-class groups Cl_3L_i are given by

$$\tau(K) \sim \left(1^3, 1^3, 21, 1^3\right), \tag{3.15}$$

then L_i has thirteen unramified bicyclic bicubic extensions $M_{i,1}, \ldots, M_{i,13}$, for $i \in \{1, 2, 4\}$, but L_3 has only four unramified abelian extensions $M_{3,1}, \ldots, M_{3,4}$ of relative degree 3^2, which are also unramified but not necessarily abelian over K, and the 3-class groups $\mathrm{Cl}_3M_{i,j}$ admit the following decision about the length ℓ_3K of the 3-class tower of K:

$$\begin{aligned} \textit{if } \tau_2(L_i) &\sim \left(2^21, (21^2)^{12}\right),\ \textit{for } i \in \{1, 2\}, \\ \tau_2(L_3) \sim \left(2^21, (2^2)^3\right), \tau_2(L_4) &\sim \left(2^21, (1^3)^3, (2^2)^3, (21)^6\right), \end{aligned} \tag{3.16}$$

then the 3-class tower $K < \mathrm{F}_3^1K < \mathrm{F}_3^2K < \mathrm{F}_3^3K = \mathrm{F}_3^\infty K$ of K has exact length $\ell_3K = 3$,

$$\begin{aligned} \textit{but if } \tau_2(L_i) &\sim \left((2^21)^4, (31^2)^9\right),\ \textit{for } i \in \{1, 2\}, \\ \tau_2(L_3) \sim \left(2^21, (32)^3\right), \tau_2(L_4) &\sim \left(2^21, (1^3)^3, (32)^3, (21)^6\right), \end{aligned} \tag{3.17}$$

then the 3-class tower $K < \mathrm{F}_3^1K < \mathrm{F}_3^2K < \mathrm{F}_3^3K \le \ldots \le \mathrm{F}_3^\infty K$ of K may have any length $\ell_3K \ge 3$.

Corollary 3.8. *A complex quadratic field K which satisfies the assumptions in Theorem 3.7, in particular the Formulas (3.14) and (3.15), has the second 3-class group*

$$\mathfrak{M} = \mathrm{G}_3^2K \simeq \langle 3^6, 45\rangle, \tag{3.18}$$

with order 729*, class* 4*, coclass* 2*, derived length* 2*, and relation rank* 4*, and, if Formula* (3.16) *is satisfied, the* 3*-class tower group*

$$G = \mathrm{G}_3^\infty K = \mathrm{G}_3^3 K \simeq \langle 3^6, 45\rangle - \#2;2, \tag{3.19}$$

with order $6\,561$*, class* 5*, coclass* 3*, derived length* 3*, and relation rank* 2*, otherwise the* 3*-class tower group is of order at least* 3^{11} *and may be any of the Schur* σ*-groups* $\langle 3^6, 45\rangle(-\#2;1-\#1;2)^j - \#2;2$ *with* $j \geq 1$ *and derived length at least* 3*.*

Proof. The statements of Theorem 3.7 and Corollary 3.8 have been proved in [41, Thm. 6.5, pp. 304–306] and [45, § 4.4, Tbl. 4]. We point out that the first layer $\tau_1(L_i) \sim ((21^2)^4, (2^2)^9)$, for $1 \leq i \leq 2$, $\tau_1(L_3) \sim (21^2, (31)^3)$, $\tau_1(L_4) \sim ((21^2)^4, (1^2)^9)$, of the iterated IPAD of second order does not admit a decision. $\square$

Example 3.9. The fundamental discriminants $-30\,000 < d < 0$ of complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$, satisfying the conditions (3.14), (3.15), and (3.16), resp. (3.17), are given by

$$d \in \{-3\,896,\ -25\,447,\ -27\,355\}, \tag{3.20}$$

resp.

$$d \in \{-6\,583,\ -23\,428,\ -27\,991\}. \tag{3.21}$$

4. Three-stage towers of 2-class fields

For historical reasons, the very first discovery of a p-class tower with three stages for $p = 2$ merits attention. It is due to Bush [19] in 2003. He investigated complex quadratic fields K with 2-class rank $\varrho = 2$, since it is relatively easy to compute the unramified 2-extensions $M|K$ in several layers with absolute degrees $[M : \mathbb{Q}] \in \{4, 8, 16\}$.

Theorem 4.1. *Let* K *be a complex quadratic field with* 2*-class group* Cl_2K *of type* (21)*, denote by* $L_{1,1}, \ldots, L_{1,3}$ *its three unramified quadratic extensions, and by* $L_{2,1}, \ldots, L_{2,3}$ *its three unramified abelian quartic extensions, such that* $L_{2,3} = \prod_{i=1}^3 L_{1,i}$ *is bicyclic biquadratic and* $L_{1,3} = \bigcap_{i=1}^3 L_{2,i}$*. If the* 2*-class groups* $\mathrm{Cl}_2L_{1,i}$*, resp.* $\mathrm{Cl}_2L_{2,i}$*, are given by*

$$\tau_1(K) = \left(2^2, 31, 1^3\right), \quad \tau_2(K) = \left(2^2, 21^2, 21^2\right), \tag{4.1}$$

then $L_{1,i}$ *has three unramified quadratic extensions* $M_{i,1}, \ldots, M_{i,3}$ *for* $1 \leq i \leq 2$*, and* $L_{1,3}$ *has seven unramified quadratic extensions* $M_{3,1}, \ldots, M_{3,7}$*, which are also unramified but not necessarily abelian over* K*, and the* 2*-class groups* $\mathrm{Cl}_2M_{i,j}$ *admit the following statement.*

$$\text{If } \tau_1(L_{1,1}) = \left((21^2)^3\right), \tau_1(L_{1,2}) = \left(21^2, 41, 41\right), \tau_1(L_{1,3}) = \left((21^2)^6, 2^2\right), \tag{4.2}$$

then the 2*-class tower* $K < \mathrm{F}_2^1K < \mathrm{F}_2^2K < \mathrm{F}_2^3K = \mathrm{F}_2^\infty K$ *of* K *has exact length* $\ell_2K = 3$*.*

Corollary 4.2. *A complex quadratic field K which satisfies the assumptions in Theorem 4.1, in particular the Formulas* (4.1) *and* (4.2)*, has the second* 2*-class group*

$$\mathfrak{M} = \mathrm{G}_2^2 K \simeq \langle 2^7, 84\rangle, \tag{4.3}$$

with order 128*, class* 4*, coclass* 3*, derived length* 2*, and relation rank* 4*, and one of the following two candidates for the* 2*-class tower group*

$$G = \mathrm{G}_2^\infty K = \mathrm{G}_2^3 K \simeq \langle 2^8, n\rangle, \quad \text{where } n \in \{426, 427\}, \tag{4.4}$$

with order 256*, class* 5*, coclass* 3*, derived length* 3*, and relation rank* 3*.*

Proof. The statements of Theorem 4.1 and Corollary 4.2 have essentially been proved by Bush in [19, Prop. 2, p. 321]. The derived subgroup of $\langle 2^7, 84\rangle$ is abelian of type (21^2) □

5. Simple and advanced tree topologies

Let p be a prime, $n > m \geq 1$ be integers, and K be a number field. Then both, the nth and mth p-class group of K, are vertices of the descendant tree $\mathcal{T}\mathrm{G}_p^1 K$ of the p-class group $\mathrm{Cl}_p K = \mathrm{G}_p^1 K$ of K. The vertex $R := \mathrm{G}_p^1 K$ is the *abelian tree root*. Several invariants describe the mutual location of the vertices $\mathrm{G}_p^n K$ and $\mathrm{G}_p^m K$ in the tree topology.

Definition 5.1. By the *class increment*, resp. *coclass increment*, we understand the difference $\Delta\mathrm{cl}(n,m) := \mathrm{cl}(\mathrm{G}_p^n K) - \mathrm{cl}(\mathrm{G}_p^m K)$, resp. $\Delta\mathrm{cc}(n,m) := \mathrm{cc}(\mathrm{G}_p^n K) - \mathrm{cc}(\mathrm{G}_p^m K)$. The biggest common ancestor of $\mathrm{G}_p^m K$ and $\mathrm{G}_p^n K$ is called their *fork*, denoted by $\mathrm{Fork}(m,n)$.

Remark 5.2. If we define the *logarithmic order* of a finite p-group G by $\mathrm{lo}(G) := \log_p \mathrm{ord}(G)$, and consider the situation in Definition 5.1, then $\Delta\mathrm{lo}(n,m) := \mathrm{lo}(\mathrm{G}_p^n K) - \mathrm{lo}(\mathrm{G}_p^m K)$, which is called the *log ord increment*, satisfies the relation $\Delta\mathrm{lo}(n,m) = \Delta\mathrm{cl}(n,m) + \Delta\mathrm{cc}(n,m)$.

The concepts actually make sense for p-class towers of length $\ell_p K \geq 3$. For two-stage towers, we have the following trivial fact.

Proposition 5.3. *For any $n \geq 2$, we have $\Delta\mathrm{cl}(n,1) = \mathrm{cl}(\mathrm{G}_p^n K) - 1$, and $\mathrm{Fork}(1,n)$ is given by the abelian tree root $R = \mathrm{G}_p^1 K$.*

In Table 1 and 2, we summarize all tree topologies currently known for three-stage p-class towers of quadratic base fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant d. We put $n = 3$ and $m = 2$, and use the abbreviations $\Delta\mathrm{lo} := \Delta\mathrm{lo}(3,2)$, $\Delta\mathrm{cl} := \Delta\mathrm{cl}(3,2)$, $\Delta\mathrm{cc} := \Delta\mathrm{cc}(3,2)$. Forks are labelled with Ascione's identifiers [3, 4], $\mathrm{N} := \langle 3^6, 45\rangle$, $\mathrm{Q} := \langle 3^6, 49\rangle$, $\mathrm{U} := \langle 3^6, 54\rangle$, $\mathrm{W} := \langle 3^6, 57\rangle$, avoiding the long symbols in angle brackets of the SmallGroups Library [10, 11]. Additionally, we define two ad hoc-identifiers $\mathrm{Y} := \langle 2^7, 84\rangle$, resp. $\mathrm{Z} := \langle 5^5, 30\rangle$, for $p = 2$, resp. $p = 5$. Four vertices on the mainline containing Q, resp. U, are denoted by $\mathrm{Q}_8 := \langle 3^7, 285\rangle - \#1;1$, $\mathrm{Q}_{10} := \langle 3^7, 285\rangle(-\#1;1)^3$, resp.

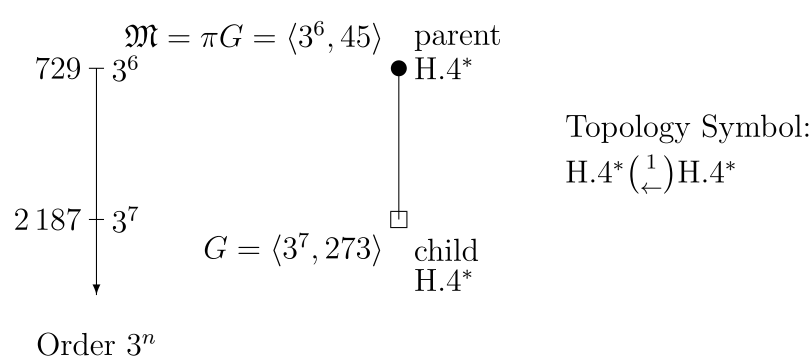
TABLE 1. Simple tree topologies of three-stage p-class towers

p	d	TKT	Δlo	Δcc	Δcl	Fork(2, 3)	Topology	G''	Proof	Ref.
2	$-1\,780$	B.8	1	0	1	Y$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[19]
3	$957\,013$	H.4*	1	0	1	N$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[41]
3	$214\,712$	G.19*	1	0	1	W$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[45]
3	$21\,974\,161$	G.19* $\uparrow$	4	1	3	W$=\mathfrak{M}=\pi^3 G$	descent	$(1,1,1,1)$	$\tau^{(2)}$	[45]
3	$-3\,896$	H.4*	2	1	1	N$=\mathfrak{M}=\pi G$	bastard	$(1,1)$	$\tau_*^{(2)}$	[41]
3	? $-6\,583$	H.4* $\uparrow$	5	2	3	N$=\mathfrak{M}=\pi^3 G$	descent	$(2,2,1)$	$\tau^{(3)}$	[45]
3	$534\,824$	c.18	1	0	1	Q$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$1\,030\,117$	c.18	1	0	1	Q$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$13\,714\,789$	c.18 $\uparrow$	1	0	1	$\mathrm{Q}_8=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$241\,798\,776$	c.18 $\uparrow^2$	1	0	1	$\mathrm{Q}_{10}=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$540\,365$	c.21	1	0	1	U$=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$1\,001\,957$	c.21 $\uparrow$	1	0	1	$\mathrm{U}_8=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]
3	$407\,086\,012$	c.21 $\uparrow^2$	1	0	1	$\mathrm{U}_{10}=\mathfrak{M}=\pi G$	child	(1)	$\tau^{(2)}$	[43]

$\mathrm{U}_8 := \langle 3^7, 303\rangle - \#1;1$, $\mathrm{U}_{10} := \langle 3^7, 303\rangle(-\#1;1)^3$, using relative ANUPQ identifiers [24].

A question mark in front of a discriminant d indicates that the result is conjectural only. In Table 2, we denote the vertex $\langle 3^7, 64\rangle$ by P_7.

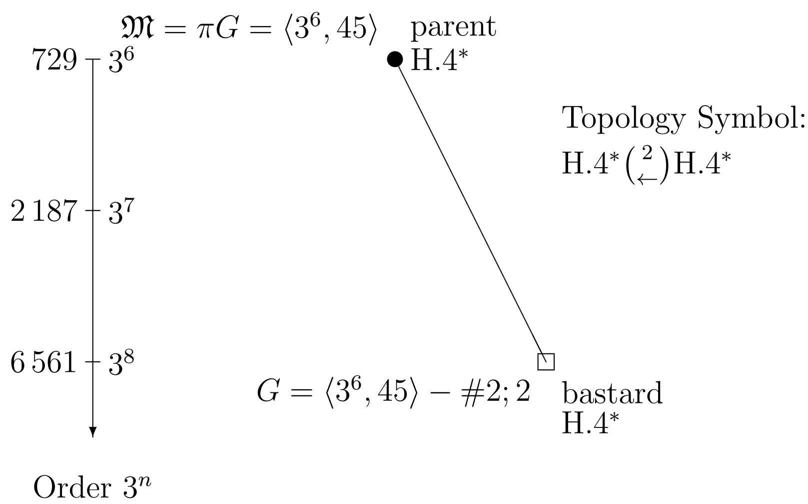
FIGURE 1. Simple tree topology of type parent – child



Example 5.4. The diagram in Figure 1 visualizes the *simple child topology* of the mutual location between the second and third 3-class group, $\mathfrak{M}=\mathrm{Gal}(\mathrm{F}_3^2K|K)$ and $G=\mathrm{Gal}(\mathrm{F}_3^3K|K)$, where $K=\mathbb{Q}(\sqrt{d})$ is the real quadratic field with discriminant $d=957\,013$. The non-metabelian 3-group $G=\langle 3^7,273\rangle$ is a child, that is an immediate descendant of step size $s=1$, of the metabelian 3-group $\mathfrak{M}=\langle 3^6,45\rangle$. The coclass remains fixed, $\mathrm{cc}(G)=\mathrm{cc}(\mathfrak{M})=2$.

Example 5.5. The diagram in Figure 2 visualizes the *simple bastard topology* of the mutual location between the second and third 3-class group, $\mathfrak{M}=$

FIGURE 2. Simple tree topology of type parent – bastard



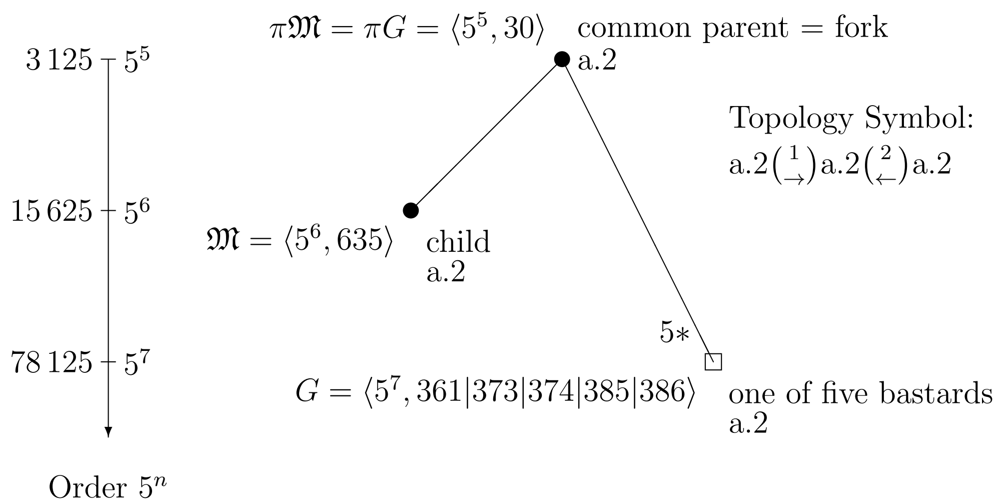
$\mathrm{Gal}(\mathrm{F}_3^2 K|K)$ and $G = \mathrm{Gal}(\mathrm{F}_3^3 K|K)$, where $K = \mathbb{Q}(\sqrt{d})$ is the complex quadratic field with discriminant $d = -3\,896$. The non-metabelian 3-group $G = \langle 3^6, 45\rangle - \#2; 2$ is a bastard, that is an immediate descendant of step size $s = 2$, of the metabelian 3-group $\mathfrak{M} = \langle 3^6, 45\rangle$. The coclass increases, $3 = \mathrm{cc}(G) > \mathrm{cc}(\mathfrak{M}) = 2$.

TABLE 2. Advanced tree topologies of three-stage p-class towers

p	d	TKT	Δlo	Δcc	Δcl	Fork(2, 3)	Topology	G''	Proof	Ref.
5	3 812 377	a.2	1	1	0	Z$= \pi\mathfrak{M} = \pi G$	sibling	(1)	$\tau^{(1)}$	[46]
3	−9 748	E.9	1	1	0	U$= \pi\mathfrak{M} = \pi G$	sibling	(1)	$\tau^{(1)}$	[20]
3	−297 079	E.9 $\uparrow$	2	2	0	U$= \pi^3\mathfrak{M} = \pi^3 G$	fork	(2)	$\tau^{(1)}$	[42]
3	−1 088 808	E.9 $\uparrow^2$	3	3	0	U$= \pi^5\mathfrak{M} = \pi^5 G$	fork	(3)	$\tau^{(1)}$	[42]
3	−11 091 140	E.9 $\uparrow^3$	4	4	0	U$= \pi^7\mathfrak{M} = \pi^7 G$	fork	(4)	$\tau^{(1)}$	[42]
3	−94 880 548	E.9 $\uparrow^4$	5	5	0	U$= \pi^9\mathfrak{M} = \pi^9 G$	fork	(5)	$\tau^{(1)}$	[42]
3	14 252 156	c.18 $\uparrow$	2	1	1	Q$= \pi^2\mathfrak{M} = \pi^3 G$	fork	(2)	$\tau^{(2)}$	[43]
3	174 458 681	c.18 $\uparrow^2$	3	2	1	Q$= \pi^4\mathfrak{M} = \pi^5 G$	fork	(3)	$\tau^{(2)}$	[43]
3	25 283 701	c.21 $\uparrow$	2	1	1	U$= \pi^2\mathfrak{M} = \pi^3 G$	fork	(2)	$\tau^{(2)}$	[43]
3	116 043 324	c.21 $\uparrow^2$	3	2	1	U$= \pi^4\mathfrak{M} = \pi^5 G$	fork	(3)	$\tau^{(2)}$	[43]
3	? 710 652	b.10	2	2	0	$\mathrm{P}_7 = \pi\mathfrak{M} = \pi G$	sibling	(1, 1)	$\tau^{(2)}$	[48]
3	? 1 535 117	d.23	2	2	0	$\mathrm{P}_7 = \pi\mathfrak{M} = \pi G$	sibling	(1, 1)	$\tau^{(2)}$	[48]
3	? 8 321 505	F.13*	1	1	0	$\mathrm{P}_7 = \pi\mathfrak{M} = \pi G$	sibling	(1)	$\tau^{(2)}$	[48]
3	? −225 299	F.7*	11	7	4	$\mathrm{P}_7 = \pi\mathfrak{M} = \pi^5 G$	fork	$(3^2, 2, 1^3)$	$\tau^{(2)}$	[48]
3	? −469 787	F.11	18	12	6	$\mathrm{P}_7 = \pi^3\mathfrak{M} = \pi^9 G$	fork	$(5, 3^3, 2^2)$	$\tau^{(2)}$	[48]

Example 5.6. The diagram in Figure 3 visualizes the *advanced siblings topology* of the mutual location between the second and third 5-class group, $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_5^2 K|K)$ and $G = \mathrm{Gal}(\mathrm{F}_5^3 K|K)$, where $K = \mathbb{Q}(\sqrt{d})$ is the real quadratic

FIGURE 3. Advanced tree topology of type siblings



field with discriminant $d = 3\,812\,377$. Exemplarily, the metabelian 5-group $\mathfrak{M} = \langle 5^6, 635\rangle$ is a child, that is an immediate descendant of step size $s = 1$, and the non-metabelian 5-group $G = \langle 5^7, 361\rangle$ is a bastard, that is an immediate descendant of step size $s = 2$, of the metabelian 5-group $\langle 5^5, 30\rangle$, which is the common parent $F := \pi\mathfrak{M} = \pi G$ of the siblings, called the *fork*. The coclass of G increases, $2 = \mathrm{cc}(G) > \mathrm{cc}(\mathfrak{M}) = \mathrm{cc}(F) = 1$.

6. BIQUADRATIC BASE FIELDS CONTAINING THE pTH ROOTS OF UNITY

6.1. **Dirichlet fields.** The first examples of fields, where a violation of the Shafarevich Theorem 1.3 in its misprinted version [54, Thm. 6, (18′)] occurred, have been found by Azizi, Zekhnini and Taous [6], the violation itself has been recognized by ourselves.

A bicyclic biquadratic field $k = \mathbb{Q}(\sqrt{-1}, \sqrt{d})$ with squarefree radicand $d > 1$ is totally complex with signature $(r_1, r_2) = (0, 2)$. Thus, the torsionfree Dirichlet unit rank of k is $r = r_1 + r_2 - 1 = 1$. The particular fields with radicand $d = p_1 p_2 q$, where $p_1 \equiv 1 \pmod 8$, $p_2 \equiv 5 \pmod 8$ and $q \equiv 3 \pmod 4$ are prime numbers such that $\left(\frac{p_1}{p_2}\right) = -1$, $\left(\frac{p_1}{q}\right) = -1$ and $\left(\frac{p_2}{q}\right) = -1$, have a 2-class group $\mathrm{Cl}_2 k$ of type (1^3) [6], and a 2-class tower of length $\ell_2 k = 2$ [6]. Thus, the 2-class rank of k is $\varrho = 3$, and, since k trivially contains the second roots of unity, we have the invariant $\theta = 1$, with respect to the even prime $p = 2$. In [42], we have identified the possible 2-class tower groups $G = \mathrm{G}_2^\infty k$ of k as $\langle 32, 35\rangle$, $\langle 64, 181\rangle$, $\langle 128, 984\rangle$, etc., visualized in the diagram [42, Fig. 2, p. 752]. A computation with the aid of MAGMA [34] shows that these metabelian 2-groups all have the maximal admissible relation rank $d_2 G = 5$, in accordance with our corrected Formula (1.1) in Theorem 1.3, $d_2(G) \le d_1 G + r + \theta = \varrho + r + \theta = 3 + 1 + 1 = 5$, whereas the misprinted formula [54, Thm. 6, (18′)] yields the contradiction $5 = d_2 G \le d_1 G + 1 = \varrho + 1 = 3 + 1 = 4$.

In contrast, no violation could be found in our previous joint paper [5], where the possible 2-class tower groups

$$G = \mathrm{G}_2^\infty k \in \{\langle 64, 180\rangle, \langle 128, 985|986\rangle, \langle 256, 6720|6721\rangle, \ldots\},$$

visualized in the diagram [5, Fig. 5, p. 1208], all have relation rank $d_2G = 4$ only.

6.2. **Eisenstein fields.** However, another series of violations showed up in our joint paper [7] on bicyclic biquadratic fields $k = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$ with squarefree radicand $d > 1$ and 3-class group Cl_3k of type (1^2), which also have torsionfree Dirichlet unit rank $r = 1$, but 3-class rank $\varrho = 2$ only. Due to the inclusion of $\sqrt{-3} \in k$, the fields contain the third roots of unity, and the invariant θ takes the value 1, with respect to the odd prime $p = 3$.

In § 7, Example 7.2 of [7], we have seen that among the 930 suitable values of the radicand in the range $0 < d < 50\,000$, there occur 197 ($\approx 21.2\%$, e.g. $d = 469$) with second 3-class group $\mathfrak{M} = \mathrm{G}_3^2k \simeq \langle 81, 9\rangle$ and 42 ($\approx 4.5\%$, e.g. $d = 7\,453$) with $\mathfrak{M} \simeq \langle 729, 95\rangle$. These 3-groups are of coclass $\mathrm{cc}(\mathfrak{M}) = 1$ and have relation rank $d_2\mathfrak{M} = 4$, as a computation by means of MAGMA [34] shows. Since their cover $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}\}$ is trivial, which means that there does not exist a finite non-metabelian 3-group H of derived length $\mathrm{dl}(H) \geq 3$ such that $\mathfrak{M}$ is isomorphic to the second derived quotient H/H'', the second 3-class group $\mathfrak{M}$ must coincide with the 3-class tower group $G = \mathrm{G}_3^\infty k$ already.

The misprinted original version of the Theorem by Shafarevich [54] (Teorema 6, p. 83, in the Russian original, resp. Theorem 6, formula (18′), p. 140, in the English translation) enforces the relation rank $d_2\mathfrak{M} = d_2G \leq d_1G + 1 = \varrho + 1 = 2 + 1 = 3$, which is obviously a contradiction to our result $d_2\mathfrak{M} = 4$ for more than a quarter (25.7%) of all fields k under investigation.

Fortunately, our corrected Formula (1.1) in Theorem 1.3, $d_2(G) \leq d_1G + r + \theta = \varrho + r + \theta = 2 + 1 + 1 = 4$, is in accordance with the fact that these metabelian 3-groups have the maximal admissible relation rank $d_2\mathfrak{M} = 4$.

Our Theorem 7.1 in [7] is the first indication of the misprint in [54] for an odd prime $p = 3$.

Since all second 3-class groups $\mathfrak{M} = \mathrm{G}_3^2k$ in Theorem 8.3 and Theorem 8.7 of [7] have relation rank $d_2\mathfrak{M} = 5$, they cannot coincide with the 3-tower group $G = \mathrm{G}_3^\infty k$, and the corresponding 3-class field tower must have length ℓ_3k at least 3 whenever the coclass is $\mathrm{cc}(\mathfrak{M}) \geq 2$.

7. Two-stage towers of p-class fields

The p-groups in the stem of Hall's isoclinism family Φ_6 [27] are two-generated metabelian groups $G = \langle x, y\rangle$ of order $|G| = p^5$, nilpotency class $\mathrm{cl}(G) = 3$ and coclass $\mathrm{cc}(G) = 2$. They do not exist for $p = 2$, but for odd prime numbers $p \geq 3$ they uniformly arise as descendants of step size 2 of the extra special group $G_0^3(0, 0)$ of order p^3 and exponent p [36, Tbl. 1, p. 483] and thus form top vertices of the coclass graph $\mathcal{G}(p, 2)$. The reason for this behaviour is the nuclear rank ν of $G_0^3(0, 0)$, which is only $\nu = 1$ for $p = 2$, but $\nu = 2$ for $p \geq 3$ giving rise to a bifurcation from $\mathcal{G}(p, 1)$ to $\mathcal{G}(p, 2)$.

The basic properties of the stem groups in Φ_6 have been discussed in [37, § 3.5, pp. 445–451], where we pointed out that the groups for $p = 3$ are irregular in the sense of Hall, but all groups for $p \geq 5$ are regular, since $\mathrm{cl}(G) = 3 < p$. In [39, pp. 1–10], we used commutator calculus for determining the transfer kernel type $\varkappa(G)$ in the regular case $p \geq 5$, for the first time. The regularity admits the simplification that all transfers uniformly map to pth powers. A summary of the group theoretic results and arithmetical applications is as follows.

Theorem 7.1. *(Artin pattern of metabelian 5-groups with order 5^5, coclass 2) The transfer kernel type (TKT) $\varkappa(G)$ of the 12 top vertices G with abelianization $G/G' \simeq (1^2)$ of the coclass graph $\mathcal{G}(5,2)$, which form the stem $\Phi_6(0)$ of 5-groups in Hall's isoclinism family Φ_6, is given by Table 3. A partial characterization is given by the counter η of fixed point transfer kernels $\kappa(i) = \mathrm{A}$, resp. abelianizations (transfer targets) $\tau(i)$ of type (1^3),*

$$\eta = \#\{1 \leq i \leq 6 \mid \kappa(i) = \mathrm{A}\} = \#\{1 \leq i \leq 6 \mid \tau(i) = (1^3)\}.$$

An asterisk after the SmallGroup identifier [11] denotes a Schur σ-group.

Proof. This is Theorem 2.2 in [39]. The characterization by the counter η is due to Theorem 3.8 and Table 3.3 of [37, § 3.2.5, pp. 423–424]. □

In Table 3, each 5-group is identified primarily with the symbol given by James [30], who used Hall's isoclinism families and regular type invariants [27], and Easterfield's characterization of maximal subgroups [21]. The property is an invariant characterization of the TKT, whereas the multiplet $\varkappa$ depends on the selection of generators and on the numeration of maximal subgroups.

TABLE 3. TKT of twelve 5-groups of order 5^5 in Φ_6

Identifier of the 5-group		Transfer kernel type (TKT)			
James	SmallGroup	η	$\varkappa$	Cycle pattern	Property
$\Phi_6(2^21)_a$	$\langle 3125, 14\rangle *$	6	(123456)	(1)(2)(3)(4)(5)(6)	identity
$\Phi_6(2^21)_{b_1}$	$\langle 3125, 11\rangle *$	2	(125364)	(1)(2)(3564)	4-cycle
$\Phi_6(2^21)_{b_2}$	$\langle 3125, 7\rangle$	2	(126543)	(1)(2)(36)(45)	two 2-cycles
$\Phi_6(2^21)_{c_1}$	$\langle 3125, 8\rangle *$	1	(612435)	(16532)(4)	5-cycle
$\Phi_6(2^21)_{c_2}$	$\langle 3125, 13\rangle *$	1	(612435)	(16532)(4)	5-cycle
$\Phi_6(2^21)_{d_0}$	$\langle 3125, 10\rangle$	0	(214365)	(12)(34)(56)	three 2-cycles
$\Phi_6(2^21)_{d_1}$	$\langle 3125, 12\rangle *$	0	(512643)	(154632)	6-cycle
$\Phi_6(2^21)_{d_2}$	$\langle 3125, 9\rangle *$	0	(312564)	(132)(456)	two 3-cycles
$\Phi_6(21^3)_a$	$\langle 3125, 4\rangle$	2	(022222)		nrl. const. with fp.
$\Phi_6(21^3)_{b_1}$	$\langle 3125, 5\rangle$	1	(011111)		nearly constant
$\Phi_6(21^3)_{b_2}$	$\langle 3125, 6\rangle$	1	(011111)		nearly constant
$\Phi_6(1^5)$	$\langle 3125, 3\rangle$	6	(000000)		constant

Theorem 7.2. *(Artin pattern of metabelian 7-groups with order 7^5, coclass 2)*

The transfer kernel type (TKT) $\varkappa(G)$ of the 14 top vertices G with abelianization $G/G' \simeq (1^2)$ of the coclass graph $\mathcal{G}(7,2)$, which form the stem $\Phi_6(0)$ of 7-groups in Hall's isoclinism family Φ_6, is given by Table 4. A partial characterization is given by the counter η of fixed point transfer kernels $\kappa(i) = \mathrm{A}$, resp. abelianizations (transfer targets) $\tau(i)$ of type (1^3),

$$\eta = \#\{1 \le i \le 8 \mid \kappa(i) = \mathrm{A}\} = \#\{1 \le i \le 8 \mid \tau(i) = (1^3)\}.$$

A star after the SmallGroup identifier [11] denotes a Schur σ-group.

Proof. This is Theorem 2.3 in [39]. The characterization by the counter η is due to Theorem 3.8 and Table 3.3 of [37, § 3.2.5, pp. 423–424] □

In Table 4, each 7-group is identified primarily with the symbol given by James [30], who used Hall's isoclinism families and regular type invariants [27], and Easterfield's characterization of maximal subgroups [21]. The property is an invariant characterization of the TKT, whereas the multiplet $\varkappa$ depends on the selection of generators and on the numeration of maximal subgroups.

TABLE 4. TKT of fourteen 7-groups of order 7^5 in Φ_6

Identifier of the 7-group		Transfer kernel type (TKT)			
James	SmallGroup	η	$\varkappa$	Cycle pattern	Property
$\Phi_6(2^21)_a$	$\langle 16807, 7\rangle *$	8	(12345678)	(1)(2)(3)(4)(5)(6)(7)(8)	identity
$\Phi_6(2^21)_{b_1}$	$\langle 16807, 11\rangle *$	2	(12753864)	(1)(2)(376845)	6-cycle
$\Phi_6(2^21)_{b_2}$	$\langle 16807, 12\rangle *$	2	(12637485)	(1)(2)(364)(578)	two 3-cycles
$\Phi_6(2^21)_{b_3}$	$\langle 16807, 13\rangle$	2	(12876543)	(1)(2)(38)(47)(56)	three transpos.
$\Phi_6(2^21)_{c_1}$	$\langle 16807, 9\rangle *$	1	(61247583)	(1657832)(4)	7-cycle
$\Phi_6(2^21)_{c_3}$	$\langle 16807, 8\rangle *$	1	(61247583)	(1657832)(4)	7-cycle
$\Phi_6(2^21)_{d_0}$	$\langle 16807, 10\rangle$	0	(21573846)	(12)(35)(47)(68)	four transpos.
$\Phi_6(2^21)_{d_1}$	$\langle 16807, 15\rangle *$	0	(41238756)	(1432)(5867)	two 4-cycles
$\Phi_6(2^21)_{d_2}$	$\langle 16807, 16\rangle *$	0	(81256347)	(18745632)	8-cycle
$\Phi_6(2^21)_{d_3}$	$\langle 16807, 14\rangle *$	0	(71283465)	(17648532)	8-cycle
$\Phi_6(21^3)_a$	$\langle 16807, 4\rangle$	2	(02222222)		nrl. const. with fp.
$\Phi_6(21^3)_{b_1}$	$\langle 16807, 6\rangle$	1	(01111111)		nearly constant
$\Phi_6(21^3)_{b_3}$	$\langle 16807, 5\rangle$	1	(01111111)		nearly constant
$\Phi_6(1^5)$	$\langle 16807, 3\rangle$	8	(00000000)		constant

For all isomorphism classes of p-groups in the stem $\Phi_6(0)$ of the isoclinism family Φ_6, some information on descendants and on the TKT can be provided independently of the prime $p \ge 5$.

Theorem 7.3. *(Descendants and Artin pattern of p-groups G with $|G| = p^5$, $\mathrm{dl}(G) = \mathrm{cc}(G) = 2$)*
The descendant tree $\mathcal{T}G$ and the transfer kernel type $\varkappa(G)$ of the p-groups G in the stem of Φ_6, that is, $p+7$ isomorphism classes of groups, can be described in the following uniform way, for any odd prime $p \ge 5$, where ν denotes the smallest positive quadratic non-residue modulo p.

(1) *The first* 4 *groups are infinitely capable vertices* G *of the coclass graph* $\mathcal{G}(p,2)$ *giving rise to infinite coclass trees* $\mathcal{T}^2G$ *of descendants. Their TKT* $\varkappa(G)$ *is nearly constant and contains at least one total transfer, indicated by a zero,* $\varkappa_i = 0$.

  (a) $\Phi_6(1^5)$ *has constant TKT* $(\overbrace{0,\ldots,0}^{p+1 \text{ times}})$*, entirely consisting of total transfers.*

  (b) $\Phi_6(21^3)_a$ *has nearly constant TKT* $(0,\overbrace{2,\ldots,2}^{p \text{ times}})$*, with fixed point* $\varkappa_2 = 2$.

  (c) $\Phi_6(21^3)_{b_r}$ *has nearly constant TKT* $(0,\overbrace{1,\ldots,1}^{p \text{ times}})$*, without fixed point, for* $r \in \{1,\nu\}$.

(2) *The next* 2 *groups are finitely capable vertices* G *of the coclass graph* $\mathcal{G}(p,2)$ *giving rise to finite trees of descendants within this graph. However, they give rise to infinitely many descendants spread over higher coclass graphs* $\mathcal{G}(p,r)$*,* $r \geq 3$*. Their TKT* $\varkappa(G)$ *is a permutation whose cycle pattern entirely consists of transpositions.*

  (a) $\Phi_6(2^21)_{b_{(p-1)/2}}$ *has TKT* $(1,2,\ldots)$ *with two fixed points, whose cycle pattern consists of* $\frac{p-1}{2}$ *transpositions.*

  (b) $\Phi_6(2^21)_{d_0}$ *has TKT* $(2,1,\ldots)$ *without fixed points, whose cycle pattern consists of* $\frac{p+1}{2}$ *transpositions, the first of them being* $(1,2)$.

(3) *The last* $p+1$ *groups are Schur* σ*-groups, in particular, they are terminal vertices of the coclass graph* $\mathcal{G}(p,2)$ *without descendants. Their TKT* $\varkappa(G)$ *is a permutation whose cycle pattern does not contain transpositions.*

  (a) $\Phi_6(2^21)_a$ *has TKT* $(1,2,\ldots,p+1)$*, the identity permutation with* $p+1$ *fixed points.*

  (b) $\Phi_6(2^21)_{b_r}$ *has TKT* $(1,2,\ldots)$ *with two fixed points, whose cycle pattern consists of cycles of length* $\frac{p-1}{\gcd(r,p-1)} > 2$*, for* $1 \leq r < \frac{p-1}{2}$.

  (c) $\Phi_6(2^21)_{c_r}$ *has TKT* $(6,1,2,4,\ldots)$ *with a single fixed point and* $\varkappa_6 = 5$*, for* $r \in \{1,\nu\}$.

  (d) $\Phi_6(2^21)_{d_r}$ *has TKT without fixed points, for* $1 \leq r \leq \frac{p-1}{2}$.

Proof. This is Theorem 2.1 in [39]. The TKT $\varkappa$ depends on the presentations given by James [30]. □

For $p+2$ isomorphism classes of stem groups in Φ_6, the TKT depends on number theoretic properties of the prime $p \geq 5$ and we have given explicit results for $p \in \{5,7\}$ in Theorem 7.1 and 7.2. For the other five, in (1)(a–c) and (3)(a), the TKT can be given uniformly for all p.

Now we come to the number theoretic harvest of Theorem 7.1, 7.2, and 7.3.

Theorem 7.4. *Let* K *be an arbitrary number field with* 5*-class group* Cl_5K *of type* (1^2)*, whose Artin pattern* $\mathrm{AP}(K) = (\tau(K),\varkappa(K))$ *is given by*

(1) *either* $\varkappa(K) = (1,2,3,4,5,6)$ *(identity) and* $\tau(K) = ((1^3)^6)$
(2) *or* $\varkappa(K) = (1,2,5,3,6,4)$ *(4-cycle) and* $\tau(K) = ((21)^4,(1^3)^2)$
(3) *or* $\varkappa(K) = (6,1,2,4,3,5)$ *(5-cycle) and* $\tau(K) = ((21)^5,(1^3))$

(4) *or* $\varkappa(K) = (5, 1, 2, 6, 4, 3)$ *(6-cycle) and* $\tau(K) = ((21)^6)$
(5) *or* $\varkappa(K) = (3, 1, 2, 5, 6, 4)$ *(two 3-cycles) and* $\tau(K) = ((21)^6)$.

Then K *has a* 5*-class field tower* $K < \mathrm{F}_5^1 K < \mathrm{F}_5^2 K = \mathrm{F}_5^\infty K$ *of exact length* $\ell_5 K = 2$.

Proof. According to Theorem 7.1 and Table 3, the five given alternatives for the transfer kernel type $\varkappa(K)$ of the number field K, which are permutations whose cycle decomposition does not contain 2-cycles, uniquely determine the Schur σ-groups $\langle 5^5, n\rangle$ with identifiers $n \in \{14, 11, 8, 13, 12, 9\}$, except for the ambiguity of the 5-cycle with two possibilities $n \in \{8, 13\}$, provided the second 5-class group $\mathfrak{M} = \mathrm{G}_5^2 K$ of K belongs to the stem of Φ_6. The latter condition is ensured by the additional assignment of the transfer target type $\tau(K)$, all of whose components are of logarithmic order $\mathrm{lo}_5 \tau_i = 3$, for $1 \le i \le 6$.

It remains to show that the metabelian Schur σ-group $\mathfrak{M}$ cannot be isomorphic to the second derived quotient G/G'' of a non-metabelian 5-group G, that is, the cover $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}\}$ is trivial. This is a consequence of [16, Lem. 4.10, p. 273], which shows that an epimorphism $G \to G/G'' \simeq \mathfrak{M}$ onto the balanced group $\mathfrak{M}$ is an isomorphism $G \simeq \mathfrak{M}$. Therefore, we have $\mathrm{G}_5^\infty K \simeq \mathrm{G}_5^2 K$ and thus $\ell_5 K = 2$. □

Theorem 7.5. *Let* K *be an arbitrary number field with* 7*-class group* $\mathrm{Cl}_7 K$ *of type* (1^2)*, whose Artin pattern* $\mathrm{AP}(K) = (\tau(K), \varkappa(K))$ *is given by*

(1) *either* $\varkappa(K) = (1, 2, 3, 4, 5, 6, 7, 8)$ *(identity) and* $\tau(K) = ((1^3)^8)$
(2) *or* $\varkappa(K) = (1, 2, 7, 5, 3, 8, 6, 4)$ *(6-cycle) and* $\tau(K) = ((12)^6, (1^3)^2)$
(3) *or* $\varkappa(K) = (1, 2, 6, 3, 7, 4, 8, 5)$ *(two 3-cycles) and* $\tau(K) = ((12)^6, (1^3)^2)$
(4) *or* $\varkappa(K) = (6, 1, 2, 4, 7, 5, 8, 3)$ *(7-cycle) and* $\tau(K) = ((12)^7, (1^3))$
(5) *or* $\varkappa(K) = (8, 1, 2, 5, 6, 3, 4, 7)$ *(8-cycle) and* $\tau(K) = ((12)^8)$
(6) *or* $\varkappa(K) = (4, 1, 2, 3, 8, 7, 5, 6)$ *(two 4-cycles) and* $\tau(K) = ((12)^8)$.

Then K *has a* 7*-class field tower* $K < \mathrm{F}_7^1 K < \mathrm{F}_7^2 K = \mathrm{F}_7^\infty K$ *of exact length* $\ell_7 K = 2$.

Proof. According to Theorem 7.2 and Table 4, the six given alternatives for the transfer kernel type $\varkappa(K)$ of the number field K, which are permutations whose cycle decomposition does not contain 2-cycles, uniquely determine the Schur σ-groups $\langle 7^5, n\rangle$ with identifiers $n \in \{7, 11, 12, 9, 8, 15, 16, 14\}$, except for the ambiguity of the 7-cycle with two possibilities $n \in \{9, 8\}$ and of the 8-cycle with two possibilities $n \in \{16, 14\}$, provided the second 7-class group $\mathfrak{M} = \mathrm{G}_7^2 K$ of K belongs to the stem of Φ_6. The latter condition is ensured by the additional assignment of the transfer target type $\tau(K)$, all of whose components are of logarithmic order $\mathrm{lo}_7 \tau_i = 3$, for $1 \le i \le 8$.

Similarly as in the proof of Theorem 7.4, we have $\mathrm{G}_7^\infty K \simeq \mathrm{G}_7^2 K$ and thus $\ell_7 K = 2$. □

Unfortunately, the TTT alone does not permit a decision about $\ell_p K$ with the aid of Theorem 7.4 or 7.5, in general. The reason is that the groups $\langle 3125, 7\rangle$ and $\langle 3125, 10\rangle$ resp. $\langle 16807, 10\rangle$ and $\langle 16807, 13\rangle$, must be identified by means of their TKT. However, an exception where the TTT suffices is given in the following Corollary, concerning $\Phi_6(2^2 1)_{c_r}$ for $r \in \{1, \nu\}$ (Theorem 7.3).

Corollary 7.6. *(Criterion for a two-stage p-class tower in terms of the TTT) Let $p \geq 5$ be a prime and let K be an arbitrary number field with p-class group $\mathrm{Cl}_p K$ of type (1^2). If the TTT of K is given by $\tau(K) = ((21)^p, (1^3))$, or equivalently, if $\eta = 1$, then $\ell_p K = 2$.*

Example 7.7. We succeeded in realizing all the metabelian Schur σ-groups in Theorem 7.4 and 7.5, with the single exception of $\langle 16807, 7\rangle$, by second p-class groups $\mathfrak{M} = \mathrm{G}_p^2 K$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with p-class tower length $\ell_p K = 2$, whose absolute discriminants $|d|$ are given in tree diagrams of the top region of the coclass graph $\mathcal{G}(5, 2)$ in Figure 14 and of $\mathcal{G}(7, 2)$ in Figure 15 of the Wikipedia article on the Artin transfer (group theory) (`https://en.wikipedia.org/wiki/Artin_transfer_(group_theory)`).

Remark 7.8. We did not touch upon 3-groups in the stem of Φ_6. We only remarked that they are irregular in the sense of Hall, which causes anomalies in the analogue of Theorem 7.3 for $p = 3$: among the 3-groups in $\Phi_6(0)$, there are, firstly, only 3 instead of 4 infinitely capable vertices of $\mathcal{G}(3, 2)$, namely $\langle 243, n\rangle$ with $n \in \{3, 6, 8\}$, and secondly, only 2 instead of 4 terminal Schur σ-groups, namely $\langle 243, n\rangle$ with $n \in \{5, 7\}$. Similarly as in Theorem 7.3, there are 2 finitely capable vertices of $\mathcal{G}(3, 2)$, namely $\langle 243, n\rangle$ with $n \in \{4, 9\}$. An analogue of Theorem 7.4 and 7.5 for the Schur σ-groups $\langle 243, n\rangle$ with $n \in \{5, 7\}$ has been proved in [37, Thm. 1.5, p. 407]. It confirms results by Scholz and Taussky [53] with a short and elegant argumentation.

8. Infinite 3-class towers

In the final section § 7 of [41], we proved that the second 3-class groups $\mathfrak{M} = \mathrm{G}_3^2 K$ of the 14 complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-10^7 < d < 0$ and 3-class group $\mathrm{Cl}_3 K$ of type (1^3) are pairwise non-isomorphic [41, Thm. 7.1, p. 307]. All these fields have an infinite 3-class tower with $\ell_3 K = \infty$, according to Koch and Venkov [31]. For the proof of this theorem in [41, § 7.3, p. 311], the IPADs of the 14 fields were insufficient, since three critical fields with discriminants

$$d \in \{-4\,447\,704, -5\,067\,967, -8\,992\,363\}$$

share the common accumulated (unordered) IPAD

$$\tau^{(1)} K = [\tau_0 K; \tau_1 K] = [1^3; (32^2 1; (21^4)^5, (2^2 1^2)^7)].$$

To complete the proof we had to use information on the occupation numbers of the accumulated (unordered) IPODs,
$\varkappa_1 K = [1, 2, 6, (8)^6, 9, (10)^2, 13]$ with maximal occupation number 6
for $d = -4\,447\,704$,
$\varkappa_1 K = [1, 2, (3)^2, (4)^2, 6, (7)^2, 8, (9)^2, 12]$ with maximal occupation number 2
for $d = -5\,067\,967$,
$\varkappa_1 K = [(2)^2, 5, 6, 7, (9)^2, (10)^3, (12)^3]$ with maximal occupation number 3
for $d = -8\,992\,363$.

In [45], we succeeded in computing the *second layer* of the transfer target type, $\tau_2 K$, for the critical fields by determining the structure of the 3-class groups

Cl_3M of the 13 unramified bicyclic bicubic extensions $M|K$ with relative degree $[M:K]=3^2$ and absolute degree 18 with the aid of the computational algebra system MAGMA [34]. In accumulated (unordered) form, the second layer of the TTTs is given by
$\tau_2K=[32^51^2;4321^5;2^51^3,(3^221^5)^2;2^41^4,32^21^5;(2^21^7)^3,(2^31^5)^3]$ for $d=-4\,447\,704$,
$\tau_2K=[3^22^21^4;(3^221^5)^3;32^21^5;(2^31^5)^8]$ for $d=-5\,067\,967$, and
$\tau_2K=[32^21^6,(3^221^5)^3;2^41^4,32^21^5;2^21^7,(2^31^5)^6]$ for $d=-8\,992\,363$.

These results admit incredibly powerful conclusions, which bring us closer to the ultimate goal of determining the precise isomorphism type of $\mathfrak{M}=\mathrm{G}_3^2K$. Firstly, they clearly show that the second 3-class groups of the critical fields are pairwise non-isomorphic, without using the IPODs. Secondly, the component with the biggest order establishes an impressively sharpened estimate for the order of $\mathfrak{M}=\mathrm{G}_3^2K$ from below.

Theorem 8.1. *(Fine estimate of the order $|\mathfrak{M}|$)*
None among the maximal subgroups of the second 3-class group $\mathfrak{M}=\mathrm{G}_3^2K$ for the critical complex quadratic fields $K=\mathbb{Q}(\sqrt{d})$, with
$d\in\{-4\,447\,704,-5\,067\,967,-8\,992\,363\}$, can be abelian.
The logarithmic order of $\mathfrak{M}$ is bounded from below by
$\mathrm{lo}_3\mathfrak{M}\ge 17$ *for* $d=-4\,447\,704$,
$\mathrm{lo}_3\mathfrak{M}\ge 16$ *for* $d=-5\,067\,967$,
$\mathrm{lo}_3\mathfrak{M}\ge 15$ *for* $d=-8\,992\,363$.

Proof. This is Theorem 6.2 in [45]. □

Example 8.2. More recently, it came to our knowledge that Leshin [32] has proved the infinitude $\ell_3N=\infty$ of the 3-class tower of the sextic S_3-field $N=\mathbb{Q}(\sqrt{-3},\sqrt[3]{D})$ with radicand $D=79\cdot 97=7\,663\equiv 4 \pmod 9$. This is the normal closure of the pure cubic field $L=\mathbb{Q}(\sqrt[3]{D})$ with three ramified primes $3,79,97$, the last two of them congruent to 1 modulo 3 and thus split in $\mathbb{Q}(\sqrt{-3})$.

However, the 3-class group Cl_3N is of type (1^5) with 3-class rank $\varrho=5$ and thus gives rise to 121 unramified cyclic cubic extensions. Thus, although we have seen that the search for the second 3-class group G_3^2K of $K=\mathbb{Q}(\sqrt{-4\,447\,704})$ with $\varrho=3$ is very tough already, it still seems to be more promising than the corresponding search for G_3^2N.

Finally, we remark that the pure cubic field $L=\mathbb{Q}(\sqrt[3]{7\,663})$ is of the type with a relative principal factorization in $N|\mathbb{Q}(\sqrt{-3})$ in the sense of Barrucand and Cohn [8], since the 3-class group Cl_3L is of type (1^3) and the class number formula $3^5=h_3N=\frac{u}{3}\cdot h_3L^2=\frac{u}{3}\cdot(3^3)^2=u\cdot 3^5$ yields the index $u=1$ of the subfield units in N.

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References

1. E. Artin, *Beweis des allgemeinen Reziprozitätsgesetzes*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 353–363, DOI 10.1007/BF02952531.
2. E. Artin, *Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 46–51, DOI 10.1007/BF02941159.
3. J. A. Ascione, G. Havas and C. R. Leedham-Green, *A computer aided classification of certain groups of prime power order*, Bull. Austral. Math. Soc. **17** (1977), 257–274, Corrigendum 317–319, Microfiche Supplement p. 320, DOI 10.1017/s0004972700010467.
4. J. A. Ascione, *On* 3*-groups of second maximal class*, Ph. D. Thesis, Australian National University, Canberra, 1979.
5. A. Azizi, A. Zekhnini, M. Taous and D. C. Mayer, *Principalization of* 2*-class groups of type* $(2,2,2)$ *of biquadratic fields* $\mathbb{Q}\left(\sqrt{p_1p_2q},\sqrt{-1}\right)$, Int. J. Number Theory **11** (2015), no. 4, 1177 – 1216, DOI 10.1142/S1793042115500645.
6. A. Azizi, A. Zekhnini and M. Taous, *Coclass of* $\mathrm{Gal}(k_2^{(2)}|k)$ *for some fields* $k = \mathbb{Q}\left(\sqrt{p_1p_2q},\sqrt{-1}\right)$ *with* 2*-class groups of type* $(2,2,2)$, J. Algebra Appl. **15** (2015), no. 2, DOI 10.1142/S0219498816500274.
7. A. Azizi, M. Talbi, M. Talbi, A. Derhem and D. C. Mayer, *The group* $\mathrm{Gal}(k_3^{(2)}|k)$ *for* $k = \mathbb{Q}\left(\sqrt{-3},\sqrt{d}\right)$ *of type* $(3,3)$, Int. J. Number Theory (2016), DOI 10.1142/S1793042116501207.
8. P. Barrucand and H. Cohn, *Remarks on principal factors in a relative cubic field*, J. Number Theory **3** (1971), no. 2, 226–239.
9. L. Bartholdi and M. R. Bush, *Maximal unramified* 3*-extensions of imaginary quadratic fields and* $\mathrm{SL}_2\mathbb{Z}_3$, J. Number Theory **124** (2007), 159–166.
10. H. U. Besche, B. Eick and E. A. O'Brien, *A millennium project: constructing small groups*, Int. J. Algebra Comput. **12** (2002), 623-644, DOI 10.1142/s0218196702001115.
11. H. U. Besche, B. Eick and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.
12. N. Blackburn, *On prime-power groups in which the derived group has two generators*, Proc. Camb. Phil. Soc. **53** (1957), 19–27
13. W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.
14. W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), *Handbook of Magma functions* (Edition 2.22, Sydney, 2016).
15. N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of imaginary quadratic fields*, to appear in Math. Annalen, 2016. (arXiv: 1111.4679v2 [math.NT] 10 Dec 2014.)
16. N. Boston and J. Ellenberg, *Random pro-p groups, braid groups, and random tame Galois groups*, Groups Geom. Dyn. **5** (2011), 265–280.
17. N. Boston and C. Leedham-Green, *Explicit computation of Galois p-groups unramified at p*, J. Algebra **256** (2002), 402–413.
18. N. Boston and H. Nover, *Computing pro-p Galois groups*, Proceedings of ANTS 2006, Lecture Notes in Computer Science **4076**, 1–10, Springer-Verlag Berlin Heidelberg, 2006.
19. M. R. Bush, *Computation of Galois groups associated to the* 2*-class towers of some quadratic fields*, J. Number Theory **100** (2003), 313–325.
20. M. R. Bush and D. C. Mayer, 3*-class field towers of exact length* 3, J. Number Theory **147** (2015), 766–777, DOI 10.1016/j.jnt.2014.08.010.
21. T. E. Easterfield, *A classification of groups of order* p^6, Ph.D. Thesis, Univ. of Cambridge, 1940.
22. C. Fieker, *Computing class fields via the Artin map*, Math. Comp. **70** (2001), no. 235, 1293–1303.
23. Ph. Furtwängler, *Beweis des Hauptidealsatzes für die Klassenkörper algebraischer Zahlkörper*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 14–36.

24. G. Gamble, W. Nickel and E. A. O'Brien, *ANU p-Quotient — p-Quotient and p-Group Generation Algorithms*, 2006, an accepted GAP package, available also in MAGMA.
25. The GAP Group, *GAP – Groups, Algorithms, and Programming — a System for Computational Discrete Algebra*, Version 4.8.4, Aachen, Braunschweig, Fort Collins, St. Andrews, 2016, (`http://www.gap-system.org`).
26. E. S. Golod and I. R. Shafarevich, *On the class field tower* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272. (English transl. in Amer. Math. Soc. Transl. (2) **48** (1965), 91–102.)
27. P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.
28. H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. Teil II: Reziprozitätsgesetz*, Jber. der DMV **6** (1930), 1–204
29. D. F. Holt, B. Eick and E. A. O'Brien, *Handbook of computational group theory*, Discrete mathematics and its applications, Chapman and Hall/CRC Press, 2005.
30. R. James, *The groups of order p^6 (p an odd prime)*, Math. Comp. **34** (1980), no. 150, 613–637.
31. H. Koch und B. B. Venkov, *Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers*, Astérisque **24–25** (1975), 57–67.
32. J. Leshin, *On infinite class field towers ramified at three primes*, New York J. Math. **20** (2014), 27–33.
33. C. McLeman, *p-Tower groups over quadratic imaginary number fields*, Ann. Sci. Math. Québec **32** (2008), no. 2, 199–209.
34. The MAGMA Group, *MAGMA Computational Algebra System*, Version 2.22-3, Sydney, 2016, (`http://magma.maths.usyd.edu.au`).
35. D. C. Mayer, *The second p-class group of a number field*, Int. J. Number Theory **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.
36. D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.
37. D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, Journal de Théorie des Nombres de Bordeaux **25** (2013), no. 2, 401–456, DOI 10.5802/jtnb.842.
38. D. C. Mayer, *Principalization algorithm via class group structure*, J. Théor. Nombres Bordeaux **26** (2014), no. 2, 415–464, DOI 10.5802/jtnb.874.
39. D. C. Mayer, *Two-stage towers of p-class fields for $p \geq 5$*, Preprint, 2014. (`https://www.researchgate.net/publication/267099298_Two-stage_towers_of_p-class_fields_for_p5`).
40. D. C. Mayer, *Periodic bifurcations in descendant trees of finite p-groups*, Adv. Pure Math. **5** (2015), no. 4, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015.
41. D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015) no. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015.
42. D. C. Mayer, *Periodic sequences of p-class tower groups*, J. Appl. Math. Phys. **3** (2015), no. 7, 746–756, DOI 10.4236/jamp.2015.37090.
43. D. C. Mayer, *New number fields with known p-class tower*, Tatra Mountains Math. Publ. **64** (2015), 21 – 57, DOI 10.1515/tmmp-2015-0040, Special Issue on Number Theory and Cryptology '15.
44. D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66 – 104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.
45. D. C. Mayer, *Index-p abelianization data of p-class tower groups, II*, submitted to Publ. Math. Debrecen. (arXiv: 1601.00179v1 [math.NT] 02 Jan 2016.)
46. D. C. Mayer, *Three-stage towers of 5-class fields*, submitted to J. Number Theory. (arXiv: 1604.06930v1 [math.NT] 23 Apr 2016.)

47. D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Phys. Math. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135.
48. D. C. Mayer and M. F. Newman, *Finite* 3*-groups with transfer kernel type* F, in preparation.
49. K. Miyake, *Algebraic investigations of Hilbert's Theorem* 94*, the principal ideal theorem and the capitulation problem*, Expo. Math. **7** (1989), 289–346.
50. M. F. Newman, *Determination of groups of prime-power order*, pp. 73–84, in: Group Theory, Canberra, 1975, Lecture Notes in Math., vol. **573**, Springer, Berlin, 1977.
51. H. Nover, *Computation of Galois groups associated to the* 2*-class towers of some imaginary quadratic fields with* 2*-class group* $C_2 \times C_2 \times C_2$, J. Number Theory **129** (2009), 231–245.
52. E. A. O'Brien, *The p-group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698, DOI 10.1016/S0747-7171(80)80082-X.
53. A. Scholz und O. Taussky, *Die Hauptideale der kubischen Klassenkörper imaginär quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluß auf den Klassenkörperturm*, J. Reine Angew. Math. **171** (1934), 19–41.
54. I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

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