Gulf Journal of Mathematics
Vol 19, Issue 2 (2025) 77-92

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v19i2.2692

MULTIPLICATION OPERATOR ON WEIGHTED LEBESGUE SEQUENCE SPACES

RENE ERLÍN CASTILLO[1], HÉCTOR CAMILO CHAPARRO[2] * AND OSCAR ALEJANDRO CHAPARRO[3]

Abstract. In this paper, we study the multiplication operator acting on the Lebesgue sequence space $\ell_{p,\omega}$, for $1 \leq p \leq \infty$, which generalizes the classical ℓ_p spaces by incorporating a weight sequence ω_n. We focus on properties such as continuity, inverse continuity, finite range, compactness and essential norm of the operator.

1. Introduction and preliminaries

Multiplication operators have been a fundamental object of study in functional analysis due to their broad range of applications in various function spaces, particularly in the study of Lebesgue and Bergman spaces. These operators have also been instrumental in advancing the theory of operators on Hilbert spaces, where they have been rigorously investigated over the past several decades. For a comprehensive treatment of these topics, we refer the reader to [1], [4], [9], and [11].

The study of multiplication operators has been extended to a variety of functional spaces, including L_p spaces [12], [16], Orlicz spaces [13], and Lorentz spaces [2], [8]. Furthermore, significant contributions have been made in the context of Lorentz sequence spaces $l_{(p,q)}$ [3] and Lorentz-Orlicz spaces [6]. Particularly relevant to our work are the investigations in Köthe spaces [7] and Köthe sequence spaces [15]. These studies underscore the versatility of multiplication operators as a tool for probing the structure and properties of various functional spaces, making them an active area of ongoing research.

Given the rich interplay between multiplication operators and the structure of functional spaces, it is both natural and compelling to extend its study to weighted Lebesgue sequence spaces. Weighted spaces arise naturally in many areas of analysis, including harmonic analysis, spectral theory, and the study of differential equations, where they provide a refined framework for understanding the behavior of operators under varying weight conditions. By investigating multiplication operators on weighted Lebesgue sequence spaces, we aim to uncover

Date: Received: Feb 11, 2025; Accepted: Mar 2, 2025.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 47B33, 43B38; Secondary 46E30.

Key words and phrases. Lebesgue sequence spaces, multiplication operator, bounded operator, compactness, essential norm.

new insights into their properties, such as boundedness, compactness and essential norm, thereby contributing to the broader understanding of operator theory in weighted settings.

We begin with some basic definitions.

Definition 1.1. We define a *weight sequence* $\omega = \omega_n$ to be a sequence such that $\omega_n > 0$ for all $n \in \mathbb{N}$ i.e., a positive sequence.

Definition 1.2. Let $1 \leq p < +\infty$. For a fixed p and a fixed weight sequence ω_n, we define the set

$$\ell_{p,\omega} := \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p \, \omega_n < \infty, x_n \in \mathbb{R}, \forall n \in \mathbb{N} \right\}$$

and its norm is given by

$$\|\mathbf{x}\|_{p,\omega} := \left(\sum_{n=1}^{\infty} |x_n|^p \, \omega_n \right)^{\frac{1}{p}}.$$

$\ell_{p,\omega}$ is called the *Weighted Lebesgue sequence space.*

By setting the sequence ω_n as the constant sequence $\omega_n = 1$ for all $n \in \mathbb{N}$, one recovers the classical Lebesgue ℓ_p sequence space. So, the $\ell_{p,\omega}$ space generalizes the ℓ_p space.

Remark 1.3. Note that for $a, b \in \mathbb{R}$, we have

$$|a + b|^p \leq 2^p(|a|^p + |b|^p).$$

Now, we use the inequality above to show that $l_{p,\omega}$ is closed under addition and scalar product. Indeed, let $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \ell_{p,\omega}$, then

$$\sum_{n=1}^{\infty} |\lambda x_n + y_n|^p \omega_n \leq 2^p \left(|\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \, \omega_n + \sum_{n=1}^{\infty} |y_n|^p \, \omega_n \right) < \infty.$$

Thus $\lambda\mathbf{x} + \mathbf{y} \in \ell_{p,\omega}$. Also, by the usual properties of $\mathbb{R}$, we conclude that $\ell_{p,\omega}$ is a *real vector space.* The Weighted Lebesgue sequence space is indeed a *separable Banach* space.

Remark 1.4. Let us define the sequence $\mathbf{e^{(n)}} = \{e_k^{(n)}\}_{k\in\mathbb{N}}$ as follows

$$e_k^{(n)} = \begin{cases} 1 \text{ if } k = n \\ 0 \text{ if } k \neq n. \end{cases}$$

We claim that $\mathbf{e^{(n)}} \in \ell_{p,\omega}$ for all weight sequences ω and $\mathbf{e^{(n)}} \cdot \mathbf{x} \in \ell_{p,\omega}$ for all $\mathbf{x} \in \ell_{p,\omega}$. In fact,

$$\sum_{k=1}^{\infty} |e_k^{(n)}|^p \omega_k = |e_n^{(n)}|^p \omega_n = 1^p \omega_n = \omega_n < \infty.$$

Therefore $\mathbf{e^{(n)}} \in \ell_{p,\omega}$ and $\mathbf{e^{(n)}} \cdot \mathbf{x} \in \ell_{p,\omega}$ for all $\mathbf{x} \in \ell_{p,\omega}$, since $\ell_{p,\omega}$ is a Banach space. Also,

$$\|\mathbf{e^{(n)}}\|_{p,\omega} = \left(\sum_{k=1}^{\infty} |e_k^{(n)}|^p \omega_k\right)^{1/p} = (\omega_n)^{1/p}. \tag{1.1}$$

The use of weights allows the convergence of some sequences which otherwise diverge. In other words, there exist sequences belonging to $\ell_{p,\omega}$ which do not belong to ℓ_p, as the following example shows.

Example 1.5. The sequence $a_n = \dfrac{1}{n^{1/p}}$ does not belong to ℓ_p, since

$$\sum_{n=1}^{\infty} |a_n|^p = \sum_{n=1}^{\infty} \left(\frac{1}{n^{1/p}}\right)^p = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

However, if we set the weight sequence $\omega_n = \dfrac{1}{n}$, then

$$\sum_{n=1}^{\infty} |a_n|^p \omega_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^{1/p}}\right)^p \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

so $a_n \in \ell_{p,\omega}$.

For a fixed weight, we would like to compare $\ell_{p,\omega}$ and $\ell_{q,\omega}$. The following example shows that this is not possible.

Example 1.6. Let $\omega = \omega_n = \dfrac{1}{n^3}$ and $a_n = n$. Then $a_n \in \ell_{1,\omega}$ but $a_n \notin \ell_{2,\omega}$.

If we set an additional condition on the weight ω_n, then it is possible to compare $\ell_{p,\omega}$ and $\ell_{q,\omega}$. The result is as follows.

Theorem 1.7. *Let $1 \leq p < q < \infty$. Then $\ell_{p,\omega} \subseteq \ell_{q,\omega}$ if and only if the weight sequence ω satisfies the condition*

$$\omega_n \geq 1 \quad (n \in \mathbb{N}).$$

We now define the multiplication operator acting on $\ell_{p,\omega}$ to itself.

Definition 1.8. Let $1 \leq p < \infty$ and $\mathbf{u} = (u_1, u_2, u_3, \ldots)$ a fixed sequence such that $\mathbf{u}\cdot\mathbf{x} \in \ell_{p,\omega}$ for all $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \ell_{p,\omega}$, where $\mathbf{u}\cdot\mathbf{x} = (u_1x_1, u_2x_2, u_3x_3, \ldots)$. Then the *multiplication operator induced by* $\mathbf{u}$ is defined as

$$\begin{aligned} M_{\mathbf{u}} : \ell_{p,\omega} &\to \ell_{p,\omega} \\ \mathbf{x} &\mapsto \mathbf{u} \cdot \mathbf{x}. \end{aligned}$$

As we stated in the Introduction, multiplication operators have been extensively studied, and it is our goal to extend its study to weighted sequence spaces. In the coming sections, we study properties such as boundedness, finite range, compactness and essential norm of this operator defined on $\ell_{p,\omega}$.

2. Continuity and continuous inverse

In this section, we characterize the boundedness of the multiplication operator $M_{\mathbf{u}}$, which is given by the boundedness of the symbol $\mathbf{u}$, as the next result shows.

Theorem 2.1. *Let $\mathbf{u}$ be a real sequence. The multiplication operator $M_{\mathbf{u}}$ is continuous on $\ell_{p,\omega}$ if and only if $\mathbf{u} \in \ell^\infty$. Moreover,*

$$\|M_{\mathbf{u}}\| = \|\mathbf{u}\|_\infty.$$

Proof. We prove first sufficiency, let $M_{\mathbf{u}}$ continuous multiplication operator on $\ell_{p,\omega}$ (therefore bounded), then there exists $K > 0$ such that

$$\|M_{\mathbf{u}}(\mathbf{x})\|_{p,\omega} \leq K\|\mathbf{x}\|_{p,\omega}$$

for all $\mathbf{x} \in \ell_{p,\omega}$. Let $n \in \mathbb{N}$ fixed, and define

$$\overline{\mathbf{e^{(n)}}} = \frac{\mathbf{e^{(n)}}}{\|\mathbf{e^{(n)}}\|_{p,\omega}}$$

remember that we defined $\mathbf{e^{(n)}}$ and its norm in (1.1), hence $\left|\left|\overline{\mathbf{e^{(n)}}}\right|\right|_{p,\omega} = 1$ and so $\overline{\mathbf{e^{(n)}}} \in \ell_{p,\omega}$. Thus

$$\left|\left|M_{\mathbf{u}}(\overline{\mathbf{e^{(n)}}})\right|\right|_{p,\omega} \leq K \left|\left|\overline{\mathbf{e^{(n)}}}\right|\right|_{p,\omega},$$

hence

$$\left|\left|M_{\mathbf{u}}(\overline{\mathbf{e^{(n)}}})\right|\right|_{p,\omega} = \left|\left|\mathbf{u} \cdot \overline{\mathbf{e^{(n)}}}\right|\right|_{p,\omega} = \frac{1}{\|\mathbf{e^{(n)}}\|_{p,\omega}}\|\mathbf{u} \cdot \mathbf{e^{(n)}}\|_{p,\omega}$$

and

$$\begin{aligned}\|\mathbf{u} \cdot \mathbf{e^{(n)}}\|_{p,\omega} &= \left(\sum_{k=1}^{\infty} |u_k e_k^{(n)}|^p \omega_k\right)^{1/p} \\ &= \left(|u_n|^p \omega_n\right)^{1/p} \\ &= |u_n| \|\mathbf{e^{(n)}}\|_{p,\omega}.\end{aligned}$$

Therefore

$$\begin{aligned}\|M_{\mathbf{u}}(\overline{\mathbf{e^{(n)}}})\|_{p,\omega} &= |u_n| \\ &\leq K\|\overline{\mathbf{e^{(n)}}}\|_{p,\omega} \\ &= K,\end{aligned}$$

we conclude that $|u_n| \leq K$ for all $n \in \mathbb{N}$, hence $\mathbf{u} \in \ell^\infty$ as we wanted.

Now, we prove necessity. For $\mathbf{x} \in \ell_{p,\omega}$, we have

$$\|M_{\mathbf{u}}(\mathbf{x})\|_{p,\omega}^p = \|\mathbf{u} \cdot \mathbf{x}\|_{p,\omega}^p = \sum_{n=1}^{\infty} |u_n x_n|^p \omega_n \leq \|\mathbf{u}\|_\infty^p \|\mathbf{x}\|_{p,\omega}^p.$$

Thus,

$$\|M_{\mathbf{u}}(\mathbf{x})\|_{p,\omega} \leq \|\mathbf{u}\|_\infty \|\mathbf{x}\|_{p,\omega},$$

and by definition of the norm of an operator

$$\|M_{\mathbf{u}}\|_{p,\omega} = \sup_{\substack{\mathbf{x}\in\ell_{p,\omega}\\ \mathbf{x}\neq 0}} \frac{\|M_{\mathbf{u}}(\mathbf{x})\|_{p,\omega}}{\|\mathbf{x}\|_{p,\omega}} \leq \|\mathbf{u}\|_{\infty}. \tag{2.1}$$

Since $\mathbf{u} \in \ell_\infty$, we conclude that the linear operator $M_{\mathbf{u}}$ is *bounded* and therefore *continuous*.

Now let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |u_n| - \varepsilon \leq |u_N|$$

since $\mathbf{u} \in \ell_\infty$ again.

Hence

$$\begin{aligned}
\left|\left|M_{\mathbf{u}}\left(\mathbf{e^{(N)}}\right)\right|\right|_{p,\omega} &= \left|\left|\mathbf{u}\cdot\mathbf{e^{(N)}}\right|\right|_{p,\omega} \\
&= \left(\sum_{n=1}^{\infty} |u_n e_n^{(N)}|^p \omega_k\right)^{1/p} \\
&= |u_N|(\omega_N)^{1/p} \\
&\geq \left(\sup_{n\in\mathbb{N}} |u_n| - \varepsilon\right)(\omega_N)^{1/p} \\
&= \left(\sup_{n\in\mathbb{N}} |u_n| - \varepsilon\right)\|\mathbf{e^{(N)}}\|_{p,\omega} \\
&= (\|\mathbf{u}\|_\infty - \varepsilon)\,\|\mathbf{e^{(N)}}\|_{p,\omega}
\end{aligned}$$

therefore ,

$$\|M_{\mathbf{u}}\|_{p,\omega} \geq \frac{\|M_{\mathbf{u}}\left(\mathbf{e^{(N)}}\right)\|_{p,\omega}}{\|\mathbf{e^{(N)}}\|_{p,\omega}} \geq \|\mathbf{u}\|_\infty - \varepsilon$$

for all $\varepsilon > 0$, thus

$$\|M_{\mathbf{u}}\|_{p,\omega} \geq \|\mathbf{u}\|_{\infty}. \tag{2.2}$$

By inequalities (2.1) and (2.2), we conclude that

$$\|M_{\mathbf{u}}\|_{p,\omega} = \|\mathbf{u}\|_{\infty}.$$

□

Our next result deals with the inverse operator of $M_{\mathbf{u}}$.

Theorem 2.2. *Let $\mathbf{u} \in \ell^\infty$. The multiplication operator $M_{\mathbf{u}} : \ell_{p,\omega} \to \ell_{p,\omega}$ has continuous inverse if and only if there exists $\delta > 0$ such that*

$$|u_n| \geq \delta \qquad \text{for all } n \in \mathbb{N}.$$

Proof. Necessity, suppose that $M_{\mathbf{u}} : \ell_{p,\omega} \to \ell_{p,\omega}$ has a continuous inverse, we claim that $u_n \neq 0$ for all $n \in \mathbb{N}$. In fact, suppose that there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = 0$. Then $\mathbf{e^{(n_0)}} = (0, 0, \ldots, \underbrace{1}_{n_0{}^{\text{th}}}, 0, 0, \ldots)$ and $M_u(\mathbf{e^{(n_0)}}) = \mathbf{u}\cdot\mathbf{e^{(n_0)}} = u_{n_0} = 0$. So $\mathbf{e^{(n_0)}} \in \ker(M_{\mathbf{u}})$ and $\ker(M_{\mathbf{u}}) \neq \emptyset$. Therefore, $M_{\mathbf{u}}$ is not injective, a contradiction.

Hence, we can consider the following sequence

$$\mathbf{v} = v_n = \frac{1}{u_n},\ n \in \mathbb{N}.$$

and $M_v(M_{\mathbf{u}}(\mathbf{x})) = \mathbf{v} \cdot (\mathbf{u} \cdot \mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \ell_{p,\omega}$. We conclude that M_v is the inverse operator of $M_{\mathbf{u}}$ and by hypotheses continuous. By Theorem 2.1, we conclude that $\mathbf{v} \in \ell^\infty$. Then there exists $M > 0$ such that

$$|v_n| \leq M \qquad \text{for all } n \in \mathbb{N},$$

then

$$\left|\frac{1}{u_n}\right| \leq M \qquad \text{for all } n \in \mathbb{N}.$$

We conclude

$$|u_n| \geq \delta \qquad \text{for all } n \in \mathbb{N}$$

where $\delta = \frac{1}{M}$ as we wanted to show.

For the reverse direction, Let $\delta > 0$ such that

$$|u_n| \geq \delta \qquad \text{for all } n \in \mathbb{N}$$

then

$$\frac{1}{\delta} \geq \left|\frac{1}{u_n}\right| = |v_n| \qquad \text{for all } n \in \mathbb{N}.$$

We conclude that $\mathbf{v} \in \ell^\infty$, and we show above that M_v is the inverse of $M_{\mathbf{u}}$ and M_v is continuous by Theorem 2.1. □

3. Finite range and compactness

Our coming result gives us a necessary and sufficient condition for $M_{\mathbf{u}}$ to have a finite range.

Theorem 3.1. *Let $\mathbf{u} \in \ell^\infty$. The multiplication operator $M_{\mathbf{u}} : \ell_{p,\omega} \to \ell_{p,\omega}$ has finite range if and only if $\mathcal{N}_u^c = \{n \in \mathbb{N} : u_n \neq 0\}$ is a finite set.*

Proof. We shall proceed by contrapositive. So let us suppose that $\mathcal{N}_u^c = \{n_1, n_{2,\ldots}\}$ is a infinite set, then for each $m \in \mathbb{N}$ we define the sequence $\mathbf{e^{(m)}}$ as we did in definition 1.8, so $\mathbf{e^{(m)}} \in \ell_{p,\omega}$ and define $\mathbf{b_m} = \mathbf{u} \cdot \mathbf{e^{(m)}} = M_{\mathbf{u}}(\mathbf{e^{(m)}})$, so $\mathbf{b_m} \in \text{Ran}\,(M_{\mathbf{u}})$.

Take scalars $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$ and suppose

$$\sum_{j=1}^{p} \alpha_j \mathbf{b^{(j)}} = 0, \qquad \mathbf{b^{(j)}} \in \{\mathbf{b_m}\}_{m\in\mathbb{N}}.$$

Let $k = 1, 2, \ldots, p$ and consider $n_k \in \mathcal{N}_u^c$. Then

$$0 = \sum_{j=1}^{p} \alpha_j b_{n_k}^{(j)} = \sum_{j=1}^{p} \alpha_j u_{n_k} e_{n_k}^{(j)} = \alpha_k u_{n_k}$$

then $\alpha_k u_{n_k} = 0$ but $u_{n_k} \neq 0$, hence $a_k = 0$ for all $k = 1, 2, \ldots, p$. This shows that the finite subset of $\{\mathbf{b_m}\}_{m\in\mathbb{N}}$, $\{\mathbf{b^{(j)}}\}_{j=1}^{p}$, is linear independent. Since the subset is arbitrary, then $\{\mathbf{b_m}\}_{m\in\mathbb{N}}$ is linear independent and $\dim(\{\mathbf{b_m}\}_{m\in\mathbb{N}}) = \infty$. We

conclude that $\dim(\mathrm{Ran}\,(M_{\mathbf{u}})) = \infty$.

For the reverse direction, suppose that $\mathcal{N}_u^c$ is a finite set, then $\mathcal{N}_u^c = \{n_1, n_2, \ldots, n_m\}$ and consider again the sequence $\mathbf{b_j} = \mathbf{u} \cdot \mathbf{e^{(j)}}$, of course $\{\mathbf{b_j}\}_{j=1}^m \subseteq \mathrm{Ran}\,(M_{\mathbf{u}})$. We are going to show that $\{\mathbf{b_j}\}_{j=1}^m$ is a basis for $\mathrm{Ran}\,(M_{\mathbf{u}})$. Let $\mathbf{z} \in \mathrm{Ran}\,(M_{\mathbf{u}})$ then there exists $\mathbf{x} \in \ell_{p,\omega}$ such that $M_{\mathbf{u}}(\mathbf{x}) = \mathbf{z}$, or $\mathbf{u} \cdot \mathbf{x} = \mathbf{z}$ and define $\alpha_j = x_{n_j}$ for each $j = 1, 2, \ldots, m$.

Take $k \in \mathcal{N}_u^c$ then $k = n_j$ for some $j = 1, 2, \ldots, m$ and

$$\begin{aligned} z_k = z_{n_j} = x_{n_j} u_{n_j} &= \alpha_j u_{n_j} \\ &= \alpha_j u_{n_j} \cdot e_{n_j}^{(j)} \\ &= \sum_{i=1}^{m} \alpha_i u_{n_j} e_{n_j}^{(i)} \\ &= \sum_{i=1}^{m} \alpha_i b_{i_k} \end{aligned}$$

since $e_{n_j}^{(i)} = 0$ for $i \neq j$.

Now, take $k \notin \mathcal{N}_u^c$ then $b_{j_k} = u_k \cdot e_k^{(j)} = 0$ for all $j = 1, 2, \ldots, m$. So

$$z_k = u_k x_k = 0 = \sum_{i=1}^{m} \alpha_i b_{i_k}.$$

In any case $\{\mathbf{b_j}\}_{j=1}^m$ span $\mathrm{Ran}\,(M_{\mathbf{u}})$ and the fact that is linear independent is analogous to the infinite case. Therefore $\{\mathbf{b_j}\}_{j=1}^m$ is a basis of $\mathrm{Ran}\,(M_{\mathbf{u}})$ and $\dim(\mathrm{Ran}\,(M_{\mathbf{u}})) = m < \infty$. □

We characterize now the compactness of $M_{\mathbf{u}}$.

Theorem 3.2. *Let $\mathbf{u} \in \ell^\infty$. The multiplication operator $M_{\mathbf{u}} : \ell_{p,\omega} \to \ell_{p,\omega}$ is a compact operator if and only if*

$$\lim_{k\to\infty} |u_k| = 0.$$

Proof. To prove necessity, suppose that $M_{\mathbf{u}}$ is a compact operator and let us show that

$$\lim_{k\to\infty} |u_k| = 0.$$

We need only to observe that $B_\varepsilon := \{k \in \mathbb{N} : |u_k| \geq \varepsilon\}$ is a finite set for every $\varepsilon > 0$. So let $\varepsilon > 0$ and define

$$\ell_{p,\omega}(\varepsilon) := \{\mathbf{x} \in \ell_{p,\omega} : x_k = 0 \text{ for all } k \in \mathbb{N} \setminus B_\varepsilon\}$$

We claim that $\ell_{p,\omega}(\varepsilon)$ is a subspace of $\ell_{p,\omega}$. It suffices to show that $\alpha\mathbf{x_1} + \beta\mathbf{x_2} \in \ell_{p,\omega}(\varepsilon)$ for $\mathbf{x_1}, \mathbf{x_2} \in \ell_{p,\omega}(\varepsilon)$ and $\alpha, \beta \in \mathbb{R}$, so for $k \in \mathbb{N} \setminus B_\varepsilon$ we have that

$$\alpha x_{1_k} + \beta x_{2_k} = \alpha \cdot 0 + \beta \cdot 0 = 0$$

showing that $\alpha\mathbf{x_1} + \beta\mathbf{x_2} \in \ell_{p,\omega}(\varepsilon)$. Now consider $\mathbf{x} \in \overline{\ell_{p,\omega}(\varepsilon)}$ then there exists $\{\mathbf{x^{(n)}}\}_{n\in\mathbb{N}} \subseteq \ell_{p,\omega}(\varepsilon)$ such that $\lim_{n\to\infty} \|\mathbf{x^{(n)}} - \mathbf{x}\|_{p,\omega} = 0$. Take $k_0 \in \mathbb{N}$ fixed and

consider the usually sequence $\mathbf{e^{(k_0)}} = (e_k^{(k_0)})_{k\in\mathbb{N}}$ defined as

$$e_k^{(k_0)} = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq k_0 \end{cases}$$

then

$$\left|x_{k_0}^{(n)} - x_{k_0}\right| e_k^{(k_0)} \leq \left|x_k^{(n)} - x_k\right| \quad \text{for all} \quad k \in \mathbb{N}.$$

Since $\mathbf{e^{(k_0)}} \in \ell_{p,\omega}$ and the inequality above

$$\left|x_{k_0}^{(n)} - x_{k_0}\right|^p \left|e_k^{(k_0)}\right|^p \omega_k \leq \left|x_k^{(n)} - x_k\right|^p \omega_k \quad \text{for all} \quad k \in \mathbb{N}.$$

$$\left|x_{k_0}^{(n)} - x_{k_0}\right|^p \sum_{k=1}^{\infty} \left|e_k^{(k_0)}\right|^p \omega_k \leq \sum_{k=1}^{\infty} \left|x_k^{(n)} - x_k\right|^p \omega_k$$

$$\left|x_{k_0}^{(n)} - x_{k_0}\right| \|\mathbf{e^{(k_0)}}\|_{p,\omega} \leq \|\mathbf{x^{(n)}} - \mathbf{x}\|_{p,\omega}$$

$$0 \leq \left|x_{k_0}^{(n)} - x_{k_0}\right| \omega^{\frac{1}{p}} \leq \|\mathbf{x^{(n)}} - \mathbf{x}\|_{p,\omega}$$

Taking $n \to \infty$ we conclude that $|x_{k_0}^{(n)} - x_{k_0}| \to 0$ for all $k_0 \in \mathbb{N}$. Particularly for every $k_0 \in N \setminus B_\varepsilon$, we have that $\lim_{n\to\infty} x_{k_0}^{(n)} = x_{k_0}$ and $x_{k_0}^{(n)} = 0$ for all $n \in \mathbb{N}$ thus $x_{k_0} = 0$ and $\mathbf{x} \in \ell_{p,\omega}(\varepsilon)$, this means that $\ell_{p,\omega}(\varepsilon)$ is a closed subspace of $\ell_{p,\omega}$.

We now show that $\ell_{p,\omega}(\varepsilon)$ is a $M_{\mathbf{u}}-$invariant subspace of $\ell_{p,\omega}$. Of course, let $\mathbf{z} \in M_{\mathbf{u}}(\ell_{p,\omega}(\varepsilon))$ then $\mathbf{z} = \mathbf{u}\cdot\mathbf{x}$ for some $\mathbf{x} \in \ell_{p,\omega}(\varepsilon)$. So $z_k = u_k x_k$ for all $k \in \mathbb{N}$. If $k \in \mathbb{N}/B_\varepsilon$ is clear that $z_k = u_k \cdot 0 = 0$, hence $\mathbf{z} \in \ell_{p,\omega}(\varepsilon)$ and $\ell_{p,\omega}(\varepsilon)$ is $M_{\mathbf{u}}-$invariant of $\ell_{p,\omega}$. Therefore $M_{\mathbf{u}} \restriction_{\ell_{p,\omega}(\varepsilon)}$ is compact since $M_{\mathbf{u}}$ is compact and $\ell_{p,\omega}(\varepsilon)$ is a $M_{\mathbf{u}}-$invariant subspace of $\ell_{p,\omega}$.

Take $\mathbf{x} \in \ell_{p,\omega}(\varepsilon)$ and define the sequence $\mathbf{z} = (z_k)_{k\in\mathbb{N}}$ as follows

$$z_k = \begin{cases} \frac{x_k}{u_k} & \text{if } k \in B_\varepsilon \\ 0 & \text{if } k \notin B_\varepsilon. \end{cases}$$

Note that if $k \in B_\varepsilon$ then $u_k \neq 0$ due to $|u_k| \geq \varepsilon$. So $\mathbf{x} = \mathbf{u}\cdot\mathbf{z}$, let us show that $\mathbf{z} \in \ell_{p,\omega}(\varepsilon)$. If $k \in B_\varepsilon$ then $|u_k| \geq \varepsilon$ and

$$|z_k| = \frac{|x_k|}{|u_k|} \leq \frac{1}{\varepsilon}|x_k|.$$

Consequently, if $k \notin B_\varepsilon$ then $z_k = 0$. In any case we will have that

$$|z_k| \leq \frac{1}{\varepsilon}|x_k|.$$

So,

$$|z_k|^p \omega_k \leq \frac{1}{\varepsilon^p}|x_k|^p \omega_k \text{ for all } k \in \mathbb{N},$$

and

$$\sum_{k=1}^{\infty} |z_k|^p \omega_k \leq \frac{1}{\varepsilon^p} \sum_{k=1}^{\infty} |x_k|^p \omega_k < \infty.$$

Therefore $\mathbf{z} \in \ell_{p,\omega}$, and we conclude that $\mathbf{x} \in \text{Ran}\left(M_{\mathbf{u}} \restriction_{\ell_{p,\omega}(\varepsilon)}\right)$, hence $\text{Ran}\left(M_{\mathbf{u}} \restriction_{\ell_{p,\omega}(\varepsilon)}\right) = \ell_{p,\omega}(\varepsilon)$. Note that $M_{\mathbf{u}} \restriction_{\ell_{p,\omega}(\varepsilon)}$ is a compact operator with closed range $\ell_{p,\omega}(\varepsilon)$, so we must have that $\ell_{p,\omega}(\varepsilon)$ is a finite dimensional subspace.

Now suppose that $B_\varepsilon = \{n_1, n_2, \ldots\}$ is a infinite set and consider the sequence $\mathbf{e_n}$ for each $n \in B_\varepsilon$, is clear that $\mathbf{e_n} \in \ell_{p,\omega}(\varepsilon)$ and $\{\mathbf{e_n}\}_{n \in B_\varepsilon}$ is linear independent set, therefore $\dim(\ell_{p,\omega}(\varepsilon)) = \infty$, a contradiction. Therefore B_ε must be finite.

To prove sufficiency, suppose that $\lim_{k\to\infty} |u_k| = 0$ then for $m \in \mathbb{N}$ define $A_m := \{1, 2, \ldots, m\}$ and $\mathbf{u_m} = \mathbf{u} \cdot \chi_{A_m}$ where

$$\mathbf{u} \cdot \chi_{A_m} = u_k \chi_{A_m}(k) = \begin{cases} u_k & \text{if } k \in A_m \\ 0 & \text{if } k \notin A_m \end{cases}$$

note that

$$\mathbf{u_m} = (u_1, u_2, \ldots, u_m, 0, 0, \ldots).$$

We claim that $\mathbf{u_m} \in \ell_{p,\omega}$. In fact, let $m \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} |u_k \chi_{A_m}(k)|^p \omega_k = \sum_{k=1}^{m} |u_k|^p \omega_k < \infty.$$

So $\mathbf{u_m} \cdot \mathbf{x} \in \ell_{p,\omega}$ for any $\mathbf{x} \in \ell_{p,\omega}$. Of course $\mathbf{u_m} \in \ell^\infty$ since $\|\mathbf{u_m}\|_\infty \leq \|\mathbf{u}\|_\infty$ and by hypothesis $\mathbf{u} \in \ell^\infty$, and also it is true $\mathcal{N}_{u_m}^c$ is a finite set. Therefore, by Theorem 2.1 and Theorem 3.1, $M_{u_m} : \ell_{p,\omega} \to \ell_{p,\omega}$ is a bounded multiplication operator of finite range, then M_{u_m} is a compact operator. It remains to show that $M_{u_m} \to M_{\mathbf{u}}$ *uniformly* to conclude that $M_{\mathbf{u}}$ is compact.

Let us show that

$$\lim_{m\to\infty} \|M_{u_m} - M_{\mathbf{u}}\| = 0.$$

By hypothesis $\lim_{k\to\infty} |u_k| = 0$ so given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|u_k| < \varepsilon \quad \text{for all} \quad k > N.$$

Let $m \geq N$ and $\mathbf{x} \in \ell_{p,\omega}$ then

$$|\left(u_k \chi_{A_m}(k) - u_k\right) x_k| = \begin{cases} 0 & \text{if } k \leq m \\ |u_k||x_k| & \text{if } m < k \end{cases}$$

Hence $|\left(u_k \chi_{A_m}(k) - u_k\right) x_k| \leq \varepsilon |x_k|$ for all $k \in \mathbb{N}$. Since $\ell_{p,\omega}$ is a linear space, $\mathbf{x} \in \ell_{p,\omega}$, and ε a scalar, we have that $\varepsilon \mathbf{x} \in \ell_{p,\omega}$ then $(\mathbf{u_m} - \mathbf{u}) \cdot \mathbf{x} \in \ell_{p,\omega}$. Therefore $\|(\mathbf{u_m} - \mathbf{u}) \cdot \mathbf{x}\|_{p,\omega} \leq \varepsilon \|\mathbf{x}\|_{p,\omega}$, this means that

$$\|M_{u_m} - M_{\mathbf{u}}\| \leq \varepsilon,$$

for all $m \geq N$. We conclude that $M_{\mathbf{u}}$ is compact. □

4. Essential norm

In this section, we study the essential norm of $M_{\mathbf{u}}$ denoted by $\|M_{\mathbf{u}}\|_e$, which is the distance from $M_{\mathbf{u}}$ to the class of the compact operators on $\ell_{p,\omega}$, that is,

$$\|M_{\mathbf{u}}\|_e := \inf\{\|M_{\mathbf{u}} - K\| : K : \ell_{p,\omega} \to \ell_{p,\omega} \text{ is compact}\},$$

where $\|M_{\mathbf{u}}\|$ denote the operator norm of $M_{\mathbf{u}}$, which depends on $\ell_{p,\omega}$.

Remark 4.1. Observe that $M_{\mathbf{u}} : \ell_{p,\omega} \to \ell_{p,\omega}$ is compact if and only if $\|M_{\mathbf{u}}\|_e = 0$.

The next result may be found in classical books of Functional Analysis (see [10], [14]) as a corollary of the Hahn-Banach Theorem for normed spaces.

Theorem 4.2. *Let X a normed space, for every $x \in X$ we have*

$$\|x\|_X = \sup_{f \in X',\ f \neq 0} \frac{|f(x)|}{\|f\|}.$$

Hence if $x_0 \in X$ is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.

We prove a Riesz representation theorem for $\ell_{p,\omega}$ spaces.

Theorem 4.3 (Riesz representation for $\ell_{p,\omega}$ space)**.** *Let $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For every bounded linear functional $f \in (\ell_{p,\omega})'$ there exists $\mathbf{z} \in \ell_{q,\omega}$ such that*

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} x_k z_k \omega_k$$

for all $\mathbf{x} \in \ell_{p,\omega}$ and $\|\mathbf{z}\|_{q,\omega} = \|f\|$.

Proof. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \ell_{p,\omega}$, we have shown (1.1) that $\mathbf{e^{(n)}} \in \ell_{p,\omega}$, then

$$\lim_{n\to\infty} \left\| \mathbf{x} - \sum_{k=1}^{n} x_k \mathbf{e^{(k)}} \right\|_{p,\omega}^{p} = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |x_k|^p \omega_k = 0.$$

This shows that each $\mathbf{x} \in \ell_{p,\omega}$ it has a unique representation of the form

$$\mathbf{x} = \sum_{k=1}^{\infty} x_k \mathbf{e^{(k)}},$$

which means that $\{\mathbf{e^{(k)}}\}_{k\in\mathbb{N}}$ is a Schauder basis of $\ell_{p,\omega}$.

Now, let $f \in (\ell_{p,\omega})'$, and define the sequence $\mathbf{z} = (z_k)_{k\in\mathbb{N}}$ as follows

$$z_k = \frac{f(\mathbf{e^{(k)}})}{\omega_k}.$$

Given that f is linear and bounded, it is continuous, and for each $\mathbf{x} \in \ell_{p,\omega}$ we have,

$$\begin{aligned} f(\mathbf{x}) &= f\left(\sum_{k=1}^{\infty} x_k \mathbf{e^{(k)}}\right) \\ &= \lim_{n\to\infty} f\left(\sum_{k=1}^{n} x_k \mathbf{e^{(k)}}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} x_k f(\mathbf{e^{(k)}}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} x_k z_k \omega_k = \sum_{k=1}^{\infty} x_k z_k \omega_k. \end{aligned}$$

We claim that $\mathbf{z} \in \ell_{q,\omega}$. Indeed, consider the sequence $\zeta^{(n)} = \left(\zeta_k^{(n)}\right)_{k\in\mathbb{N}}$ defined as

$$\zeta_k^{(n)} = \begin{cases} \frac{|z_k|^q}{z_k} & \text{if } k \leq n \text{ and } z_k \neq 0 \\ 0 & \text{if } n < k \text{ or } z_k = 0 \end{cases}$$

then $\zeta^{(n)} \in \ell_{p,\omega}$ (since it has a finite number of nonzero elements).

So

$$f(\zeta^{(n)}) = \sum_{k=1}^{\infty} \zeta_k^{(n)} z_k \omega_k = \sum_{k=1}^{n} |z_k|^q \omega_k,$$

and

$$\begin{aligned} |f(\zeta^{(n)})| &\leq \|f\| \|\zeta^{(n)}\|_{p.\omega} \\ &= \|f\| \left(\sum_{k=1}^{\infty} |\zeta_k^{(n)}|^p \omega_k\right)^{1/p} \\ &= \|f\| \left(\sum_{k=1}^{n} |z_k|^q \omega_k\right)^{1/p}, \end{aligned}$$

hence

$$\begin{aligned} |f(\zeta^{(n)})| = \sum_{k=1}^{n} |z_k|^q \omega_k &\leq \|f\| \left(\sum_{k=1}^{n} |z_k|^q \omega_k\right)^{1/p} \\ \left(\sum_{k=1}^{n} |z_k|^q \omega_k\right)^{1-1/p} &\leq \|f\| \\ \left(\sum_{k=1}^{n} |z_k|^q \omega_k\right)^{1/q} &\leq \|f\|. \end{aligned}$$

By taking the limit as $n \to \infty$, we conclude that

$$\|\mathbf{z}\|_{q,\omega} \leq \|f\|, \tag{4.1}$$

and so $\mathbf{z} \in \ell_{q,\omega}$.

It remains to show that $\|\mathbf{z}\|_{q,\omega} = \|f\|$.
Since f is linear and

$$\begin{aligned} |f(x)| &= \left| \sum_{k=1}^{\infty} x_k z_k \omega_k \right| \\ &\leq \sum_{k=1}^{\infty} |x_k z_k| \omega_k \\ &\leq \|\mathbf{x}\|_{p,\omega} \|\mathbf{z}\|_{q,\omega} \end{aligned}$$

by weighted Hölder inequality. Therefore

$$\frac{|f(\mathbf{x})|}{\|\mathbf{x}\|_{p,\omega}} \leq \|\mathbf{z}\|_{q,\omega} \ ; \ \mathbf{x} \neq \mathbf{0}.$$

so

$$\|f\| \leq \|\mathbf{z}\|_{q,\omega}$$

and we had recently shown (4.1) that $\|\mathbf{z}\|_{q,\omega} \leq \|f\|$. We conclude that

$$\|\mathbf{z}\|_{q,\omega} = \|f\|.$$

□

We arrive to the main result of this section, which tells us how to calculate the essential norm of $M_{\mathbf{u}}$ in terms of $\mathbf{u}$. The result reads as follows.

Theorem 4.4. *Let $\mathbf{u} \in \ell^{\infty}$. Then*

$$\|M_{\mathbf{u}}\|_e = \limsup_{n\to\infty} |u_n|.$$

Proof. Since $\mathbf{u} = \{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence, define $A_N := \{1, 2, 3, \ldots, N\}$ for each $N \in \mathbb{N}$ and consider the sequence $\{\mathbf{u_N}\}_{N\in\mathbb{N}} = \{u_n \chi_{A_N}(n)\}_{n\in\mathbb{N}}$, that is, $u_{N_1} = (u_1, u_2, \ldots, u_{N_1}, 0, 0, 0, \ldots)$. Note that

$$\lim_{N\to\infty} |u_n \chi_{A_N}(n)| = 0$$

by Theorem 3.2 the operator M_{u_N} is a compact operator and

$$\|M_{\mathbf{u}}\|_e \leq \|M_{\mathbf{u}} - M_{u_N}\| = \|M_{u-u_N}\|.$$

Now, let $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \in \ell_{p,\omega}$ such that $\mathbf{x} \neq 0$. If $n \leq N$ then

$$|(u_n - u_{Nn})x_n| = 0 \leq \sup_{k\in\mathbb{N}} |u_k||x_k|.$$

If $n \geq N$ then

$$|(u_n - u_{Nn})x_n| = |u_k||x_k| \leq \sup_{k\in\mathbb{N}} |u_k||x_k|.$$

In any case, we have

$$|(u_n - u_{Nn})x_n| = |u_k||x_k| \leq \sup_{k\in\mathbb{N}} |u_k||x_k|$$

and $\mathbf{u} \in \ell_{p,\omega}, \mathbf{u_N} \in \ell_{p,\omega}, \mathbf{x} \in \ell_{p,\omega}$ then $(\mathbf{u} - \mathbf{u_N})\,\mathbf{x} \in \ell_{p,\omega}$. Hence

$$|(u_n - u_{Nn})x_n|^p\omega_n \leq \sup_{k\in\mathbb{N}} |u_k|^p |x_k|^p \omega_n \quad ; \quad n \in \mathbb{N}, \omega_n > 0.$$

$$\left(\sum_{n=1}^{\infty} |(u_n - u_{Nn})x_n|^p \omega_n\right)^{1/p} \leq \left(\sup_{k\in\mathbb{N}} |u_k|^p \sum_{n=1}^{\infty} |x_k|^p \omega_n\right)^{1/p}$$

$$\|\,(\mathbf{u} - \mathbf{u_N})\,\mathbf{x}\|_{p,\omega} \leq \sup_{k\in\mathbb{N}} |u_k| \|\mathbf{x}\|_{p,\omega}$$

$$\|M_{u-u_N}(\mathbf{x})\|_{p,\omega} \leq \sup_{k\in\mathbb{N}} |u_k| \|\mathbf{x}\|_{p,\omega}.$$

So,

$$\|M_{u-u_N}\| = \sup_{\substack{\mathbf{x}\in\ell_{p,\omega} \\ \mathbf{x}\neq 0}} \frac{\|M_{u-u_N}(\mathbf{x})\|_{p,\omega}}{\|\mathbf{x}\|_{p,\omega}} \leq \sup_{k\in\mathbb{N}} |u_k|$$

then

$$\|M_{\mathbf{u}}\|_e \leq \sup_{k\in\mathbb{N}} |u_k|.$$

Therefore

$$\|M_{\mathbf{u}}\|_e \leq \limsup_{n\to\infty} |u_n|. \tag{4.2}$$

Consider now the usual sequence $\mathbf{e^{(n)}} = \{e_k^{(n)}\}_{k\in\mathbb{N}}$ defined as

$$e_k^{(n)} = \begin{cases} 1 \text{ if } k = n \\ 0 \text{ if } k \neq n \end{cases}$$

and also consider

$$\overline{\mathbf{e^{(n)}}} = \frac{\mathbf{e^{(n)}}}{\|\mathbf{e^{(n)}}\|_{p,\omega}}$$

remember that $\overline{\mathbf{e^{(n)}}} \in \ell_{p,\omega}$, $\|\overline{\mathbf{e^{(n)}}}\|_{p,\omega} = 1$ and $\|\mathbf{e^{(n)}}\|_{p,\omega} = \omega_n^{1/p}$. We are going to show that

$$\lim_{n\to\infty} \|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega} = 0$$

where $K : \ell_{p,\omega} \to \ell_{p,\omega}$ is a compact operator. Suppose the equation does not hold, i.e., $\lim_{n\to\infty} \|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega} \neq 0$ so there exists $\delta > 0$ and a subsequence $\{\overline{\mathbf{e^{(n_m)}}}\}_{m\in\mathbb{N}}$ of $\{\overline{\mathbf{e^{(n)}}}\}_{n\in\mathbb{N}}$ such that

$$\|\overline{\mathbf{e^{(n_m)}}}\|_{p,\omega} \geq \delta \quad \text{for all} \quad m \in \mathbb{N}.$$

Since $K : \ell_{p,\omega} \to \ell_{p,\omega}$ is compact and $\{\overline{\mathbf{e^{(n)}}}\}$ is bounded in $\ell_{p,\omega}$ then we could suppose that $\{\overline{\mathbf{e^{(n_m)}}}\}_{m\in\mathbb{N}}$ is convergent in $\ell_{p,\omega}$ and therefore $\{K(\overline{\mathbf{e^{(n_m)}}})\}_{m\in\mathbb{N}}$ converge to some element of $\ell_{p,\omega}$, namely $\mathbf{y} \in \ell_{p,\omega}$. Hence,

$$\lim_{m\to\infty} \|K\left(\overline{\mathbf{e^{(n_m)}}}\right) - \mathbf{y}\|_{p,\omega} = 0.$$

Let us see that $\mathbf{y}=0$. By Theorem 4.2, it sufficient to show that $f(\mathbf{z})=0$ for all $f\in(\ell_{p,\omega})'$. Let $f\in(\ell_{p,\omega})'$ then $f\circ K\in(\ell_{p,\omega})'$ and due to Riesz representation Theorem for $\ell_{p,\omega}$, Theorem 4.3, there exists $\mathbf{z}\in\ell_{q,\omega}$, where $\frac{1}{p}+\frac{1}{q}=1$, such that

$$f\circ K(\mathbf{x})=\sum_{k=1}^{\infty}x_k z_k\omega_k$$

for all $\mathbf{x}\in\ell_{p,\omega}$. Particularly for $\overline{\mathbf{e^{(n_m)}}}\in\ell_{p,\omega}$. So

$$\begin{aligned}\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right| &= \left|\sum_{k=1}^{\infty}\frac{\overline{\mathbf{e^{(n_m)}}}}{\|\overline{\mathbf{e^{(n_m)}}}\|_{p,\omega}}z_k\omega_k\right|\\ &\leq \frac{|z_{n_m}|\omega_{n_m}}{\|\overline{\mathbf{e^{(n_m)}}}\|_{p,\omega}}\\ &= \frac{|z_{n_m}|\omega_{n_m}}{\omega_{n_m}^{1/p}}\\ &= |z_{n_m}|\omega_{n_m}^{1-1/p}=|z_{n_m}|\omega_{n_m}^{1/q}\end{aligned}$$

then

$$\lim_{m\to\infty}\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|\leq\lim_{m\to\infty}|z_{n_m}|\omega_{n_m}^{1/q}$$

and $\lim_{m\to\infty}|z_{n_m}|\omega_{n_m}^{1/q}=0$ since $\mathbf{z}\in\ell_{q,\omega}$. We conclude that

$$\lim_{m\to\infty}\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|=0.$$

Hence

$$\begin{aligned}|f(\mathbf{y})| &\leq \left|f(\mathbf{y})-f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|+\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|\\ &\leq \|f\|\|\mathbf{y}-K(\overline{\mathbf{e^{(n_m)}}})\|_{p,\omega}+\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|\end{aligned}$$

taking limits, we have

$$\begin{aligned}|f(\mathbf{y})| &\leq \|f\|\lim_{m\to\infty}\|\mathbf{y}-K(\overline{\mathbf{e^{(n_m)}}})\|_{p,\omega}+\lim_{m\to\infty}\left|f\circ K(\overline{\mathbf{e^{(n_m)}}})\right|\\ &= \|f\|\cdot 0+0\end{aligned}$$

then $|f(\mathbf{y})|=0$ and it is hold for all $f\in(\ell_{p,\omega})'$, therefore $\mathbf{y}=0$. We conclude that

$$\lim_{n\to\infty}\|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}=0.$$

Now, note that

$$\begin{aligned}\|M_{\mathbf{u}}-K\| &\geq \|M_{\mathbf{u}}(\overline{\mathbf{e^{(n)}}})-K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}\\ &\geq \left|\|M_{\mathbf{u}}(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}-\|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}\right|\\ &= \left|\frac{1}{\|\mathbf{e^{(n)}}\|_{p,\omega}}\|\mathbf{u}\cdot\mathbf{e^{(n)}}\|_{p,\omega}-\|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}\right|\\ &= \left||u_n|-\|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega}\right|\end{aligned}$$

Therefore, taking $n \to \infty$ we have

$$\|M_{\mathbf{u}} - K\| \geq \lim_{n\to\infty} \left| |u_n| - \|K(\overline{\mathbf{e^{(n)}}})\|_{p,\omega} \right| \geq \limsup_{n\to\infty} |u_n|.$$

Since we take $K : \ell_{p,\omega} \to \ell_{p,\omega}$ compact arbitrary then

$$\|M_{\mathbf{u}}\|_e \geq \limsup_{n\to\infty} |u_n|. \tag{4.3}$$

By equations (4.2) and (4.3) we could finally conclude that

$$\|M_{\mathbf{u}}\|_e = \limsup_{n\to\infty} |u_n|.$$

□

5. Conclusions

In conclusion, we can highlight the great importance of the space $\ell_{p,\omega}$, firstly, as a generalization of the classical space ℓ_p and also as the discrete case of the space $L_{p,\omega}$. In this article, we have shown the characterizations of continuity, continuous inverse, finite range, compactness, and essential norm when considering the multiplication operator defined on the space $\ell_{p,\omega}$. These results leave the door open for the study of other properties such as closed range and Fredholm-type properties. Furthermore, they pave the way for further studies, such as the variable p case and other related sequence spaces (Lorentz sequence spaces or Banach sequence spaces [5]).

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgments. The authors extend their gratitude to the Editor and the anonymous reviewers for their valuable suggestions and comments, which significantly contributed to enhancing the quality of the submitted manuscript.

References

1. M. B. Abrahamse, *Multiplication operators*, Hilbert space operators (Proc. Conf., Calif. State Univ., Long Beach, Calif., 1977), Lecture Notes in Math., vol. 693, Springer, Berlin, 1978, pp. 17–36.
2. S.C. Arora, G. Datt, and S. Verma, *Multiplication operators on Lorentz spaces*, Indian J. Math. **48** (2006), no. 3, 317–329.
3. ———, *Operators on Lorentz sequence spaces*, Math. Bohem. **134** (2009), no. 1, 87–98. http://dx.doi.org/10.21136/MB.2009.140643
4. A. Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. **336** (1982), 26–44. https://doi.org/10.1515/crll.1982.336.26
5. M. Candan and E. E. Kara, *A study of topological and geometrical characteristics of new Banach sequence spaces*, Gulf J. Math. **3** (2015), no. 4, https://doi.org/10.56947/gjom.v3i4.50
6. R. E. Castillo, H. C. Chaparro, and J. C. Ramos Fernández, *Orlicz-Lorentz spaces and their multiplication operators*, Hacet. J. Math. and Stat. **44** (2015), no. 5, 991–1009. https://dergipark.org.tr/en/pub/hujms/issue/49537/524272

7. R. E. Castillo, H. Rafeiro, J. C. Ramos Fernández and M. Salas-Brown, *Multiplication Operator on Köthe Spaces: Measure of Non-compactness and Closed Range*, Bull. Malays. Math. Sci. Soc. **42** (2019), 1523–1534. https://doi.org/10.1007/s40840-017-0562-0
8. R.E. Castillo, R. León, and E. Trousselot, *Multiplication operator on $L_{(p,q)}$ spaces*, PanAmer. Math. J. **19** (2009), no. 1, 37–44.
9. R. G. Douglas, *Banach algebra techniques in operator theory*, second ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998. https://doi.org/10.1007/978-1-4612-1656-8
10. P. Habala, P. Hájek, and V. Zizler, *Introduction to banach spaces*, no. v. 1, Matfyzpress, Czech Republic, 1996.
11. P. R. Halmos, *A Hilbert space problem book*, second ed., Graduate Texts in Mathematics, vol. 19, Springer-Verlag, New York-Berlin, 1982. https://doi.org/10.1007/978-1-4684-9330-6
12. H. Hudzik, R. Kumar, and R. Kumar, *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math., vol. 116, Springer India, 2006, pp. 71–81. https://doi.org/10.1007/BF02829740
13. B.S. Komal and S. Gupta, *Multiplication operators between Orlicz spaces*, Integral Equations Operator Theory **41** (2001), no. 3, 324–330. https://doi.org/10.1007/BF01203174
14. E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, Windsor, 1978. https://www.wiley.com/en-us/Introductory+Functional+Analysis+with+Applications-p-9780471504597
15. J. C. Ramos-Fernández and M. Salas-Brown, *On multiplication operators acting on Köthe sequence spaces*, Afr. mate. **28** (2017), no. 3, 661–667. https://doi.org/10.1007/s13370-016-0475-3
16. H. Takagi and K. Yokouchi, *Multiplication and composition operators between two L^p-spaces*, Function spaces (Edwardsville, IL, 1998), Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 321–338. http://dx.doi.org/10.1090/conm/232/03408

[1] Departamento de Matemáticas, Universidad Nacional De Colombia, Bogotá, Colombia
Email address: recastillo@unal.edu.co

[2] Programa de Matemáticas, Universidad de Cartagena, Cartagena de Indias, Colombia
Email address: hchaparrog@unicartagena.edu.co

[3] Departamento de Ciencias Básicas, Universidad Manuela Beltrán, Bogotá, Colombia
Email address: oscar.chaparro@docentes.umb.edu.co