Gulf Journal of Mathematics ISSN: 2309-4966
Vol 19, Issue 2 (2025) 304-314 https://doi.org/10.56947/gjom.v19i2.2697

ON σ-DERIVATIONS OF KRASNER HYPERRINGS

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Abstract. We present a significant extension of the derivation concept to Krasner hyperrings, namely the σ-derivation, which is defined utilizing a self-mapping σ on the hyperring R. We investigate several important properties of these σ-derivations and explore their behavior in prime Krasner hyperrings. Furthermore, we establish conditions under which a Krasner hyperring becomes commutative.

1. Introduction

The Krasner hyperring [7], a well-known type of hyperrings, was introduced in 1983 by M. Krasner. Krasner hyperring is an algebraic structure that generalizes the classical ring theory where the addition operation is defined as a hyperoperation, meaning the sum of two elements is not a single element but a set of elements. This hyperstructure provides a more general framework for algebraic structures, enabling the exploration of properties that extend beyond traditional ring theory. Some reviews of the hyperstructure theory can be found in [1, 3, 4, 5, 10, 11, 12].

Several types of hyperrings have been introduced and studied. Ameri and Norouzi [1] examined homomorphisms of hyperrings and the extension and contraction of hyperideals in commutative hyperrings. Z. Kou et al. [8] explored the connection between ordered semihyperrings.

In [2], Asokkumar initiated the study of derivations on Krasner hyperrings, followed by investigations into the relationships between derivations and hyperring structures by other authors (see [6, 13, 14, 15, 16]). Recently, Leerawat and Sanguansat [9] extended this work by introducing (f, g)-semi-derivation on Krasner hyperring R defined by using self-mappings f and g, and explored their resulting properties.

In this paper, we define σ-derivations on Krasner hyperrings, generalizing traditional derivations, and simplify the product rule of (f, g)-semi-derivations. A σ-derivation is a map that satisfies a modified version of the Leibniz rule, governed by an auxiliary function σ, which controls the interaction between hyperring elements during differentiation. We investigate some properties of σ-derivations and

Date: Received: Feb 13, 2025; Accepted: Mar 2, 2025.
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2010 *Mathematics Subject Classification.* $20N20, 16Y99$.
Key words and phrases. derivation, hyperring, Krasner hyperring, prime hyperring.

their behavior in prime Krasner hyperrings. Furthermore, we provide conditions that determine when a Krasner hyperring becomes commutative.

2. Preliminaries

This section reviews definitions and concepts related to hyperstructures, which will be used throughout. Readers seeking more comprehensive information are directed to Davvaz and Leoreanu-Fotea [5] and Asokummar [2].

Definition 2.1 ([5])**.** A hypergroupoid $(H, \circ)$ is a nonempty set H equipped with a hyperoperation $\circ$, which is a map $\circ : H \times H \to P^{\star}(H)$, where $P^{\star}(H)$ is the set of all nonempty subsets of H. For any $a, b \in H$, the hyperproduct of a and b is denoted by $a \circ b$. For nonempty subsets A and B of H, and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,$$

and $A \circ x = A \circ \{x\}, x \circ B = \{x\} \circ B$.

A hypergroupoid $(H, \circ)$ is called commutative if $a \circ b = b \circ a$ for all $a, b \in H$. A semihypergroupoid is a hypergroupoid $(H, \circ)$ such that $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$. A semihypergroupoid $(H, \circ)$ is called a hypergroup if $a \circ H = H = H \circ a$ for all $a \in H$.

Definition 2.2 ([5])**.** A commutative hypergroup $(H, \circ)$ is called a canonical hypergroup if

(i) $(H, \circ)$ has a scalar identity, which means that there is an element $e \in H$ such that $e \circ x = \{x\}$ for all $x \in H$,
(ii) for all x of H there exists a unique x^{-1} in H such that $e \in x \circ xs^{-1}$,
(iii) if $x \in y \circ z$, then there exist the inverse y^{-1} of y and z^{-1} of z, such that $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Note that a scalar identity is unique since if e and e' are scalar identities of a hypergroupoid $(H, \circ)$, then $\{e\} = e \circ e' = \{e'\}$, so that $e = e'$.

Definition 2.3 ([5])**.** A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1. $(R, +)$ is a canonical hypergroup, that is,
   (i) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$,
   (ii) $x + y = y + x$ for all $x, y \in R$,
   (iii) there exists $0 \in R$ such that $0 + x = \{x\}$, for all $x \in R$, such an element 0 is called an identity,
   (iv) for each $x \in R$, there exists a unique element, denoted by $-x \in R$ such that $0 \in x + (-x)$, such element $-x$ is called the inverse of x,
   (v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$ for all $x, y, z \in R$,
2. $(R, \cdot)$ is a semigroup having zero, which is the identity of $(R, +)$, as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$,
3. the multiplication is distributive with respect to the hyperoperation $+$, that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for every $x, y, z \in R$.

A Krasner hyperring $(R, +, \cdot)$ is commutative if $x \cdot y = y \cdot x$ for all $x, y \in R$ and an identity of R is an element, say 1, in R such that $1 \cdot x = x \cdot 1 = x$ for all $x \in R$.

For conciseness, we write xy instead of $x \cdot y$ for all $x, y \in R$, and we use $0+x = x$ instead of $0 + x = \{x\}$ for all $x \in R$.

Throughout this paper, by a hyperring we mean a Krasner hyperring.

For nonempty subset A of a hyperring R, let $-A = \{-a | a \in A\}$. These elementary facts are direct consequences of the axioms:

(i) $-(-x) = x$ for all $x \in R$,
(ii) $-(x + y) = -x - y$ for all $x \in R$,
(iii) $-(xy) = (-x)y = x(-y)$ for all $x, y \in R$,
(iv) $(a + b)(c + d) \subseteq ac + bc + ad + bd$ for all $a, b, c, d \in R$.

Example 2.4 ([12])**.** Let $R = \{0, 1, 2, 3\}$. Define the hyperoperation $+$ and the binary operation $\cdot$ as follows:

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2,3\}$	$\{1,2\}$	$\{1,3\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,3\}$	$\{2,3\}$
3	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{0,1,2\}$

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

.

Then $(R, +, \cdot)$ is a commutative hyperring with 1 as its identity.

Example 2.5 ([11])**.** Let R be the unit interval $[0, 1]$. Define the hyperoperation $+$ on R by

$$x + y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

Then $(R, +, \cdot)$ is a hyperring where $\cdot$ is the usual multiplication. Note that 0 is the zero of $(R, +, \cdot)$ and for each $x \in R, x$ is the inverse of x in $(R, +)$.

Definition 2.6. A nonempty subset S of a hyperring $(R, +, \cdot)$ is called a subhyperring of R if $(S, +, \cdot)$ is itself a hyperring containing 0.

A subhyperring I of a hyperring R is a left (respectively, right) hyperideal of R if $ra \in I$ (respectively, $ar \in I$) for all $r \in R$ and $a \in I$. Also, I is called a hyperideal of R if I is both a left and a right hyperideal of R.

Lemma 2.7 ([5])**.** *A nonempty subset I of a hyperring R is a left (respectively, right) hyperideal of R if and only if for all $a, b \in I$ and $r \in R, a - b \in I, ra \in I(ar \in I)$.*

Definition 2.8. Let $(R, +, \cdot)$ and $(S, \oplus, \odot)$ be hyperrings and σ a mapping from R into S. Then σ is called a homomorphism if

(i) $\sigma(a + b) \subseteq \sigma(a) \oplus \sigma(b)$,
(ii) $\sigma(a \cdot b) = \sigma(a) \odot \sigma(b)$,

for all $a, b \in R$.

If σ is a homomorphism from a hyperring $(R, +, \cdot)$ into a hyperring $(S, \oplus, \odot)$. The kernel of σ, written $\ker(\sigma)$, is defined to be the set

$$\ker(\sigma) = \{a \in R | \sigma(a) = 0\},$$

where 0 is the zero of $(S, \oplus, \odot)$.

A homomorphism σ from R into S is said to be a good homomorphism if

$$\sigma(a + b) = \sigma(a) \oplus \sigma(b),$$

for all $a, b \in R$.

An isomorphism from a hyperring $(R, +, \cdot)$ into a hyperring $(S, \oplus, \odot)$ is a bijective good homomorphism from R into S.

Example 2.9 ([11]). Let $([0, 1], +, \cdot)$ be the hyperring defined as in Example 2.5. Define $\sigma : [0, 1] \to [0, 1]$ by $\sigma(x) = 1$, for all $x \in [0, 1]$. Since $1 + 1 = [0, 1]$ and $1 \cdot 1 = 1$, it follows that σ is a homomorphism from $[0, 1]$ into itself. Note that $\ker(\sigma) = \emptyset$.

Theorem 2.10 ([11]). *Let σ be a homomorphism from a hyperring $(R, +, \cdot)$ into a hyperring $(S, \oplus, \odot)$ such that $\ker(\sigma) \neq \emptyset$. Then the following statements hold:*

(i) $\sigma(0) = 0$,
(ii) $\sigma(-x) = -\sigma(x)$ *for all* $x \in R$,
(iii) $\ker(\sigma)$ *is a hyperideal of* R,
(iv) *if* σ *is 1-1, then* $\ker(\sigma) = \{0\}$,
(v) *if* σ *is a good homomorphism and* $\ker(\sigma) = \{0\}$, *then* σ *is 1-1,*
(vi) *if* σ *is a good homomorphism, then* $\sigma(R)$ *is a subhyperring of* S.

Lemma 2.11. *Let σ be a good homomorphism from a hyperring $(R, +, \cdot)$ onto a hyperring $(S, \oplus, \odot)$ such that $\ker(\sigma) \neq \emptyset$. If I is a hyperideal of R then $\sigma(I)$ is a hyperideal of S.*

Proof. Assume that I is a hyperideal of R. Let $x, y \in \sigma(I)$ and $s \in S$. Then there exist two elements $a, b \in I$ such that $\sigma(a) = x$ and $\sigma(b) = y$. Since σ is onto S, there exists $r \in R$ such that $\sigma(r) = s$. It follows that

$$\begin{aligned} x + y = \sigma(a) + \sigma(b) &= \sigma(a + b) \subseteq \sigma(I), \\ -x = -\sigma(a) &= \sigma(-a) \subseteq \sigma(I), \\ sx = \sigma(r)\sigma(a) &= \sigma(ra) \subseteq \sigma(I), \text{ and} \\ xs = \sigma(a)\sigma(r) &= \sigma(ar) \subseteq \sigma(I). \end{aligned}$$

Therefore, $\sigma(I)$ is a hyperideal of S. □

Definition 2.12 ([16]). A hyperring R is prime if for any $a, b \in R, aRb = \{0\}$ implies $a = 0$ or $b = 0$.

A hyperring R is 2-torsion free if for any $x \in R, 0 \in x + x$ implies $x = 0$.

Lemma 2.13 ([16]). *Let R be a hyperring. For all $x, y \in R, [x, y]$ denotes the set $xy - yx$ Then, for all $x, y, z \in R$, we have*

(i) $[x + y, z] = [x, z] + [y, z]$,
(ii) $[xy, z] \subseteq x[y, z] + [x, z]y = [x, z]y + x[y, z]$.

Lemma 2.14 ([6]). *Let I be a non-zero hyperideal of a prime hyperring R. Then, for all $x, y \in R$,*

(i) *if $Ix = \{0\}$ or $xI = \{0\}$, then $x = 0$,*
(ii) *fi $xIy = \{0\}$, then $x = 0$ or $y = 0$.*

Definition 2.15 ([2]). Let R be a hyperring. A map $d : R \to R$ is said to be a derivation of R if d satisfies

(i) $d(x + y) \subseteq d(x) + d(y)$,
(ii) $d(xy) \in d(x)y + xd(y)$ for all $x, y \in R$.

If the map d is such that $d(x + y) = d(x) + d(y)$, for all $x, y \in R$ and satisfies the second condition, then d is called a strong derivation of R.

3. σ-Derivations in Krasner Hyperrings

In this section, we introduce the concept of a σ-derivation on a hyperring, extending the idea of derivation from ring theory to the broader framework of hyperrings.

Definition 3.1. Let R be a hyperring and let $\sigma : R \to R$ be a mapping. A map $d : R \to R$ is called a σ-derivation of R if it satisfies the following conditions for all $x, y \in R$:

(i) $d(x + y) \subseteq d(x) + d(y)$,
(ii) $d(xy) \in d(x)\sigma(y) + \sigma(x)d(y)$,
(iii) $d(\sigma(x)) = \sigma(d(x))$.

Example 3.2. Let $R = \{0, a, b\}$ be a hyperring under the hyperoperation $+$ and the binary operation $\cdot$ as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a, b\}$	R
b	$\{b\}$	R	$\{a, b\}$

and

$\cdot$	0	a	b
0	0	0	0
a	0	b	a
b	0	a	b

.

Define $d : R \to R$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = b, \\ b & \text{if } x = a, \end{cases}$$

and define $\sigma : R \to R$ by $\sigma(x) = x$ for all $x \in R$. It is straightforward to check that the map d is a σ-derivation of R.

In what follows, let R denote a hyperring unless otherwise specified.

Definition 3.3. Let d be a σ-derivation of R. For any positive integer n, the nth σ-derivation of d, denoted by d^n, is obtained by composing d with itself n times. Additionally, we define $d^0(x)$ and $\sigma^0(x)$ as $d^0(x) = x$ and $\sigma^0(x) = x$, respectively.

Next, we show that the nth σ-derivation of d satisfies a form of the Leibniz rule.

Theorem 3.4. *Let R be a commutative hyperring and let $\sigma : R \to R$ be a good homomorphism. If d is a σ-derivation of R, then*

(i) $d(x^n) \in n(\sigma^{n-1}(x)d(x))$ *for all $x \in R$ and $n \geq 2$,*

(ii) $d^n(xy) \in \sum_{i=0}^{n} \binom{n}{i} d^{n-i}(\sigma^i(x))\sigma^{n-i}(d^i(y))$ *for all $x, y \in R$ and $n \in \mathbb{N}$.*

Proof. (i) We proceed by induction on n. It is clearly true for $n = 2$. Let $k \geq 2$ and assume that $d(x^k) \in k(\sigma^{k-1}(x)d(x))$ holds for all $x \in R$. Now, we have

$$\begin{aligned} d(x^{k+1}) &= d(x^k \cdot x) \\ &\in d(x^k)\sigma(x) + \sigma^k(x)d(x) \\ &\subseteq k(\sigma^{k-1}(x)d(x))\sigma(x) + \sigma^k(x)d(x) \\ &= k(\sigma^k(x)d(x)) + \sigma^k(x)d(x) \\ &= (k+1)(\sigma^k(x)d(x)). \end{aligned}$$

Hence, $d(x^{k+1}) \in (k+1)(\sigma^k(x)d(x))$.

(ii) Let $x, y \in R$. We use induction on n. For $n = 1$, the identity clearly holds. Let $k \geq 1$ and assume that

$$d^k(xy) \in \sum_{i=0}^{k} \binom{k}{i} d^{k-i}(\sigma^i(x))\sigma^{k-i}(d^i(y)).$$

Now, we have

$$\begin{aligned} &d^{k+1}(xy) \\ &= d(d^k(xy)) \\ &\in d\left(\sum_{i=0}^{k} \binom{k}{i} d^{k-i}(\sigma^i(x))\sigma^{k-i}(d^i(y))\right) \\ &\subseteq \sum_{i=0}^{k} d\left(\binom{k}{i} d^{k-i}(\sigma^i(x))\sigma^{k-i}(d^i(y))\right) \\ &\in \sum_{i=0}^{k} \binom{k}{i} d^{k-i+1}\left(\sigma^i(x)\right)\sigma^{k-i+1}\left(d^i(y)\right) + \sum_{i=0}^{k} \binom{k}{i} d^{k-i}\left(\sigma^{i+1}(x)\right)\sigma^{k-i}\left(d^{i+1}(y)\right) \\ &= \binom{k}{0} d^{k+1}\left(\sigma^0(x)\right)\sigma^{k+1}\left(d^0(y)\right) + \binom{k}{1} d^k\left(\sigma^1(x)\right)\sigma^k\left(d^1(y)\right) \\ &+ \binom{k}{2} d^{k-1}\left(\sigma^2(x)\right)\sigma^{k-1}\left(d^2(y)\right) + \cdots + \binom{k}{k} d^1\left(\sigma^k(x)\right)\sigma^1\left(d^k(y)\right) \\ &+ \binom{k}{0} d^k\left(\sigma^1(x)\right)\sigma^k\left(d^1(y)\right) + \binom{k}{1} d^{k-1}\left(\sigma^2(x)\right)\sigma^{k-1}\left(d^2(y)\right) \\ &+ \binom{k}{2} d^{k-2}\left(\sigma^3(x)\right)\sigma^{k-2}\left(d^3(y)\right) + \cdots + \binom{k}{k} d^0\left(\sigma^{k+1}(x)\right)\sigma^0\left(d^{k+1}(y)\right) \end{aligned}$$

$$
\begin{aligned}
&= \binom{k}{0} d^{k+1}\left(\sigma^0(x)\right)\sigma^{k+1}\left(d^0(y)\right)\\
&+ \left[\binom{k}{1} d^{k}\left(\sigma^1(x)\right)\sigma^{k}\left(d^1(y)\right) + \binom{k}{0} d^{k}\left(\sigma^1(x)\right)\sigma^{k}\left(d^1(y)\right)\right]\\
&+ \left[\binom{k}{2} d^{k-1}\left(\sigma^2(x)\right)\sigma^{k-1}\left(d^2(y)\right) + \binom{k}{1} d^{k-1}\left(\sigma^2(x)\right)\sigma^{k-1}\left(d^2(y)\right)\right] + \cdots\\
&+ \left[\binom{k}{k} d^{1}\left(\sigma^k(x)\right)\sigma^{1}\left(d^k(y)\right) + \binom{k}{k-1} d^{1}\left(\sigma^k(x)\right)\sigma^{1}\left(d^k(y)\right)\right]\\
&+ \binom{k}{k} d^{0}\left(\sigma^{k+1}(x)\right)\sigma^{0}\left(d^{k+1}(y)\right)\\
&= \binom{k+1}{0} d^{k+1}\left(\sigma^0(x)\right)\sigma^{k+1}\left(d^0(y)\right) + \left[\binom{k}{1}+\binom{k}{0}\right] d^{k}\left(\sigma^1(x)\right)\sigma^{k}\left(d^1(y)\right)\\
&+ \left[\binom{k}{2}+\binom{k}{1}\right] d^{k-1}\left(\sigma^2(x)\right)\sigma^{k-1}\left(d^2(y)\right) + \cdots\\
&+ \left[\binom{k}{k}+\binom{k}{k-1}\right] d^{1}\left(\sigma^k(x)\right)\sigma^{1}\left(d^k(y)\right) + \binom{k+1}{k+1} d^{0}\left(\sigma^{k+1}(x)\right)\sigma^{0}\left(d^{k+1}(y)\right)\\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} d^{k+1-i}\left(\sigma^i(x)\right)\sigma^{k+1+i}\left(d^i(y)\right).
\end{aligned}
$$

Hence,

$$
d^{k+1}(xy) \in \sum_{i=0}^{k+1} \binom{k+1}{i} d^{k+1-i}\left(\sigma^i(x)\right)\sigma^{k+1+i}\left(d^i(y)\right).
$$

□

Theorem 3.5. *Let R be a commutative hyperring and let d be a σ-derivation of R, where $\sigma : R \to R$ is a good homomorphism. Suppose that there exists the smallest positive integer $n > 1$ such that $d^n(R) = \{0\}$. Then, for each $y \in R, d(y) = 0$ or there exists positive integer $k < n$ and $x_0 \in R - \{0\}$ such that $0 \in nd^{n-1}\left(\sigma(x_0)\right) d^k\left(\sigma^{n-1}(y)\right)$.*

Proof. Suppose that there exists the smallest positive integer $n > 1$ such that $d^n(R) = \{0\}$. Then there exists $x_0 \in R - \{0\}$ such that $d^{n-1}(x_0) \neq 0$.

Now, let $y \in R$. If $d(y) = 0$, then there is nothing to prove. Suppose that $d(y) \neq 0$. Then there is a positive integer $k < n$ such that $d^k(y) \neq 0$ but $d^{k+1}(y) = 0$. By Theorem 3.4 (ii), we have

$$
\begin{aligned}
0 &= d^n\left(x_0 d^{k-1}(y)\right)\\
&\in \sum_{i=0}^{n} \binom{n}{i} d^{n-i}(\sigma^i(x_0))\sigma^{n-i}(d^i(d^{k-1}(y)))\\
&= d^n(x_0)\sigma^n(d^{k-1}(y)) + nd^{n-1}(\sigma(x_0))\sigma^{n-1}(d^k(y)) + \sum_{i=2}^{n} \binom{n}{i} d^{n-i}(\sigma^i(x_0))\sigma^{n-i}(d^{k+i-1}(y))
\end{aligned}
$$

$$= nd^{n-1}(\sigma(x_0))\sigma^{n-1}(d^k(y))$$
$$= nd^{n-1}(\sigma(x_0))d^k(\sigma^{n-1}(y)).$$

This implies $0 \in nd^{n-1}(\sigma(x_0))d^k(\sigma^{n-1}(y))$. □

Lemma 3.6. *Let R be a prime hyperring and let I be a non-zero hyperideal of R. Let d be a σ-derivation of R, where σ is an isomorphism of R.*

(i) *If $d(I) = \{0\}$, then $d(R) = \{0\}$.*
(ii) *If $rd(I) = \{0\}$ or $d(I)r = \{0\}$, for any $r \in R$, then $r = 0$ or $d(R) = \{0\}$.*

Proof. (i) Assume that $d(x) = 0$ for all $x \in I$. For all $x \in R$, and $u \in I$, we have

$$0 = d(ux) \in d(u)\sigma(x) + \sigma(u)d(x) = \sigma(u)d(x).$$

Hence, $\sigma(u)d(x) = 0$ for all $x \in R$, and $u \in I$. Since I is a non-zero hyperideal of $R, \sigma(I)$ is a non-zero hyperideal of R. Thus, we have $\sigma(I)d(x) = 0$ for all $x \in R$. By Lemma 2.14, it follows that $d(R) = \{0\}$.

(ii) Suppose that $rd(I) = \{0\}$, where $r \in R$. Let $x \in R$ and $y \in I$. Then $yx \in I$, and we have

$$0 = rd(yx) \in r(d(y)\sigma(x) + \sigma(y)d(x)) = rd(y)\sigma(x) + r\sigma(y)d(x).$$

This implies that $r\sigma(y)d(x) = 0$ for all $x \in R$ and $y \in I$. Since I is a non-zero hyperideal of $R, \sigma(I)$ is a non-zero hyperideal of R. Hence, $r\sigma(I)d(x) = \{0\}$ for all $x \in R$. By Lemma 2.14, we have $r = 0$ or $d(R) = \{0\}$.

An analogous argument establishes the case where $d(I)r = \{0\}$. This concludes the proof of the lemma. □

Corollary 3.7. *Let R be a prime hyperring and let d be a σ-derivation of R, where σ is an isomorphism of R. If $rd(R) = \{0\}$ or $d(R)r = \{0\}$, for any $r \in R$, then $r = 0$ or $d(R) = \{0\}$.*

Theorem 3.8. *Let I be a non-zero hyperideal of a 2-torsion free prime hyperring R. Let $\sigma : R \to R$ be an isomorphism. If d is a σ-derivation of R such that $d^2(I) = \{0\}$, then $d(R) = \{0\}$.*

Proof. Let d be a σ-derivation of R such that $d^2(x) = 0$ for all $x \in I$. For any $x, y \in I$, we have

$$\begin{aligned} 0 = d^2(xy) = d(d(xy)) \\ &\in d(d(x)\sigma(y) + \sigma(x)d(y)) \\ &= d^2(x)\sigma^2(y) + \sigma(d(x))d(\sigma(y)) + d(\sigma(x))\sigma(d(y)) + \sigma^2(x)d^2(y). \end{aligned}$$

It follows that

$$0 \in \sigma(d(x))d(\sigma(y)) + d(\sigma(x))\sigma(d(y)) = \sigma(d(x))\sigma(d(y)) + \sigma(d(x))\sigma(d(y)).$$

Since R is a 2-torsion free prime hyperring, $\sigma(d(x))\sigma(d(y)) = 0$ for all $x, y \in I$. Since $\sigma : R \to R$ is an isomorphism, $d(x)d(y) = 0$ for all $x, y \in I$. Then, by Lemma 3.6, we have $d(R) = \{0\}$. □

Theorem 3.9. *Let I be a non-zero hyperideal of a 2-torsion free prime hyperring R. Let $\sigma : R \to R$ be an isomorphism. Suppose that d_1 and d_2 are σ-derivations of R such that $d_1(I) \subseteq I$ and $d_2(I) \subseteq I$. If $d_1d_2(I) = \{0\}$, then either $d_1(R) = \{0\}$ or $d_2(R) = \{0\}$.*

Proof. Suppose that $d_1d_2(I) = \{0\}$. We have $d_1d_2(xy) = 0$ for all $x, y \in I$. Then

$$\begin{aligned}0 &\in d_1(d_2(x)\sigma(y) + \sigma(x)d_2(y))\\ &= d_1d_2(x)\sigma^2(y) + \sigma(d_2(x))d_1(\sigma(y)) + d_1(\sigma(x)\sigma(d_2(y)) + \sigma^2(x)d_1(d_2(y))\\ &= \sigma(d_2(x))d_1(\sigma(y)) + d_1(\sigma(x))\sigma(d_2(y)).\end{aligned}$$

Hence $0 \in \sigma(d_2(x))\sigma(d_1(y)) + \sigma(d_1(x))\sigma(d_2(y))$ for all $x, y \in I$. Since $d_2(I) \subseteq I, d_2(y) \in I$ for all $y \in I$. Replacing y by $d_2(y)$ in the above expression, we get

$$0 \in \sigma(d_2(x))\sigma(d_1(d_2(y))) + \sigma(d_1(x))\sigma(d_2^2(y)).$$

It follows that $\sigma(d_1(x)d_2^2(y)) = \sigma(d_1(x))\sigma(d_2^2(y)) = 0$ for all $x, y \in I$. Since $\sigma : R \to R$ is an isomorphism, $d_1(x)d_2^2(y) = 0$ for all $x, y \in I$. By using Lemma 3.6 (ii), we get $d_1(x) = 0$ for all $x \in I$ or $d_2^2(y) = 0$ for all $y \in I$. Again, by using Lemma 3.6 (i) and Theorem 3.8, we obtain that $d_1(R) = \{0\}$ or $d_2(R) = \{0\}$. □

Theorem 3.10. *Let R be a 2-torsion free prime hyperring and let d be a σ-derivation of R, where $\sigma : R \to R$ is an isomorphism. If $[d(x), y] = \{0\}$ for all $x, y \in R$ then $d(x) = 0$ for all $x \in R$ or R is commutative.*

Proof. Assume $[d(x), y] = \{0\}$ for all $x, y \in R$. Substituting x by xz, where $z \in R$, in the equation yields

$$\begin{aligned}0 \in [d(xz), y] &\subseteq [d(x)\sigma(z), y] + [\sigma(x)d(z), y]\\ &= d(x)[\sigma(z), y] + [d(x), y]\sigma(z) + \sigma(x)[d(z), y] + [\sigma(x), y]d(z).\end{aligned}$$

It follows that $\{0\} \subseteq d(x)[\sigma(z), y] + [\sigma(x), y]d(z)$.

Now, replacing x with $d(x)$ in the above expression, results in

$$\begin{aligned}\{0\} &\subseteq d^2(x)[\sigma(z), y] + [\sigma(d(x)), y]d(z)\\ &= d^2(x)[\sigma(z), y] + [d(\sigma(x)), y]d(z)\\ &= d^2(x)[\sigma(z), y].\end{aligned}$$

Therefore, we have $\{0\} \subseteq d^2(x)[\sigma(z), y]$ for all $x, y, z \in R$. Since σ is surjective, $\{0\} \subseteq d^2(x)[z, y]$ for all $x, y, z \in R$. Then, there exists $r \in [z, y]$ such that $d^2(x)r = 0$ for all $x \in R$. By Corollary 3.7, we get $d^2(R) = \{0\}$ or $r = 0$. If $d^2(R) = \{0\}$, then using Theorem 3.8, we obtain $d(x) = \{0\}$. On the other hand, if $r = 0$, then $0 \in [z, y]$ for all $y, z \in R$, which means R is commutative. □

4. Conclusion

This paper has presented a novel generalization of derivations on Krasner hyperrings, namely σ-derivations, and examined their fundamental properties. By analyzing their behavior in prime hyperrings and establishing commutativity criteria, we have demonstrated the significance of this new concept. Future research could explore applications of σ-derivations in areas such as hyperfield theory and hypermodule theory, as well as investigate further structural properties and classifications of Krasner hyperrings admitting σ-derivations.

Acknowledgement: This research was supported by International SciKU Branding (ISB), and Department of Mathematics, Faculty of Science, Kasetsart University, Thailand.

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