

ON THE CATENARIAN PROPERTY AND MINIMAL PRIME IDEALS OF AMALGAMATIONS OF RINGS

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ABSTRACT. In this paper, we investigate the transfer of the notion of catenarian rings to amalgamated duplication of a ring along an ideal; our result provides original illustrating examples. Then, we establish necessary and sufficient conditions under which $A \bowtie^f J$ is a MP -ring. By the way, we study the minimality and the height of special prime ideals of amalgamations of rings.

1. INTRODUCTION

All rings considered in this paper are commutative with unit and all subrings and homomorphisms are unital.

If R is a ring and I a proper ideal of R , the amalgamated duplication of R along I is known to be the subring of $R \times R$, with unity $(1, 1)$, defined by $R \bowtie I := \{(r, r + i) \mid r \in R \text{ and } i \in I\}$. It was first proposed by D'anna and Fontana [8], as an extension for idealization which was originally introduced in [12].

In recent years, there have been several investigations to find properties of the amalgamated duplication of a ring along a proper ideal [7, 8, 9, 3, 4, 10, 11]. On the other hand, a more general construction have been introduced and studied (cf. [5] and [9]), as follows :

let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. The subring, $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, \text{ and } j \in J\}$, of $A \times B$ is called *the amalgamation of A with B along J with respect to f* . Which in turn generalizes classical pullback constructions such as $D + M$, $A + XB[X]$, $A + XB[[X]]$, Nagata's idealization [12] and CPI extensions of Boisen and sheldon [1].

The aim of this paper is to complete the circle of investigations about these constructions. Particularly, we pose to study some properties related to minimal prime ideals and the catenarian ring property in amalgamations of commutative rings.

Let R be a commutative ring, if $P \in \text{Spec}(R)$, the height of P , denoted htP , is the maximum of lengths of the maximal chains of prime ideals ending at P .

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Then the Krull dimension of R , in short $\dim R$, is defined to be the maximum of heights of prime ideals of R , that is $\dim R = \max\{ht P; P \in \text{Spec}(R)\}$. The concept of “Krull dimension” is a remarkable tool for the classification of rings by means of their prime ideals. Particularly, it is known that for each $P \subseteq Q$ in $\text{Spec}(R)$, $ht Q/P \leq ht Q - ht P$. If equality holds for all pairs of prime ideals of R , with R is locally finite dimensional, then R is said to be catenarian [2]. The polynomial ring $R[X]$ needs not be catenarian even when R is. Thus R is said to be universally catenarian if $R[X_1, \dots, X_n]$ is catenarian for each integer $n \geq 0$.

On the other hand, a commutative ring R is called a MP -ring if every prime ideal of R contains only a unique minimal prime ideal of R .

In Section 1, we establish necessary and sufficient conditions for, an amalgamated duplication, $A \bowtie^f J$ to be catenarian. It allows us also to construct original examples of non-valuation catenarian rings. Section 2 devoted to the transference of the property of being a MP -ring among A , B and $A \bowtie^f J$. Also, we are led to study the minimality of the prime ideal $J_0 := 0 \times J$ of $A \bowtie^f J$, when A is a domain. As a consequence, we generalize [8, Proposition 3.1]. Also we compute a lower bound of the height of $\overline{J}^f$, when J is a prime ideal of B .

2. AMALGAMATED DUPLICATIONS AND THE CATENARIAN RING PROPERTY

A locally finite dimensional ring R is catenarian if for each pair $P \subset Q$ of primes, all the saturated chains of prime ideals of R between P and Q have the same finite length. Any ring of dimension ≤ 2 is catenarian.

An integral domain R , is catenarian if, and only if, for any consecutive prime ideals $P \subset Q$ in R , $ht Q = ht P + 1$. Moreover, a commutative ring R is catenarian if and only if R/P is catenarian for each minimal prime ideal P of R .

Next, we give necessary and sufficient conditions for, an amalgamated duplication, $A \bowtie I$ to be catenarian.

Proposition 2.1. *Let A be a ring and I an ideal of A . If $A \bowtie I$ is a catenarian ring then so is A . Conversely, if A is a catenarian ring and $I \subset \text{Nil}(A)$, then $A \bowtie I$ is a catenarian ring.*

To prove this Proposition we recall the following lemma from [8]:

Lemma 2.2. [8, Proposition 2.2] *Let I be an ideal of a ring A . Let P be a prime ideal of A and set*

$$P_0 = \{(p, p+i) \mid p \in P, i \in I \cap P\},$$

$$P_1 = \{(p, p+i) \mid p \in P, i \in I\},$$

$$P_2 = \{(p+i, p) \mid p \in P, i \in I\}.$$

(a) *If $I \subseteq P$, then $P_0 = P_1 = P_2$ is a prime ideal of $A \bowtie I$ and it is the unique prime ideal of $A \bowtie I$ lying over P .*

(b) *If $I \not\subseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$, and P_1 and P_2 are the only prime ideals of $A \bowtie I$ lying over P .*

Furthermore, every prime ideal of $A \bowtie I$ has the form of P_1 or the form of P_2 for some prime P of the ring A .

Proof. of Proposition 2.1

Assume that $A \bowtie I$ is a catenarian ring. Since catenarity is stable under quotient, by [7, Proposition 3.1], A is a catenarian ring.

Conversely, assume that A is a catenarian ring and $I \subset \text{Nil}(A)$. Let $P' \subsetneq Q'$ be prime ideals of $A \bowtie I$ and let $P' = P'_0 \subsetneq P'_1 \subsetneq \dots \subsetneq P'_m = Q'$ be a saturated chain of primes between P' and Q' . Since $I \subset \text{Nil}(A)$, then, by the previous Lemma, $P' := P \bowtie I = \{(p, p+i) \mid p \in P, i \in I\}$ and $Q' := Q \bowtie I = \{(q, q+i) \mid q \in Q, i \in I\}$ for some $P, Q \in \text{Spec}(A)$. It is easy to see that $P \subsetneq Q$, and by assumption all the saturated chains of primes between P and Q have the same length, say n . So let $P = Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_n = Q$ be a saturated chain of primes between P and Q . Hence $P' = Q'_0 \subsetneq Q'_1 \subsetneq \dots \subsetneq Q'_n = Q'$, where $Q'_k = Q_k \bowtie I$ for every $k \in \{0, 1, \dots, n\}$, is a chain of prime ideals between P' and Q' and so $m \geq n$.

On the other hand, with the help of Lemma 2.2, we have $P'_k = P_k \bowtie I$ for every $k \in \{1, \dots, m-1\}$ and thus $P \subsetneq P_1 \subsetneq \dots \subsetneq P_{m-1} \subsetneq Q$. Then $m \leq n$ and so we are done. $\square$

Recall that a ring A is called a valuation ring if the set of all ideals of A is totally ordered by set inclusion. This notion extends the notion of valuation domains to a ring with zero-divisors. A valuation ring is always catenarian.

As an illustration of Proposition 2.1, we give an example of a non-valuation catenarian ring.

Example 2.3. Let A be a valuation ring which is not integral, and P its minimal prime ideal. Then $A \bowtie P$ is a non-valuation catenarian ring.

Proof. Since A is a valuation ring (and also a catenarian ring) and P its minimal prime ideal, then $\text{Nil}(A) = P$, and by Proposition 2.1, $A \bowtie P$ is a catenarian ring. By [10, Example 3.2.5], $A \bowtie P$ is not a valuation ring ($P \neq (0)$). $\square$

3. AMALGAMATED DUPLICATION AND MP -RINGS

Recall that a ring R is called a MP -ring if every prime ideal of R contains a unique minimal prime ideal of R . The main purpose of this section is to provide necessary and sufficient conditions for an amalgamation of rings to be a MP -ring.

We start by a description of the minimal prime ideals of $A \bowtie^f J$ established by D'Anna et al:

Lemma 3.1. [6, Corollary 2.8] *Let A and B be two rings, J an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. Set $J_0 := \{0\} \times J$ and $\Lambda := \bigcup_{Q \in \text{Spec}(B) \setminus V(J)} V(f^{-1}(Q + J))$. The following properties hold.*

(1) *The map $Q \rightarrow \overline{Q}^f$ establishes a homeomorphism of $\text{Min}(B) \setminus V(J)$ with $\text{Min}(A \bowtie^f J) \setminus V(J_0)$.*

(2) *The map $P \rightarrow P^f$ establishes a homeomorphism of $\text{Min}(A) \setminus \Lambda$ with $\text{Min}(A \bowtie^f J) \cap V(J_0)$.*

Theorem 3.2. *Let A and B be two rings, let J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. With the same notations as in Lemma 3.1, assume that for each prime ideal P of A , $P \notin \Lambda$, then:*

$A \bowtie^f J$ is a MP -ring if, and only if, A is a MP -ring and for each prime ideal Q of B , not containing J , B_Q has at most one minimal prime ideal.

Proof. Assume that $A \bowtie^f J$ is a MP -ring. If there exists a prime ideal $Q \in \text{Spec}(B) \setminus V(J)$ such that B_Q has two minimal prime ideals $Q_1 B_Q$ and $Q_2 B_Q$, where Q_1 and Q_2 are minimal prime ideals of B contained in Q . Obviously, Q_1 and Q_2 do not contain J . Thus, $\overline{Q_1}^f$ and $\overline{Q_2}^f$ are minimal prime ideals of $A \bowtie^f J$ contained in $\overline{Q}^f$, a contradiction.

Now, If A is not a MP -ring, so there exists a prime ideal P of A that contains two minimal prime ideals $P_{\min 1}$ and $P_{\min 2}$ of A . Since $P_{\min 1} \notin \Lambda$ and $P_{\min 2} \notin \Lambda$, $P_{\min 1}'^f$ and $P_{\min 2}'^f$ are minimal prime ideals of $A \bowtie^f J$. Hence, P'^f contains two minimal prime ideals of $A \bowtie^f J$.

Conversely, Let $I \in \text{Spec}(A \bowtie^f J)$. To avoid a trivial case, we assume that I is not minimal. Two cases are then possible:

Case 1. $I = \overline{Q}^f$, with $Q \in \text{Spec}(B) \setminus V(J)$.

By assumption B_Q has at most one minimal prime ideal $Q_{\min} B_Q$ of B such that $Q_{\min} \subset Q$, and it is clear that $Q_{\min}$ doesn't contain J . hence, $\overline{Q_{\min}}^f$ is the unique minimal prime ideal of $A \bowtie^f J$ contained in I , keeping in mind that a prime ideal of the type P'^f cannot be contained in a prime of the type $\overline{Q}^f$.

Case 2. $I = P'^f$ with $P \in \text{Spec}(A)$. Since A is a MP -ring, there exists a unique minimal prime ideal $P_{\min}$ of A such that $P_{\min} \subset P$. Since $P_{\min} \notin \Lambda$, $P_{\min}'^f$ is a minimal prime ideal of $A \bowtie^f J$. Since $P \notin \Lambda$, no prime ideal of the type $\overline{Q}^f$ can be contained in P'^f and so $P_{\min}'^f$ is the unique minimal prime ideal contained in I .

It follows that every prime ideal of $A \bowtie^f J$ contains a unique minimal prime ideal of $A \bowtie^f J$. Hence, $A \bowtie^f J$ is a MP -ring and the proof is complete. $\square$

4. MINIMALITY OF J_0 AND THE HEIGHT OF $\overline{J}^f$

We start by a useful lemma, whose proof is straightforward.

Lemma 4.1. *Let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . Let $P \in \text{Spec}(A)$ and $Q \in \text{Spec}(B) \setminus V(J)$, then $\overline{Q}^f \subset P'^f$ if and only if $f^{-1}(Q + J) \subset P$.*

Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . Set $Y_{(f, J)} := \{Q \in \text{Spec}(B) \setminus V(J) \text{ such that } f^{-1}(Q + J) = \{0\}\}$.

Proposition 4.2. *Let A be a domain and let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . Then the following assertions are equivalent:*

- (i) $J_0 := \{0\} \times J$ is a minimal prime ideal of $A \bowtie^f J$;
- (ii) $Y_{(f, J)} = \emptyset$.

Proof. Since A is a domain, $J_0 (= (0)'^f)$ is a prime ideal of $A \bowtie^f J$.

(ii) $\Rightarrow$ (i) It is obvious that no prime of the form P'^f can be contained in J_0 . Now, suppose that there exists a prime ideal $Q \in \text{Spec}(B) \setminus V(J)$ such that $\overline{Q}^f \subset J_0$. By Lemma 4.1, $f^{-1}(Q + J) \subset \{0\}$ and hence $f^{-1}(Q + J) = \{0\}$. That is $Q \subset Y_{(f, J)}$, a contradiction.

(i) $\Rightarrow$ (ii) Suppose that $Y_{(f, J)} \neq \emptyset$. So there exists $Q \in \text{Spec}(B) \setminus V(J)$ such that $f^{-1}(Q + J) = \{0\}$. Hence, by Lemma 4.1, $\overline{Q}^f \subset (0)^{f'} = J_0$ and so J_0 is not minimal. $\square$

The previous result generalizes [7, Proposition 3.1], which we state as:

Corollary 4.3. *Let A be a domain and I a non-zero ideal of A . $\{0\} \times I$ is a minimal prime ideal of the simple amalgamation $A \bowtie I$.*

Proof. It is clear that $Y_{(id_A, I)} = \emptyset$, and the result follows immediately by applying Proposition 4.2. $\square$

In what follows, we give a lower bound for the height of $\overline{J}^f$, in the case J is a prime ideal of B .

Proposition 4.4. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a prime ideal of B . Then, $ht_{A \bowtie^f J} \overline{J}^f \geq ht_B J$. The equality holds if $f^{-1}(J) \in \text{Min}(A)$.*

Proof. $J \in \text{Spec}(B)$ implies $\overline{J}^f = (f^{-1}(J))^{f'} \in \text{Spec}(A)$. Now let $\overline{Q}_0 \subsetneq \overline{Q}_1 \subsetneq \dots \subsetneq \overline{Q}_{n-1} \subsetneq \overline{Q}_n = J$ be a saturated chain of prime ideals in B , then $\overline{Q}_0 \subsetneq \overline{Q}_1 \subsetneq \dots \subsetneq \overline{Q}_{n-1} \subsetneq \dots \subseteq \overline{J}^f = (f^{-1}(J))^{f'}$ is not necessarily a saturated chain of prime ideals of $A \bowtie^f J$. Since it may happen that there exists a prime ideal of the form $P^{f'}$ between $\overline{Q}_{n-1}$ and $(f^{-1}(J))^{f'}$. then, $ht_{A \bowtie^f J} \overline{J}^f \geq ht_B J = n$.

The extra assumption $f^{-1}(J) \in \text{Min}(A)$ ensures the saturation of the chain $\overline{Q}_0 \subsetneq \overline{Q}_1 \subsetneq \dots \subsetneq \overline{Q}_{n-1} \subsetneq \overline{J}^f = (f^{-1}(J))^{f'}$. In fact, suppose that there exists a prime ideal $P^{f'}$ of $A \bowtie^f J$ between $\overline{Q}_{n-1}$ and $(f^{-1}(J))^{f'}$, so $P \subsetneq f^{-1}(J)$ which contradicts the fact that $f^{-1}(J)$ is a minimal prime ideal of B and we are done. $\square$

Remark 4.5. If A is a domain, then $ht_{A \bowtie^f J} \overline{J}^f = ht_B J$ if, and only if, $f^{-1}(J) = \{0\}$.

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