

APPROXIMATION BY NEW BASKAKOV OPERATORS

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ABSTRACT. In this paper, we discuss a uniform convergence and the error estimate for the second-order Baskakov operator with the maximal function of the Lipschitz type. A new type of Baskakov-type operator is constructed. We present some results on the convergence rate and Voronovskaja Theorem of the new operator. Finally, we give some numerical examples to demonstrate how three different Baskakov operators converge.

1. INTRODUCTION

In 1957, Baskakov [2] introduced a sequence of positive linear operators, called Baskakov operators as follows

$$B_s(g; u) = \sum_{j=0}^{\infty} P_{s,j}(u) g\left(\frac{j}{s}\right), \quad P_{s,j}(u) = \binom{s+j-1}{j} \frac{u^j}{(1+u)^{s+j}}. \quad (1.1)$$

A non-negative real parametric generalization of the above sequences was proposed by Aral and Erbay [1] and it is known as the α -Baskakov operator $L_{s,a}(g; u)$. Cheng [3] created a new modification of Baskakov operators $V_s(g; u)$ based on the second central moment of the classical Baskakov operators. Several researchers applied generalizations to the Baskakov operators. A few of the results are shown in the appropriate direction in ([4, 6, 8, 9]). In a recent paper, Jabbar and Hassan [5] introduced two new types of modified Baskakov operators as follows

$$B_s^1(g; u) = \sum_{j=0}^{\infty} P_{s,j}^1(u) g\left(\frac{j}{s}\right), \quad (1.2)$$

$$P_{s,j}^1(u) = \psi(u, s) P_{s+1,j}(u) + \psi(-1-u, s) P_{s+1,j-1}(u),$$

$$\psi(u, s) = a(s) + b(s)u,$$

for any $g \in C_\gamma(I) = \{g \in C(I) : |g(u)| \leq M(1+u^\gamma), \text{ for some } \gamma > 0\}$ where $I = [0, \infty)$ and M is a positive constant dependent on f .

$$B_s^2(g; u) = \sum_{j=0}^{\infty} P_{s,j}^2(u) g\left(\frac{j}{s}\right), \quad (1.3)$$

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$$P_{s,j}^2(u) = A(u, s)P_{s+2,j}(u) + F(u, s)P_{s+2,j-1}(u) + A(-1-u, s)P_{s+2,j-2}(u),$$

$$A(u, s) = d_0(s) + d_1(s)u + d_2(s)u^2, \quad F(u, s) = d_3(u)u(1+u).$$

For the second modification of the Baskakov operators $B_s^2(g; u)$ the order of approximation $O\left(\frac{1}{s^2}\right)$ was obtained in [5]. However, this case lacks the positivity of the operators. The objective of this paper is to generalize the technique used in the paper [5] to obtain an approximation order $O\left(\frac{1}{s^3}\right)$.

2. RESULTS ON $B_s^2(g; u)$ OPERATOR

In this section, we prove uniform convergence and degree of approximation of the second order Baskakov operator.

Let $I_h = [0, v]$ be a compact subset of I and $C(I_h)$ be the set of all continuous functions on I_h such that

$$\|g\|_{C(I_h)} = \sup\{|g(u)| : u \in I_h\}.$$

Theorem 2.1. *Let $g \in C_\gamma^2(I)$. Then $\lim_{s \rightarrow \infty} B_s^2(g; u) = g(u)$, uniformly on I_n .*

Proof. Since

$$\begin{aligned} B_s^2(g; u) = & \sum_{j=0}^{\infty} \left[\left(1 + \left(1 + \frac{s}{2}\right)u - \frac{s}{2}u^2\right) P_{s+2,j}(u) + (su + su^2)P_{s+2,j-1}(u) \right. \\ & \left. - \left(u + \frac{s}{2}u + \frac{s}{2}u^2\right) P_{s+2,j-2}(u) \right] g\left(\frac{j}{s}\right). \end{aligned} \quad (2.1)$$

It is easy to check that,

$$\begin{aligned} B_s^2(g; u) = & \frac{-(su + su^2)}{2} \sum_{j=0}^{\infty} P_{s+2,j}(u) \left(g\left(\frac{j}{s}\right) - 2g\left(\frac{j+1}{s}\right) + g\left(\frac{j+2}{s}\right) \right) \\ & + (1+u) \sum_{j=0}^{\infty} P_{s+2,j}(u) g\left(\frac{j}{s}\right) - u \sum_{j=0}^{\infty} P_{s+2,j}(u) g\left(\frac{j+2}{s}\right). \end{aligned} \quad (2.2)$$

Since $g \in C_\gamma^2(I)$, using the mean value theorem guarantees that there exist $\frac{j}{s} < M_1 < \frac{j+1}{s}$, $\frac{j+1}{s} < M_2 < \frac{j+2}{s}$, $M_1 < M_3 < M_2$ such that:

$$\begin{aligned} g\left(\frac{j+1}{s}\right) - g\left(\frac{j}{s}\right) &= \frac{1}{s}g'(M_1), \quad g\left(\frac{j+2}{s}\right) - g\left(\frac{j+1}{s}\right) = \frac{1}{s}g'(M_2), \\ g\left(\frac{j}{s}\right) - 2g\left(\frac{j+1}{s}\right) + g\left(\frac{j+2}{s}\right) &= \frac{M_2 - M_1}{s}g''(M_3), \quad 0 < M_2 - M_1 < \frac{2}{s}. \end{aligned}$$

Hence, if we set $M_4 = \max_{u \in [0, v]} |g''(u)|$, then for $\varepsilon > 0$ we have

$$\begin{aligned} |B_s^2(g; u) - g(u)| &\leq \frac{-u(1+u)}{s} M_4 + (1+u) \sum_{j=0}^{\infty} P_{s+2,j}(u) \left| g\left(\frac{j}{s}\right) - g(u) \right| \\ &\quad - u \sum_{j=0}^{\infty} P_{s+2,j}(u) \left| g\left(\frac{j+2}{s}\right) - g(u) \right| \leq \varepsilon. \end{aligned}$$

Thus, the proof is complete. $\square$

Lenze [7] provides the Lipschitz type maximum function as

$$\omega_\varepsilon(g, u) = \sup_{\substack{z \neq u, \\ z, u \in I}} \frac{|g(z) - g(u)|}{|z - u|^\varepsilon}, \quad u \in I \text{ and } \varepsilon \in (0, 1]. \quad (2.3)$$

The following theorem gives us a local direct approximation for the operators described in (1.3).

Theorem 2.2. *Let $g \in C_\gamma(I)$ and $\varepsilon \in (0, 1]$. Then, for any $u \in I$, we have*

$$|B_s^2(g; u) - g(u)| \leq \omega_\varepsilon(g, u) \left(\frac{-2u(1+u)}{s^2} \right)^{\frac{\varepsilon}{2}}.$$

Proof. We know that

$$|B_s^2(g; u) - g(u)| \leq B_s^2(|g(t) - g(u)|; u).$$

Thus, from equality (2.3), we obtain

$$|B_s^2(g; u) - g(u)| \leq \omega_\varepsilon(g, u) B_s^2(|t - u|^\varepsilon; u).$$

Using the Holder's inequality for $p = \frac{2}{\varepsilon}$ and $q = \frac{2}{2-\varepsilon}$, we get

$$|B_s^2(g; u) - g(u)| \leq \omega_\varepsilon(g, u) (B_s^2((t - u)^2; u))^{\frac{\varepsilon}{2}},$$

which gives the desired result. $\square$

The modulus of continuity for $g \in C(I_n)$ and $\delta > 0$ is given by

$$w(g; \delta) = \sup\{|g(t) - g(u)| : t, u \in I_n \text{ and } |t - u| \leq \delta\}. \quad (2.4)$$

Theorem 2.3. *Let $g \in C(I_n)$. Then,*

$$|B_s^2(g; u) - g(u)| \leq E(u, s)(1 + u + u^2)w\left(g; \frac{1}{\sqrt{s}}\right), \quad (2.5)$$

where $E(u, s) = |A(u, s)| + |F(u, s)| + |A(-1 - u, s)|$.

Proof.

$$\begin{aligned} |B_s^2(g; u) - g(u)| &= \left| \sum_{j=0}^{\infty} P_{s,j}^2(u) g\left(\frac{j}{s}\right) - (A(u, s) + F(u, s) + A(-1 - u, s)) g(u) \right| \\ &\leq \sum_{j=0}^{\infty} A(u, s) P_{s+2,j}(u) \left| g\left(\frac{j}{s}\right) - g(u) \right| \\ &\quad + \sum_{j=0}^{\infty} F(u, s) P_{s+2,j-1}(u) \left| g\left(\frac{j}{s}\right) - g(u) \right| \\ &\quad + \sum_{j=0}^{\infty} A(-1 - u, s) P_{s+2,j-2}(u) \left| g\left(\frac{j}{s}\right) - g(u) \right|. \end{aligned}$$

Using the following well-known relationship for the first-order modulus of continuity $|g(t) - g(u)| \leq w(g; |t - u|) \leq (1 + \sqrt{s}|t - u|)w\left(g; \frac{1}{\sqrt{s}}\right)$, we obtain

$$\left|g\left(\frac{j}{s}\right) - g(u)\right| \leq w\left(g; \left|\frac{j}{s} - u\right|\right) \leq \left(1 + \sqrt{s}\left|\frac{j}{s} - u\right|\right) w\left(g; \frac{1}{\sqrt{s}}\right).$$

Hence,

$$\begin{aligned} |B_s^2(g; u) - g(u)| &\leq w\left(g; \frac{1}{\sqrt{s}}\right) \left[|A(u, s)| \left(1 + \sqrt{s} \sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j}(u)\right) \right. \\ &\quad + w\left(g; \frac{1}{\sqrt{s}}\right) \left[|F(u, s)| \left(1 + \sqrt{s} \sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j-1}(u)\right) \right. \\ &\quad \left. \left. + w\left(g; \frac{1}{\sqrt{s}}\right) \left[|A(-1 - u, s)| \left(1 + \sqrt{s} \sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j-2}(u)\right) \right] \right] \right]. \end{aligned} \quad (2.6)$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j}(u) &\leq \left(\sum_{j=0}^{\infty} \left(\frac{j}{s} - u\right)^2 P_{s+2,j}(u)\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} P_{s+2,j}(u)\right)^{\frac{1}{2}} \\ &= \left(\frac{(s+2)(s+3)u^2 + (s+2)u}{s^2} + \frac{2u^2(s+2)}{s^2}\right)^{\frac{1}{2}} \\ &= \left(\frac{2u(1+3u)}{s^2} + \frac{u(1+u)}{s}\right)^{\frac{1}{2}} \leq \frac{u(1+u)}{\sqrt{s}}. \end{aligned} \quad (2.7)$$

Similarly, we obtain

$$\sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j-1}(u) = \sum_{j=0}^{\infty} \left|\frac{j}{s} - u\right| P_{s+2,j-2}(u) \leq \frac{u(1+u)}{\sqrt{s}}. \quad (2.8)$$

Substituting (2.7) and (2.8) in (2.6), we arrive

$$|B_s^2(g; u) - g(u)| \leq E(u, s)(1 + u + u^2)w\left(g; \frac{1}{\sqrt{s}}\right).$$

Thus, the proof is complete. $\square$

3. THE BASKAKOV OPERATOR OF THE THIRD ORDER

We create a third-order approximation formula in this section. Let's look at the following operator adjustment.

$$B_s^3(g, u) = \sum_{j=0}^{\infty} P_{s,j}^3(u) g\left(\frac{j}{s}\right), \quad (3.1)$$

where

$$\begin{aligned} P_{s,j}^3(u) = & \phi(u, s)P_{s+4,j}(u) + \lambda(u, s)P_{s+4,j-1}(u) + \theta(u, s)P_{s+4,j-2}(u) \\ & + \lambda(-1-u, s)P_{s+4,j-3}(u) + \phi(-1-u, s)P_{s+4,j-4}(u), \end{aligned} \quad (3.2)$$

and moreover

$$\phi(u, s) = x_0(s) + x_1(s)u + x_2(s)u^2 + x_3(s)u^3 + x_4(s)u^4, \quad (3.3)$$

$$\lambda(u, s) = y_0(s) + y_1(s)u + y_2(s)u^2 + y_3(s)u^3 + y_4(s)u^4, \quad (3.4)$$

$$\theta(u, s) = z_0(s)(u + u^2)^2. \quad (3.5)$$

We note that $x_i(s), y_i(s), i = 0, \dots, 4$, and $z_0(s)$ are some unidentified sequences that are identified in this manner. Additionally, we note the fact that operator (3.1), with $x_0(s) = x_4(s) = 1, x_1(s) = x_3(s) = 4, x_2(s) = 6, y_0(s) = 0, y_1(s) = y_4(s) = -4, y_2(s) = y_3(s) = -12$ and $z_0(s) = 6$ reduces to the Baskakov operator (1.1).

In order to study the rate of convergence, we set $B_s^3(1; u) = 1$, and this yields

$$2(x_0(s) + y_0(s)) - (x_1(s) + y_1(s)) + (x_2(s) + y_2(s)) - (x_3(s) + y_3(s)) + (x_4(s) + y_4(s)) = 1, \quad (3.6)$$

$$2(x_2(s) + y_2(s)) - 3(x_3(s) + y_3(s)) + 4(x_4(s) + y_4(s)) = 0, \quad (3.7)$$

$$z_0(s) + 2(x_4(s) + y_4(s)) = 0. \quad (3.8)$$

To get $\lim_{s \rightarrow \infty} B_s^3(t; u) = u$, we consider the sequences $x_i(s)$ and $y_i(s), i = 0, 3, 4$ to verify the conditions

$$\lim_{s \rightarrow \infty} \frac{x_0(s)}{s} = 0 \text{ and } \lim_{s \rightarrow \infty} \frac{y_0(s)}{s} = 0, \quad 2(2x_4(s) + y_4(s)) - (2x_3(s) + y_3(s)) = 0. \quad (3.9)$$

In order to have $\lim_{s \rightarrow \infty} B_s^3(t^i; u) = u^i, i = 2, 3$ we consider the sequences $x_i(s)$ and $y_i(s), i = 1, 3, 4$ to verify the conditions

$$\lim_{s \rightarrow \infty} \frac{x_1(s)}{s^2} = 0, \lim_{s \rightarrow \infty} \frac{x_3(s)}{s^3} = 0 \text{ and } \lim_{s \rightarrow \infty} \frac{x_4(s)}{s^3} = 0, \quad 4x_4(s) + y_4(s) = 0.$$

We propose our analysis for the cases $x_0(s) = 1, y_0(s) = 0, x_1(s) = 4 - \frac{7}{12}s, x_3(s) = 4 - \frac{14}{3}s + \frac{1}{4}s^2$ and $x_4(s) = 1 - \frac{23}{12}s + \frac{1}{8}s^2$, therefore

$$x_2(s) = 6 - \frac{10}{3}s + \frac{1}{8}s^2, y_1(s) = s - 4, y_2(s) = -12 + \frac{31}{3}s - \frac{1}{2}s^2, \quad (3.10)$$

$$y_3(s) = -12 + 17s - \frac{3}{2}s^2 \text{ and } y_4(s) = -4 + \frac{23}{3}s - \frac{1}{2}s^2.$$

Lemma 3.1. *For the operator $B_s^3(g; u)$, there holds the estimates*

- (1) $B_s^3(1; u) = 1$,
- (2) $B_s^3(t; u) = u$,
- (3) $B_s^3(t^2; u) = u^2$,
- (4) $B_s^3(t^3; u) = u^3$,
- (5) $B_s^3(t^4; u) = u^4 + \frac{12u(1+u)}{s^3}$,
- (6) $B_s^3(t^5; u) = u^5 + \frac{20u^5 + 100u^4 + 140u^3 + 80u^2}{s^3} + \frac{96u^5 + 240u^4 - 240u^2 - 96u}{s^4}$,

$$\begin{aligned}
(7) \quad B_s^3(t^6; u) &= u^6 + \frac{255u^6+645u^5+705u^4+315u^3}{s^3} \\
&+ \frac{194u^6+870u^5+2660u^4+2910u^3+926u^2}{s^4} \\
&- \frac{2280u^6+6840u^5+4800u^4-1800u^3-2580u^2+540u}{s^5}, \\
(8) \quad B_s^3(t^7; u) &= \frac{1}{s^8} (s^7u^7 + 945s^4u^7 + 2415s^4u^6 + 4424s^3u^7 + 2415s^4u^5 + 14805s^3u^6 \\
&- 6300s^2u^7 + 945su^4 + 23345s^3u^5 - 14070s^2u^6 - 39600su^7 + 17955s^3u^4 \\
&+ 16730s^2u^5 - 138600su^6 + 4991s^3u^3 + 53900s^2u^4 - 152040su^7 + 36610s^2u^3 \\
&- 33600su^4 + 7210s^2u^2 + 37800su^3 + 21000su^2 + 2630su), \\
(9) \quad B_s^3(t^8; u) &= \frac{1}{s^9} (s^8u^8 + 2660s^5u^8 + 6860s^5u^7 + 24899s^4u^8 + 6580s^5u^6 + 81900s^4u^7 \\
&+ 41860s^3u^8 + 2380s^5u^5 + 114170s^4u^6 + 192640s^3u^7 - 222300s^2u^8 + 76860s^4u^5 \\
&+ 447650s^3u^6 - 730800s^2u^7 - 630000su^8 + 19691s^4u^4 + 521150s^3u^5 - 507500s^2u^6 \\
&- 2520000su^7 + 275730s^3u^4 + 534660s^2u^5 - 3570000su^6 + 51450s^3u^3 \\
&+ 848680s^2u^4 - 1890000su^5 + 362460s^2u^3 + 134400su^4 + 47440s^2u^2 + 478800su^3 \\
&+ 146412su^2 + 12012su).
\end{aligned}$$

Lemma 3.2. *The moments of the operator (3.1) are given by*

$$\begin{aligned}
(1) \quad B_s^3(t-u; u) &= B_s^3((t-u)^2; u) = B_s^3((t-u)^3; u) = 0, \\
(2) \quad B_s^3((t-u)^4; u) &= \frac{12u(1+u)}{s^3}, \\
(3) \quad B_s^3((t-u)^5; u) &= \frac{20u^2(2u^3+5u^2+4u+1)}{s^3} + \frac{24u(4+10u-10u^3-4u^3)}{s^4}, \\
(4) \quad B_s^3((t-u)^6; u) &= \frac{15(u+u^2)^3}{s^3} + \frac{5u(70u+294u^2+532u^3+462u^4+154u^5)}{s^4} \\
&+ \frac{5u(108+516u+360u^2-960u^3-1368u^4-456u^5)}{s^5}, \\
(5) \quad B_s^3((t-u)^8; u) &= \frac{7u(45u^7+180u^6+270u^5+180u^4+45u^3)}{s^4} \\
&+ \frac{7u(4060u^7+16240u^6+25630u^5+20050u^4+7870u^3+1270u^2)}{s^5} \\
&+ \frac{7u(13500u^7+54000u^6+101260u^5+114780u^4+78040u^3+27780u^2+3760u)}{s^6} \\
&+ \frac{-7u \times 10^4(9u^7+36u^6+51u^5+27u^4)+7u(19200u^3+68400u^2+20916u+1716)}{s^7}.
\end{aligned}$$

The asymptotic order of approximation of $B_s^3(g; u)$ to f as n approaches infinity is discussed in the following theorem.

Theorem 3.3. *Let $g^{(6)} \in C_\gamma(I)$. Then*

$$\lim_{s \rightarrow \infty} s^3(B_s^3(g; u) - g(u)) = \frac{u(1+u)}{2}g^{(4)}(u) + \frac{u^2(2u^3+5u^2+4u+1)}{6}g^{(5)}(u) + \frac{(u+u^2)^3}{48}g^{(6)}(u).$$

Proof. Applying the operator $B_s^3(g; u)$ to the Taylor's formula, we obtain

$$\begin{aligned}
B_s^3(g; u) - g(u) &= B_s^3((t-u); u)g'(u) + \frac{1}{2}B_s^3((t-u)^2; u)g''(u) \\
&+ \frac{1}{6}B_s^3((t-u)^3; u)g^{(3)}(u) + \frac{1}{24}B_s^3((t-u)^4; u)g^{(4)}(u) \\
&+ \frac{1}{120}B_s^3((t-u)^5; u)g^{(5)}(u) + \frac{1}{720}B_s^3((t-u)^6; u)g^{(6)}(u) \\
&+ B_s^3(0(t, u)(t-u)^6; u),
\end{aligned}$$

where $0(t, u) \rightarrow 0$ as $t \rightarrow u$.

From the Lemma (3.2), we have

$$\lim_{s \rightarrow \infty} s^3 (B_s^3(g; u) - g(u)) = \frac{u(1+u)}{2} g^{(4)}(u) + \frac{u^2(2u^3 + 5u^2 + 4u + 1)}{6} g^{(5)}(u) + \frac{(u+u^2)^3}{48} g^{(6)}(u) \\ + \lim_{s \rightarrow \infty} B_s^3(0(t, u)(t-u)^6; u).$$

Now, we want to show that $\lim_{s \rightarrow \infty} s^3 B_s^3(\theta(t, u)(t-u)^6; u) = 0$.

For given $\varepsilon > 0$ there is $\delta > 0$ such that if $|\frac{j}{s} - u| < \delta \rightarrow |\theta(t, u)| < \varepsilon$ and for $|\frac{j}{s} - u| \geq \delta$, then there is a constant $C > 0$ such that $|\theta(t, u)| < C$.

$$s^3 |B_s^3(\theta(t, u)(t-u)^6; u)| \leq s^3 \sum_{|\frac{j}{s} - u| < \delta} P_{s,j}^3(u) |\theta(t, u)| \left| \frac{j}{s} - u \right|^6 \\ + s^3 \sum_{|\frac{j}{s} - u| \geq \delta} P_{s,j}^3(u) |\theta(t, u)| \left| \frac{j}{s} - u \right|^6 \\ \leq n^3 \varepsilon \sum_{k=0}^{\infty} P_{s,j}^3(u) \left(\frac{j}{s} - u \right)^6 + \frac{n^3 C}{\delta^2} \sum_{k=0}^{\infty} P_{s,j}^3(u) \left(\frac{j}{s} - u \right)^8 \\ \leq s^3 \varepsilon O\left(\frac{1}{s^3}\right) + \frac{s^3 C}{\delta^2} O\left(\frac{1}{s^4}\right) = o(1).$$

The proof of this theorem is completed. $\square$

4. NUMERICAL RESULTS

The theoretical findings from the earlier sections will be analyzed in this section using numerical examples.

Example 4.1. $g_1(u) = \frac{\cos(2\pi u)}{e^u}$, $u \in [0, 3]$.

Example 4.2. $g_2(u) = \sin(4u)$, $u \in [0, 4]$.

Example 4.3. $g_3(u) = \sin(3\pi u) + 3 \sin\left(\frac{1}{3}\pi u\right)$, $u \in [0, 2]$.

Example 4.4. $g_4(u) = e^{5u}$, $u \in [0, 1]$.

Example 4.5. $g_5(u) = \cos(2u)$, $u \in [0, 6]$.

Example 4.6. $g_6(u) = (u+2)(u+3)$, $u \in [0, 1]$.

Example 4.7. $g_7(u) = \frac{\sin(2\pi u)}{e^u}$, $u \in [0, 3]$.

Remark 4.8. The behaviors of the approximation $B_s^1(g; u)$, $B_s^2(g; u)$ and $B_s^3(g; u)$ are illustrated in Figures 1, 2 and 3.

Remark 4.9. Figures 4 and 5 illustrate the convergence of operators $B_s^1(g; u)$, $B_s^2(g; u)$, $B_s^3(g; u)$ and $L_{n,a}(g; u)$ to functions $g_4(u)$ and $g_5(u)$, respectively.

Remark 4.10. Figures 4 and 5 illustrate the convergence of operators $B_s^1(g; u)$, $B_s^2(g; u)$, $B_s^3(g; u)$ and $V(g; u)$ to functions $g_6(u)$ and $g_7(u)$, respectively.

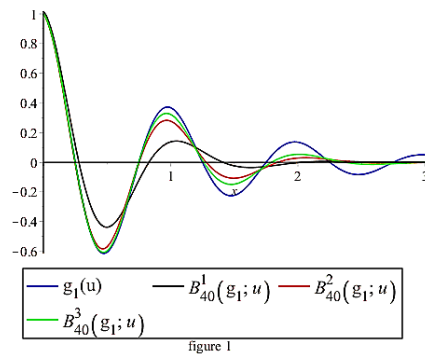


figure 1

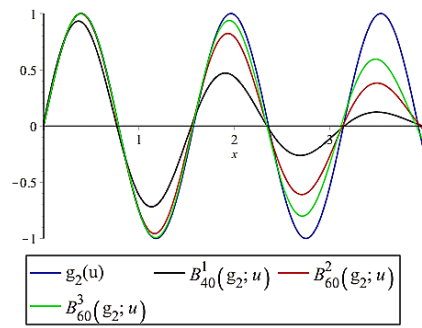


figure 2

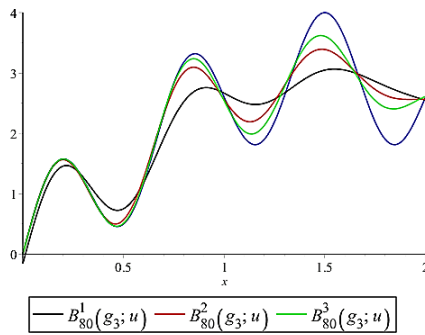


figure 3

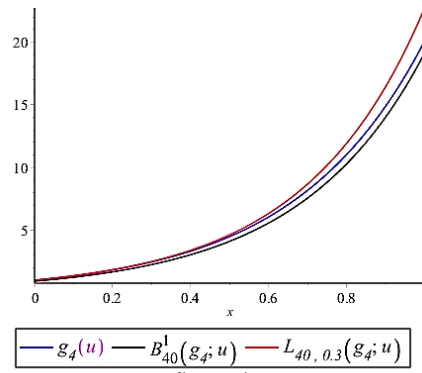


figure 4

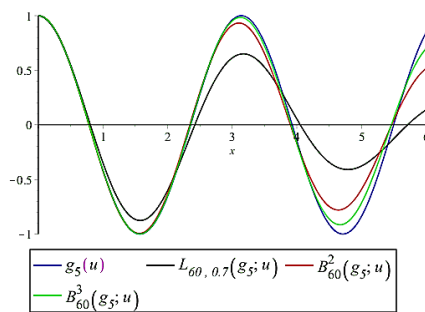


figure 5

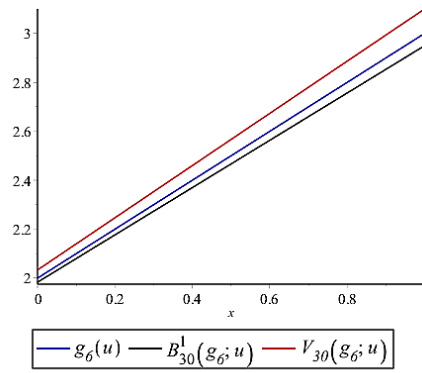


figure 6

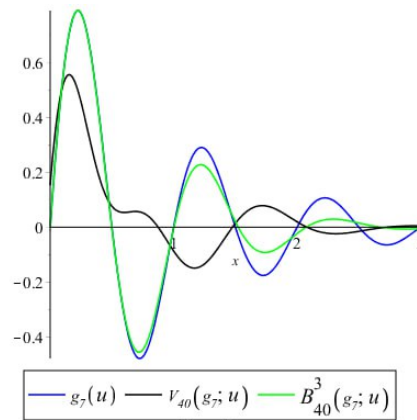


figure 7

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