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APPLICATIONS OF $m-$FOLD$-p-$VALENT SYMMETRIC FUNCTIONS ASSOCIATED WITH A GENERALIZED RIEMANN$-$LIOUVILLE FRACTIONAL INTEGRAL OPERATOR

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Abstract. This work addresses a new definition of a fractional integral operator that acts on functions which are $m-$fold$-p-$valent symmetric in the unit disk. We provide examples of the considered geometric functions of the $m-$fold$-p-$valent symmetric type. Also, using the Hadamard product, we prove several results, including the fractional integral operator associated with the well-known Fox-Wright function. Finally, some properties based on the Briot-Bouquet differential subordination are investigated.

1. Introduction

When we discuss analytic functions, we use the term "m-fold symmetry" to describe functions that display a particular kind of rotational invariance in the complex plane. If, for every z in its domain, the value of an analytic function $f(z) = f(e^{2\pi i/m}z)$, $m \in \mathbb{N} := \{1, 2, 3, ...\}$ remains unchanged when the input z is rotated by an angle of $2\pi/m$ radians, then it is said to have m-fold symmetry. For example, $f(z) = z^\kappa$ is $m-$fold whenever κ is a multiple of m. Another concept in this investigation is a $p-$valent, also called a $p-$fold, analytic function in complex analysis, which is a function that, when multiplicities are taken into account, requires on each value exactly p times in a given domain where it is analytic (holomorphic). These functions expand the idea of univalent functions when the equation $f(z) = u$ has exactly p elements of a domain $\mathscr{D}$ placed at every specific value of the range, counting multiplicities. For each u in the range of f, an analytic function $f(z)$ generated on a domain $\mathscr{D} \subset$ C is said to be $p-$valent. Fractional integral operators of analytic functions offer a foundation for exploring classical integral operators to non-integer orders, providing valuable insights for theoretical and applied mathematics. Numerous areas of mathematics, including fractional calculus, complex dynamics, and the study of fractional iterations of analytic functions, employ fractional integral operators of analytic functions(see [1, 2, 4, 5, 7, 11, 19]). We will combine these concepts to study a

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class of analytic functions acting on the fractional integral operator. This paper introduces a new fractional integral operator that includes functions of type $m-$fold$-p-$valent symmetric in $\mathscr{D}$. For example, we present several prominent geometric functions of the $m-$fold$-p-$valent symmetric correlated with this new operator. Furthermore, we investigate the substances of this operator in various geometric contexts, including its applications with the widely known *Fox-Wright* function and the subordination of the Briot-Bouquet differential.

2. The class of $m-$fold symmetric

Let the set of all functions defined analytically on the unit disk $\mathscr{D} = \{z \in \mathrm{C} : |z| < 1\}$ symbolized by $H(\mathscr{D})$. And $\mathscr{A}$ indicates the subclass of $H(\mathscr{D})$ that has a Taylor-Maclaurin series expansion as follows:

$$f(z) = z + \sum_{w=2}^{\infty} a_w z^w, \quad (z \in \mathscr{D}) \tag{2.1}$$

while $\mathscr{S}$ is said to be the subclass of $\mathscr{A}$ containing the univalent functions in $\mathscr{D}$ and satisfying (2.1). Let $\mathscr{S}(p)$, $(p = 1, 2, 3, \cdots)$ be the class of p-valent functions in $\mathscr{D}$ satisfies the following structure:

$$f(z) = z^p + \sum_{w=1}^{\infty} a_{w+p} z^{w+p} \tag{2.2}$$

where $z \in \mathscr{D}$. A domain $\mathscr{D}$ is said to be m-fold symmetric if a rotation of $\mathscr{D}$ about the origin through an angle $2\pi/m$, $(m \in \mathbb{N})$ carries $\mathscr{D}$ on itself. For $m \in \mathbb{N}$, a function f holomorphic in $\mathscr{D}$ is said to be $m-$fold symmetric if

$$f(e^{i(2\pi/m)} z) = e^{i(2\pi/m)} f(z)$$

this demonstrates that each function $f \in \mathscr{S}$, can be presented as

$$g(z) = [f(z^m)]^{1/m} = z \left(\frac{f(z^m)}{z^m} \right)^{1/m}$$

which is univalent and maps the unit disk into a region with $m-$fold symmetry. let $\mathscr{S}_m$ denote the subclass of $\mathscr{S}$ whose functions are normalized defined by the $m-$fold symmetric formula:

$$\chi_m(z) = z + \sum_{w=1}^{\infty} a_{mw+1} z^{mw+1}, \quad (z \in \mathscr{D}) \tag{2.3}$$

and represents the class of $m-$fold symmetric univalent functions in $\mathscr{D}$. Note that the class $\mathscr{S}$ functions are $1-$fold symmetric.

Fig. 1: Magnitude of $\chi_m(z)$ in the Unit Disk

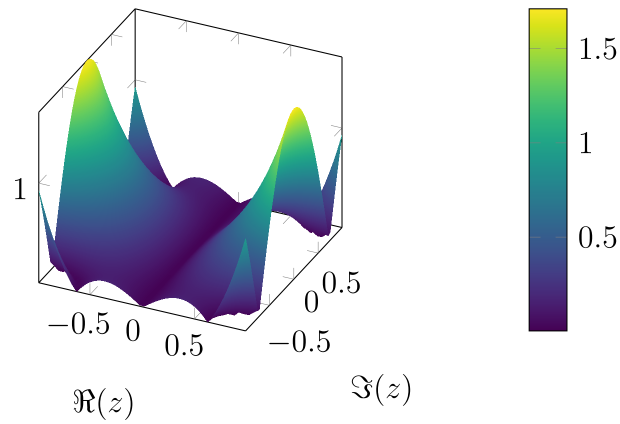

Figure 1 illustrates the magnitude $|\chi_m(z)|$ in $\mathscr{D}$, where the coefficients are stated as $a_1 = 1$, $a_2 = 0.5$, along with $a_3 = 0.25$, etc. The sum has been simplified at $w = 3$.

Example 2.1. For $m \geq 1$, let $z \in \mathscr{D}$ and

$$T_m(z) = \frac{1 + z^m}{1 - z^m} \tag{2.4}$$

in the class $\mathscr{S}_m$.

Proof. To prove that the function T_m is univalent (one−to−one) in $\mathscr{D}$, we need to prove that $T_m(z_1) = T_m(z_2) \Rightarrow z_1 = z_2$. By assuming that there exist z_1 and $z_2 \in \mathscr{D}$, such that

$$\frac{1 + z_1^m}{1 - z_1^m} = \frac{1 + z_2^m}{1 - z_2^m}.$$

After simplifying, we have

$$2z_1^m = 2z_2^m \Rightarrow z_1^m = z_2^m,$$

since $m \geq 1$, we get $z_1 = z_2 e^{i(2q\pi/m)}$, for some integer q. To ensure uniformity, we check the derivative of the function T_m using the quotient rule:

$$\{T_m(z)\}' = \frac{mz^{m-1}(1 - z^{2m})}{(1 - z^m)^2},$$

since $|z| < 1$, we obtain $|z^m| < 1$, which means $1 - z^{2m} > 0$ and $(1 - z^m)^2 > 0$. Also, for $z \neq 0$, we have $mz^{m-1} \neq 0$. Then, the function $\{T_m(z)\}' \neq 0$ for each $z \in \mathscr{D} \setminus \{0\}$. Consequently, $T_m(z)$ is a conformal map that is (one−to−one and analytic) in $\mathscr{D}$. Since,

$$\begin{aligned}\mathscr{R}\{T_m(z)\} =& \mathscr{R}\left\{\frac{1+z^m}{1-z^m}\right\} \\ =& \frac{1}{2}\left[\frac{1+z^m}{1-z^m}+\overline{\left(\frac{1+z^m}{1-z^m}\right)}\right] \\ =& \frac{1}{2}\left(\frac{1+z^m}{1-z^m}+\frac{1+\overline{z^m}}{1-\overline{z^m}}\right) \\ =& \frac{1-|z^m|^2}{|1-z^m|^2}>0\end{aligned}$$

for $|z|<1$ and $T_m(0)=1$. It is clear that T_m introduces a special case of the Janowski function combined with symmetry m, as follows:

$$\begin{aligned}T_m(z) :=& \frac{1+z^m}{1-z^m} \\ =& 1+2z^m+2z^{2m}+\ldots. \\ =& 1+2\sum_{w=1}^{\infty} z^{mw} \end{aligned} \tag{2.5}$$

where $z \in \mathscr{D}$. □

For $p, m \in \mathbb{N}$, let function χ_m of the form:

$$\chi_m(z) = z^p + \sum_{w=1}^{\infty} a_{mw+p} z^{mw+p}, \quad (z \in \mathscr{D}) \tag{2.6}$$

is said to be in the subclass $\mathscr{S}_m(p)$ of $\mathscr{S}(p)$ whose function is presented by $m-$fold symmetric series, where $\mathscr{S}(p)$ is the class of $p-$valent functions of the form (2.2). This extension function in the class $\mathscr{S}_m(p)$ is widely studied by Koepf [8] and Mishra and Choudhury [16]. The Hadamard product (or convolution) of two functions belonging to the class $\mathscr{S}_m(p)$ is given by

$$(\chi_m * \hbar_m)(z) = z^p + \sum_{w=1}^{\infty} a_{mw+p} b_{mw+p} z^{mw+p}, \quad (z \in \mathscr{D}) \tag{2.7}$$

where

$$\hbar_m(z) = z^p + \sum_{w=1}^{\infty} b_{mw+p} z^{mw+p} \in \mathscr{S}_m(p)$$

and χ_m is defined in (2.6).

3. Fractional integration operator

This section investigates a new definition of a generalized fractional operator specified by the integral form of functions defined in the class $\mathscr{S}_m(p)$ in $\mathscr{D}$. Motivated by this new definition, we study some of its univalence properties and given examples associated with related functions. Some applications involving special functions in class $\mathscr{S}_m(p)$ are also considered.

Definition 3.1. For $z \in \mathscr{D}$, the new generalized fractional integral operator $\aleph_p^{\varsigma,\tau}$ of function $\chi_m \in \mathscr{S}_m(p)$, is defined by:

$$\aleph^{\varsigma}_{\tau,p}\chi_m(z) = \binom{\frac{p}{\tau+1}+\varsigma}{\frac{p}{\tau+1}} \frac{\varsigma(1+\tau)}{z^{1+\tau}} \int_0^z \left(1-\frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1} s^{\tau}\chi_m(s)d(s)$$
$$(1>\varsigma\geq 0,\ \tau>-1, p\geq 1,\ m\geq 1) \tag{3.1}$$

where $\binom{\mu}{\vartheta} = \frac{(\mu)!}{(\vartheta)!(\mu-\vartheta)!}$.

Under the parameter's conditions in Definition 3.1, the fractional integral operator $\aleph^{\varsigma}_{\tau,p}\chi_m(z) : \mathscr{S}_m(p) \to \mathscr{S}_m(p)$, and the image of the power series

$$\chi_m(z) = z^p + \sum_{w=1}^{\infty} a_{mw+p} z^{mw+p}$$

have the following form

$$\aleph^{\varsigma}_{\tau,p}\chi_m(z) = z^p + \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)} \sum_{w=1}^{\infty} \frac{\Gamma(\frac{mw+p}{\tau+1}+1)}{\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)} a_{mw+p} z^{mw+p}, \quad (z\in\mathscr{D}) \tag{3.2}$$

and

$$\aleph^{0}_{0,1}\chi_1(z) = z + \sum_{w=1}^{\infty} a_{w+1} z^{w+1}.$$

Remark 3.2. The fractional integral operator $\aleph^{\varsigma}_{\tau,p}$ fulfills the following recurrence condition:

$$z(\aleph^{\varsigma}_{\tau,p}\chi_m(z))' = \big(p+\varsigma(1+\tau)\big)\aleph^{\varsigma-1}_{\tau,p}\chi_m(z) - \varsigma(1+\tau)\aleph^{\varsigma}_{\tau,p}\chi_m(z). \tag{3.3}$$

3.1. **Some Examples.** The geometric investigation of univalent functions in $\mathscr{D}$ is essential in comprehending complex analysis, mathematical physics, engineering applications, and pure mathematical research. Their effects show deep outcomes in function theory, conformal mapping, and affiliated fields. Here, we consider studying the geometric behavior of a fractional integral operator $\aleph^{\varsigma}_{\tau,p}$ with several $m-$fold$-p-$valent symmetric functions belonging to the class $\mathscr{S}_m(p)$.

Example 3.3. For $p,\ m \geq 1$, let $\psi_m \in \mathscr{S}_m(p)$

$$\begin{aligned}\psi_m(z) &= z^p e^{z^m}\\ &= z^p + z^{m+p} + \frac{1}{2}z^{2m+p} + \frac{1}{6}z^{3m+p} + \frac{1}{24}z^{4m+p} + \cdots\\ &= z^p + \sum_{w=1}^{\infty}\frac{z^{mw+p}}{w!}, \quad (z\in\mathscr{D})\end{aligned} \tag{3.4}$$

we define

$$\aleph^{\varsigma}_{\tau,p}\psi_m(z) = z^p + \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)} \sum_{w=1}^{\infty} \frac{\Gamma(\frac{mw+p}{\tau+1}+1)}{\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)w!} z^{mw+p}. \tag{3.5}$$

Proof. For $z \in \mathscr{D}$, let the fractional integral operator $\aleph^{\varsigma}_{\tau,p}$ define in (3.1) of function ψ_m given by (3.4) presented as

$$\aleph^{\varsigma}_{\tau,p}\psi_m(z) = \binom{\frac{p}{\tau+1}+\varsigma}{\frac{p}{\tau+1}} \frac{\varsigma(1+\tau)}{z^{1+\tau}} \int_0^z \left(1 - \frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1} s^{\tau}\psi_m(s)d(s). \tag{3.6}$$

By setting $\sigma = \frac{s^{1+\tau}}{z^{1+\tau}}$ in (3.6) so that $z^{1+\tau}d\sigma = (1+\tau)s^{\tau}ds$, hence we have

$$\aleph^{\varsigma}_{\tau,p}\psi_m(z) = \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)\Gamma(\varsigma)} \int_0^1 (1-\sigma)^{\varsigma-1}\,\psi_m(z^p\sigma^{\frac{p}{1+\tau}})d(\sigma) \tag{3.7}$$

which is equivalent to

$$\aleph^{\varsigma}_{\tau,p}\psi_m(z) = \frac{1}{B(\frac{p}{\tau+1}+1,\varsigma)} \int_0^1 (1-\sigma)^{\varsigma-1}\,\psi_m(z^p\sigma^{\frac{p}{1+\tau}})d(\sigma). \tag{3.8}$$

Then by utilizing (3.4) in (3.8), we deduce that

$$\aleph^{\varsigma}_{\tau,p}\psi_m(z) = \frac{1}{B(\frac{p}{\tau+1}+1,\varsigma)} \left(\int_0^1 (1-\sigma)^{\varsigma-1}\left[z^p\sigma^{\frac{p}{1+\tau}} + \sum_{w=1}^{\infty}\frac{z^{mw+p}}{w!}\sigma^{\frac{mw+p}{1+\tau}}\right]d(\sigma)\right),$$

now applying the Beta function

$$B(t,c) = \int_0^1 \varrho^{t-1}(1-\varrho)^{c-1}d\varrho = \frac{\Gamma(t)\Gamma(c)}{\Gamma(t+c)}. \tag{3.9}$$

Therefore, the desired result in (3.5) is achieved. $\square$

By following the same methods in proving Example 3.3, we have:

Example 3.4. For $p,\ m \geq 1$ and $0 < \frac{p}{m} < 1$, let $\Xi_m \in \mathscr{S}_m(p)$ given by

$$\begin{aligned}\Xi_m(z) &= \frac{z^p}{(1-z^m)^{p/m}} \\ &= z^p + \frac{p}{m}z^{m+p} + \frac{p(p+m)}{2m^2}z^{2m+p} + \frac{p(p+2m)(p+m)}{6m^3}z^{3m+p} + \ldots \\ &= z^p + \sum_{w=1}^{\infty}\frac{z^{mw+p}}{w!}\prod_{n=0}^{w-1}\left(\frac{p}{m}+n\right),\ (z \in \mathscr{D})\end{aligned} \tag{3.10}$$

we define

$$\aleph^{\varsigma}_{\tau,p}\Xi_m(z) = \binom{\frac{p}{\tau+1}+\varsigma}{\frac{p}{\tau+1}} \frac{\varsigma(1+\tau)}{z^{1+\tau}} \int_0^z \left(1 - \frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1} \Xi_m(s)s^{\tau}d(s) \tag{3.11}$$

and by applying (3.10) in (3.11), we obtain

$$\aleph^{\varsigma}_{\tau,p}\Xi_m(z) = z^p + \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)} \sum_{w=1}^{\infty} \frac{\Gamma(\frac{mw+p}{\tau+1}+1)\prod_{n=0}^{w-1}\left(\frac{p}{m}+n\right)}{\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)\ w!} z^{mw+p}. \tag{3.12}$$

Example 3.5. For p, $m \geq 1$ and $0 < \frac{2p}{m} < 1$, let $\Omega_m \in \mathscr{S}_m(p)$ defined by

$$\begin{aligned}\Omega_m(z) &= \frac{z^p}{(1-z^m)^{2p/m}} \\ &= z^p + \frac{2p}{m} z^{m+p} + \frac{p(m+2p)}{m^2} z^{2m+p} + \frac{2p(m+p)(m+2p)}{3m^3} z^{3m+p} + ... \\ &= z^p + \sum_{w=1}^{\infty} \frac{z^{mw+p}}{w!} \prod_{n=0}^{w-1} \left(\frac{2p}{m} + n\right), \ (z \in \mathscr{D}) \end{aligned} \tag{3.13}$$

we define

$$\aleph^{\varsigma}_{\tau,p} \Omega_m(z) = \binom{\frac{p}{\tau+1} + \varsigma}{\frac{p}{\tau+1}} \frac{\varsigma(1+\tau)}{z^{1+\tau}} \int_0^z \left(1 - \frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1} s^{\tau} \Omega_m(s) d(s) \tag{3.14}$$

and by involving (3.13) in (3.14), we get

$$\aleph^{\varsigma}_{\tau,p} \Omega_m(z) = z^p + \frac{\Gamma(\frac{p}{\tau+1} + \varsigma + 1)}{\Gamma(\frac{p}{\tau+1} + 1)} \sum_{w=1}^{\infty} \frac{\Gamma(\frac{mw+p}{\tau+1} + 1) \prod_{n=0}^{w-1} \left(\frac{2p}{m} + n\right)}{\Gamma(\frac{mw+p}{\tau+1} + \varsigma + 1) \ w!} z^{mw+p}. \tag{3.15}$$

3.2. **Special function.** Researchers continue to focus on the *Fox-Wright* function ${}_n\Psi_s$ and its geometrical characteristics within a unit disk $\mathscr{D}$ (see [13, 9, 12, 20]). Studying some geometrical characteristics, such as convexity and starlikeness, requires an understanding of how to manipulate this function. The basic definition of this function is expressed as follows:
For complex parameters

$$\kappa_1 \dots \kappa_n \ (\frac{\kappa_i}{a_i} \neq \{0,1,2,..\}; \ a_i > 0 \text{ for all } i = \{1, \dots, n\})$$

and

$$\varepsilon_1 \dots \varepsilon_s \ (\frac{\varepsilon_i}{b_i} \neq \{0,1,2,..\}; \ b_i > 0 \text{ for all } i = \{1, \dots, s\}).$$

In the following structure, the function is defined:

$${}_n\Psi_s \left[\begin{array}{c} (\kappa_1, a_1), \dots, (\kappa_n, a_n) \\ (\varepsilon_1, b_1), \dots, (\varepsilon_s, b_s) \end{array} |z \right] = \sum_{w=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma(\kappa_i + a_i w)}{\prod_{i=1}^{s} \Gamma(\varepsilon_i + b_i w)} \frac{z^w}{w!}. \tag{3.16}$$

Based on the known asymptotic of the *Euler Gamma* function, which occurs across the complex plane C, the series at the right-hand side of (3.16) converges when

$$\mathfrak{W} = \sum_{i=1}^{s} b_i - \sum_{i=1}^{n} a_i > -1$$

and (3.16) converges for every bounded $|z| < c$ and $|z| = c$ if $\mathfrak{W} = -1$, provided that $\mathscr{R}(d) > \frac{1}{2}$, (see [13] for further information) where

$$c := \left(\prod_{i=1}^{n} a_i^{-a_i}\right) \times \left(\prod_{i=1}^{s} b_i^{b_i}\right), \ d := \left(\sum_{i=1}^{s} \varepsilon_i - \sum_{i=1}^{n} \kappa_i\right) + \left(\frac{n-s}{2}\right).$$

The series (3.16) is not a member of the class $\mathscr{A}$, as the definition above demonstrates. The *Fox-Wright* function that we investigate has the following normalization definition:

$$ {}_n\Psi_s\left[\begin{array}{c}(\kappa_1,a_1),\ldots,(\kappa_n,a_n)\\(\varepsilon_1,b_1),\ldots,(\varepsilon_s,b_s)\end{array}|z\right]=\frac{\prod_{i=1}^{s}\Gamma(\varepsilon_i)}{\prod_{i=1}^{n}\Gamma(\kappa_i)}\sum_{w=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma(\kappa_i+a_iw)}{\prod_{i=1}^{s}\Gamma(\varepsilon_i+b_iw)}\frac{z^{w+1}}{w!}. \quad (3.17)$$

If we set $a_1=\cdots=a_n=1$ and $b_1=\cdots=b_s=1$ in (3.17), then we get:

$$ {}_nF_s\left[\begin{array}{c}\kappa_1,\ldots,\kappa_n\\\varepsilon_1,\ldots,\varepsilon_s\end{array}|z\right]=\frac{\prod_{i=1}^{s}\Gamma(\varepsilon_i)}{\prod_{i=1}^{n}\Gamma(\kappa_i)}\sum_{w=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma(\kappa_i+w)}{\prod_{i=1}^{s}\Gamma(\varepsilon_i+w)}\frac{z^{w+1}}{w!} \quad (3.18)$$

where ${}_nF_s$ is the generalized hypergeometric function.

To study and discuss certain results related to the function (3.17), we used the Hadamard product method. In what follows, the univalence behavior of the fractional integral operator of a function defined in $\mathscr{S}_m(p)$ will be considered.

Theorem 3.6. Let $\chi_m(z)\in\mathscr{S}_m(p)$ and

$$\Theta_m(z)=\chi_m(z)*\Xi_m(z)$$

where $\Xi_m(z)$ is given in (3.10). Then

$$z^{-p}\aleph_{\tau,p}^{\varsigma}\Theta_m(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,\frac{p}{m}),(1+\frac{p}{\tau+1},\frac{m}{\tau+1})\\(1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|\cdot\right]*\chi_m\right)(z),\ \ (z\in\mathscr{D}) \quad (3.19)$$

where

$$\begin{aligned}{}_2\Psi_1\left[\begin{array}{c}(1,\frac{p}{m}),(1+\frac{p}{\tau+1},\frac{m}{\tau+1})\\(1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|\cdot\right](z)&={}_2\Psi_1\left[\begin{array}{c}(1,\frac{p}{m}),(1+\frac{p}{\tau+1},\frac{m}{\tau+1})\\(1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|z^m\right]\\&=\frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)}\sum_{w=0}^{\infty}\frac{\Gamma(\frac{p}{m}+w)\Gamma(\frac{mw+p}{\tau+1}+1)z^m}{\Gamma(\frac{p}{m})\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)(w)!},\end{aligned}$$

is the normalization *Fox-Wright* function.

Proof. Let $\Theta_m(z)\in\mathscr{S}_m(p)$ given in (3.10) and the fractional integral operator $\aleph_{\tau,p}^{\varsigma}$ defined as

$$\begin{aligned}\aleph_{\tau,p}^{\varsigma}\Theta_m(z)=&\binom{\frac{p}{\tau+1}+\varsigma}{\frac{p}{\tau+1}}\frac{\varsigma(1+\tau)}{z^{1+\tau}}\int_0^z\left(1-\frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1}s^{\tau}\left(\Xi_m*\chi_m\right)(s)d(s)\\=&\binom{\frac{p}{\tau+1}+\varsigma}{\frac{p}{\tau+1}}\frac{\varsigma(1+\tau)}{z^{1+\tau}}\int_0^z\left(1-\frac{s^{1+\tau}}{z^{1+\tau}}\right)^{\varsigma-1}s^{\tau}\left(\frac{s^p}{(1-s^m)^{\frac{p}{m}}}*\chi_m(s)\right)d(s)\end{aligned}$$

and when we apply (3.10) to (3.2), we have

$$\begin{aligned}\aleph^{\varsigma}_{\tau,p}\Theta_m(z) =& z^p + \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)}\sum_{w=1}^{\infty}\frac{\Gamma(\frac{p}{m}+w)\Gamma(\frac{mw+p}{\tau+1}+1)}{\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}\frac{a_{mw+p}}{(w)!}z^{mw+p}\\ =& \frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)}\sum_{w=0}^{\infty}\frac{\Gamma(\frac{p}{m}+w)\Gamma(\frac{mw+p}{\tau+1}+1)}{\Gamma(\frac{p}{m})\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}\frac{a_{mw+p}}{(w)!}z^{mw+p}\\ =& z^p{}_2\Psi_1\left[\begin{array}{c}(1,\frac{p}{m}),(1+\frac{p}{\tau+1},\frac{m}{\tau+1})\\ (1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|z^m\right]*\chi_m(z).\end{aligned}$$

□

Corollary 3.7. Upon setting $p=1$ and $m=1$ in (3.19), it can be observed that $\aleph^{\varsigma}_{\tau,p}\Theta_m(z)$ is an extension of the fractional integral operator derived by Ibrahim [6]

$$\mathscr{V}^{\varsigma}_{z}f(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\frac{1}{1+\tau},\frac{1}{1+\tau})\\ (1+\varsigma+\frac{1}{1+\tau},\frac{1}{1+\tau}))\end{array}|z\right]*f\right)(z),\quad (z\in\mathscr{D}).$$

Also, for $z\in\mathscr{D}$, let $p=1, m=1$ and $\tau=0$ in (3.19), is an extension of the Riemann-Liouvill fractional integral operator, which Srivastava and Owa [17, 18] studied in the complex plane as

$$\mathscr{I}^{\varsigma}_{z}f(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,1),(2,1)\\ (2+\varsigma,1)\end{array}|z\right]*f\right)(z)$$

with $\varsigma=1$, we present Libra operator, which was showed in [10], as

$$\mathscr{L}f(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,1),(2,1)\\ (3,1)\end{array}|z\right]*f\right)(z).$$

In fractional calculus, these operators are essential for studying the geometric functions in complex planes.

Moreover, for $z\in\mathscr{D}$, let $p=1, m=1, \tau=0$ and $\varsigma=0$ in (3.19), it showed that $\aleph^{\varsigma}_{\tau,p}\Theta_m(z)$ is a convex function denoted by $\mathscr{C}$ and presented as follows:

$$\mathscr{C}(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,1),(2,1)\\ (2,1)\end{array}|z\right]*f\right)(z)$$

and, we obtain

$$\mathscr{K}(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,1),(2,1)\\ (2,1)\end{array}|z\right]*z(f)'\right)(z)$$

which is the normalization Koebe function $\mathscr{K}$ defined in $\mathscr{D}$.

Using the same techniques used for proving Theorem 3.6, we obtain:

Theorem 3.8. For $z\in\mathscr{D}$, let $\chi_m(z)\in\mathscr{S}_m(p)$ and

$$\Lambda_m(z)=\chi_m(z)*\Omega_m(z)$$

where $\Omega_m(z)$ is given in (3.13). Then

$$z^{-p}\aleph^{\varsigma}_{\tau,p}\Lambda_m(z)=\left({}_2\Psi_1\left[\begin{array}{c}(1,\frac{2p}{m}),(1+\frac{p}{\tau+1},\frac{m}{\tau+1})\\ (1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|\cdot\right]*\chi_m\right)(z) \tag{3.20}$$

where

$$ {}_2\Psi_1\left[\begin{array}{c}(1,\frac{2p}{m}),(1+\frac{2p}{\tau+1},\frac{m}{\tau+1})\\(1+\varsigma+\frac{p}{\tau+1},\frac{m}{\tau+1})\end{array}|z^m\right]=\frac{\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)}\sum_{w=0}^{\infty}\frac{\Gamma(\frac{2p}{m}+w)\Gamma(\frac{mw+p}{\tau+1}+1)z^m}{\Gamma(\frac{2p}{\tau+1})\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)(w)!} $$

is the normalized form of (3.17).

Corollary 3.9. Upon setting $\varsigma, m, p = 1$ and $\tau = 0$ in (3.20), it can be observed that

$$ \mathscr{V}_z^{\varsigma} f(z)=\left({}_2\Psi_1\left[\begin{array}{c}(2,1),(2,1)\\(3,1)\end{array}|z\right]*f\right)(z),\ \ (z\in\mathscr{D}). $$

4. Subordination properties

Differential subordination is a key technique in geometric function theory that has been used by several scholars to produce very original results. The idea of differential subordination was published by Miller and Mocanu ([15] and [14]). Their goal was to apply the idea of inequality to the complex plane instead of just the real numbers, which resulted in the theory of differential subordination. Within the context of geometric function theory, differential subordination is investigated with particular attention to classes of functions defined in $\mathscr{D}$. The rotational, reflectional, and inversional symmetries that the unit disk displays help examine the geometric characteristics of functions in this symmetrical domain, such as convexity and starlikeness. This section presents a novel class of functions defining in $\mathscr{S}_m(p)$ associated with the fractional integral operator $\aleph^{\varsigma}_{\tau,p}$. The differential subordination technique defines our class.

Let f, φ be two analytic functions in H. We state that f is subordinate to φ, if there exists a *schwarz* function ω, analytic in $\mathscr{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = \varphi(\omega(z))$ for each $z \in \mathscr{D}$. In such a case, we observe that $f \prec \varphi$ or $f(z) \prec \varphi(z)(z \in \mathscr{D})$. If φ is univalent in $\mathscr{D}$, then $f(z) \prec \varphi(z) \Leftrightarrow f(0) = \varphi(0)$ and $f(\mathscr{D}) \subset \varphi(\mathscr{D})$.

To demonstrate the results we obtained in this section, the following lemma will be needed:

Lemma 4.1. [14] *Let Ω be analytic and convex univalent function in $\mathscr{D}$ with $\Omega(0) = 1$, and $\hbar(z) = 1 + q_w z^w + q_{w+1}z^{w+1} + \cdots$ be analytic in $\mathscr{D}$. If*

$$ \hbar(z)+\frac{z\hbar'(z)}{d}\prec\Omega(z),\ \ (d\neq 0,\mathscr{R}\{d\}\geq 0) \tag{4.1} $$

then

$$ \hbar(z)\prec\frac{d}{w}z^{-d/w}\int_0^z \ell^{d/w-1}\Omega(\ell)d(\ell). \tag{4.2} $$

For any A and B such that $-1 \leq B < A \leq 1$, we put

$$ \Omega(A,B;z)=\frac{1+Az}{1+Bz},\ \ (z\in\mathscr{D}) \tag{4.3} $$

and when $-1 \leq B \leq 1$, the well-known function in (**??**) is the conformal map of the unit disk onto the disk symmetrical respect to the real axis having the center

$(1-AB)(1-B^2)^{-1}$ and radius $(A-B)(1-B^2)^{-1}$ and the boundary circle cuts the real axis at the point $(1-A)(1-B)^{-1}$ and $(1+A)(1+B)^{-1}$.

In this section, we will use the differential subordination technique to obtain certain subordination properties of the fractional integral operator $\aleph^{\varsigma}_{\tau,p}$ of $m-$fold$-p-$valent symmetric function.

Theorem 4.2. For $-1 \le B < A \le 1$ and $0 < \Lambda < 1$, let

$$\chi_m(z) = z^p + \sum_{w=1}^{\infty} a_{mw+p}\, z^{mw+p}$$

be a function of class $\mathscr{S}_m(p)$ $(w \ge m;\ p, m \in \mathrm{N} = \{1, 2, 3, ...\})$, and let

$$\sum_{w=1}^{\infty} \varrho_{mw} \mid a_{mw+p} \mid \le 1 \tag{4.4}$$

where

$$\varrho_{mw} = \frac{(1-B)[\varsigma(\tau+1)+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(A-B)(\varsigma(\tau+1)+p))\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}. \tag{4.5}$$

(i)- If $-1 \le B < 0$, then

$$(1-\Lambda)\frac{\aleph^{\varsigma-1}_{\tau,p}\chi_m(z)}{z^p} + \Lambda\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} \prec \Omega(A, B; z). \tag{4.6}$$

(ii)- If $-1 \le B < 0$ and $\lambda \ge 0$, then

$$\mathscr{R}\left\{\left(\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p}\right)^{1/\lambda}\right\} > \left\{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}\int_0^1 t^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1}\left(\frac{1-At}{1-Bt}\right)dt\right\}^{1/\lambda} \tag{4.7}$$

the result is sharp.

Proof. To prove (i), we let

$$\Delta = (1-\Lambda)\frac{\aleph^{\varsigma-1}_{\tau,p}\chi_m(z)}{z^p} + \Lambda\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p}$$

and by applying (3.2), we have

$$\Delta = 1 + \sum_{w=1}^{\infty}\frac{[\varsigma(\tau+1)+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(\varsigma(\tau+1)+p)(\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)} a_{mw+p}z^{mw+1}. \tag{4.8}$$

For $-1 \le B < 0$ and $z \in \mathscr{D}$, it satisfies from (4.5) that

$$\begin{aligned}
&\left|\frac{\Delta-1}{A-B\Delta}\right| \\
&=\left|\frac{\sum_{w=1}^{\infty}\frac{[\varsigma(\tau+1)+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(\varsigma(\tau+1)+p))\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}a_{mw+p}z^{mw+1}}{A-B+B\sum_{w=1}^{\infty}\frac{[\varsigma(\tau+1)+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(\varsigma(\tau+1)+p))\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}a_{mw+p}z^{mw+1}}\right| \\
&\leq\frac{\sum_{w=1}^{\infty}\varrho_{mw}|a_{mw+p}|}{1-B+B\sum_{w=1}^{\infty}\varrho_{mw}|a_{mw+p}|} \\
&\leq 1, \qquad (4.9)
\end{aligned}$$

which leads to

$$(1-\Lambda)\frac{\aleph_{\tau,p}^{\varsigma-1}\chi_m(z)}{z^p}+\Lambda\frac{\aleph_{\tau,p}^{\varsigma}\chi_m(z)}{z^p}\prec\Omega(A,B;z).$$

Now to prove (ii), we set

$$\hbar(z)=\frac{\aleph_{\tau,p}^{\varsigma}\chi_m(z)}{z^p}, \tag{4.10}$$

where $\chi_m\in\mathscr{S}_m(p)$, this leads to $\hbar(z)=1+q_{mw}z^{mw}+q_{mw+1}z^{mw+1}+\cdots$ being analytic in $\mathscr{D}$. By utilizing (3.3) and (4.10), we obtain

$$\frac{\aleph_{\tau,p}^{\varsigma-1}\chi_m(z)}{z^p}=\hbar(z)+\frac{1}{(\varsigma(\tau+1)+p)}z\hbar'(z). \tag{4.11}$$

From (4.6), (4.10) and (4.11), then we have

$$(1-\Lambda)\frac{\aleph_{\tau,p}^{\varsigma-1}\chi_m(z)}{z^p}+\Lambda\frac{\aleph_{\tau,p}^{\varsigma}\chi_m(z)}{z^p}=\hbar(z)+\frac{1-\Lambda}{(\varsigma(\tau+1)+p)}\hbar'(z)\prec\Omega(A,B;z). \tag{4.12}$$

Therefore, by using Lemma 4.1, we get

$$\hbar(z)\prec\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}z^{-\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}}\int_0^z s^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1}\left(\frac{1+As}{1+Bs}\right)ds$$

or

$$\hbar(z)=\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}\int_0^1 t^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1}\left(\frac{1+At\omega(z)}{1+Bt\omega(z)}\right)dt \tag{4.13}$$

where $\omega(z)$ is analytic in $\mathscr{D}$ and $\omega(0)=0$, we obtain $|\omega(z)|<1$ for $z\in\mathscr{D}$. Given $-1\leq B<A\leq 1$, $\varsigma(\tau+1)>-p$ and from (4.13), we have

$$\mathscr{R}\left\{\frac{\aleph_{\tau,p}^{\varsigma}\chi_m(z)}{z^p}\right\}>\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}\int_0^1 t^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1}\left(\frac{1-At}{1-Bt}\right)dt,\ (z\in\mathscr{D}). \tag{4.14}$$

Since $\mathscr{R}\left(\omega^{1/\lambda}\right)\geq(\mathscr{R}\,\omega)^{1/\lambda}$ for $\mathscr{R}\,\omega>0$ and $\lambda\geq 1$, by applying (4.14) then inequality (4.7) holds.

Now, we want to prove the sharpness property, first, we take $\chi_m(z) \in \mathscr{S}_m(p)$ and define by

$$\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} = \frac{\varsigma(\tau+1)+p}{m(1-\Lambda)} \int_0^1 t^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1} \left(\frac{1+Atz^m}{1+Btz^m}\right) dt,$$

we get

$$(1-\Lambda)\frac{\aleph^{\varsigma-1}_{\tau,p}\chi_m(z)}{z^p} + \Lambda\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} = \frac{1+Az^m}{1+Bz^m}$$

and as $z \to e^{i\pi/m}$, then we obtain

$$\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} \to \frac{\varsigma(\tau+1)+p}{m(1-\Lambda)} \int_0^1 t^{\frac{\varsigma(\tau+1)+p}{m(1-\Lambda)}-1} \left(\frac{1-At}{1-Bt}\right) dt\ .$$

Consequently, the proof of Theorem 4.2 has been achieved. □

Next, we derive the following result of partial sums:

4.1. **Partial sums.**

Theorem 4.3. For $-1 \le B < A \le 1$ and $0 < \Lambda < 1$, let

$$\chi_m(z) = z^p + \sum_{w=1}^{\infty} a_{mw+p}\, z^{mw+p} \in \mathscr{S}_m(p),\ \ u_{m,0}(z) = z^p$$

and its sequence of the partial sums

$$u_{m,l}(z) = z^p + \sum_{w=1}^{l-1} a_{mw+p}\, z^{mw+p},\ \ (l \ge 1).$$

If the $\{\varrho_{mw}\}$ is non-decreasing sequence with

$$\varrho_{mw} \ge \frac{(1-B)[\varsigma(\tau+1)+p+mw(1-\Lambda)]}{(A-B)(\varsigma(\tau+1)+p)}\,(m \ge 1) \tag{4.15}$$

where ϱ_{mw} is proven by (4.5) and satisfied the following condition

$$\mathscr{R}\left\{\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{u_{m,l}(z)}\right\} > 0. \tag{4.16}$$

The expression (4.16) is the best possible for $l \in \mathrm{N}$.

Proof. To prove the bound in (4.16), we let

$$\hbar(z) = \frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{u_{m,l}(z)}\,(\chi_m \in \mathscr{S}_m(p); z \in \mathscr{D}).$$

By utilizing (3.2), we set

$$\hbar(z) = 1 + \frac{\sum_{w=1}^{l-1}(\Bbbk_{mw}-1)a_{mw+p}z^{mw} + \sum_{w=l}^{\infty}\Bbbk_{mw}a_{mw+p}z^{mw}}{1+\sum_{w=1}^{l-1}a_{m+p}z^{mw}} \tag{4.17}$$

where

$$\Bbbk_{mw} = \frac{\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}.$$

It is easily noted from (4.5) and (4.15) that $\varrho_{mw} > 1$ and

$$\Bbbk_{mw} = \frac{(A-B)(\varsigma(\tau+1)+p))}{(1-B)[\varsigma(\tau+1)+p+mw(1-\Lambda)]}\varrho_{mw} \geq 1. \tag{4.18}$$

Hence, by using (4.4) and (4.18), we obtain

$$\begin{aligned}\left|\frac{\hbar(z)-1}{\hbar(z)+1}\right| &= \left|\frac{\sum_{w=1}^{l-1}(\Bbbk_{mw}-1)a_{mw+p}z^{mw}+\sum_{w=l}^{\infty}\Bbbk_{mw}a_{mw+p}z^{mw}}{2+\sum_{w=1}^{l-1}(\Bbbk_{mw}+1)a_{mw+p}z^{mw}+\sum_{w=l}^{\infty}\Bbbk_{mw}a_{mw+p}z^{mw}}\right| \\ &\leq \frac{\sum_{w=1}^{l-1}(\Bbbk_{mw}-1)|a_{mw+p}|+\sum_{w=l}^{\infty}\Bbbk_{mw}|a_{mw+p}|}{2-\sum_{w=1}^{l-1}(\Bbbk_{mw}+1)|a_{mw+p}|-\sum_{w=l}^{\infty}\Bbbk_{mw}|a_{mw+p}|} \\ &\leq 1, \ (z \in \mathscr{D}) \end{aligned} \tag{4.19}$$

which readily yields the inequality (4.16).
If we set $\chi_m(z) = z^p - z^{ml+p}$, then

$$\frac{\chi_m(z)}{u_{m,l}(z)} = 1 - z^{ml} \to 0 \text{ as } z \to 1^- \tag{4.20}$$

where the bound of (4.20) is given by $\frac{1}{l}$, for $l \geq 1$ and $|z| < 1$. This indicates that for any $l \geq 1$, the bound in (4.16) is the best possible.

□

4.2. **Special integral transform.** Now, by utilizing the function $\chi_m \in \mathscr{S}_m(p)$ and $\epsilon > -p$, we consider defining the generalized Bernard-Livera-Livingston integral operator: $\mathscr{L}_{\varepsilon,p} : \mathscr{S}_m(p) \longrightarrow \mathscr{S}_m(p)$ by:

$$\begin{aligned}\mathscr{L}_{\varepsilon,p}(\chi_m)(z) &= \frac{(\epsilon+p)}{z^{\epsilon}}\int_0^z s^{\epsilon-1}\chi_m(s)ds \\ &= (\epsilon+p)\int_0^z \left(\frac{s}{z}\right)^{\epsilon}(s)^{p-1}ds + (\epsilon+p)\sum_{w=1}^{\infty}a_{mw+p}\int_0^z\left(\frac{s}{z}\right)^{\epsilon}s^{mw+p-1}ds,\end{aligned} \tag{4.21}$$

by letting $t = \frac{s}{z}$ such that $zdt = ds$ in (4.21), we obtain

$$\mathscr{L}_{\varepsilon,p}(\chi_m)(z) = (\epsilon+p)\left\{\int_0^1 (u)^{\epsilon+p-1}z^p du + \sum_{w=1}^{\infty}a_{mw+p}\int_0^1 (u)^{mw+\epsilon+p-1}z^{mw+p}du\right\}$$

and by recalling the definition of the beta function in (3.9), we give the generalized Bernard-Livera-Livingston integral operator definition:

$$\mathscr{L}_{\varepsilon,p}(\chi_m)(z) = \left(z^p + \sum_{w=1}^{\infty}\frac{\epsilon+p}{mw+\epsilon+p}z^{mw+p}\right) * \chi_m(z), \ \ (z \in \mathscr{D}). \tag{4.22}$$

by taking $m = 1$ in (4.22), we obtain the original integral operator(see[3, 10, 18]).

Theorem 4.4. For $-1 \leq B < A \leq 1$ and $0 < \Lambda < 1$, let

$$\chi_m(z) = z^p + \sum_{w=1}^{\infty}a_{mw+p}\, z^{mw+p}$$

be a function of class $\mathscr{S}_m(p)$ suppose that

$$\sum_{w=1}^{\infty} b_{mw}|a_{mw}| \leq 1 \tag{4.23}$$

where

$$b_{mw} = \frac{1-B}{A-B}\frac{[\epsilon+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(\epsilon+p+mw)(\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)}. \tag{4.24}$$

(i)- If $-1 \leq B < 0$, then

$$(1-\Lambda)\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} + \Lambda\frac{\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)}{z^p} \prec \Omega(A,B;z). \tag{4.25}$$

(ii)- If $-1 \leq B < 0$ and $\lambda \geq 1$, then

$$\mathscr{R}\left\{\left(\frac{\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)}{z^p}\right)^{1/\lambda}\right\} > \left\{\frac{\epsilon+p}{m(1-\Lambda)}\int_0^1 t^{\frac{\epsilon+p}{m(1-\Lambda)}-1}\left(\frac{1-At}{1-Bt}\right)dt\right\}^{1/\lambda} \tag{4.26}$$

for $z \in \mathscr{D}$, the result is sharp.

Proof. To prove (i), let

$$\Delta = (1-\Lambda)\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} + \Lambda\frac{\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)}{z^p}$$

from (3.2) and (4.22), we obtain

$$\Delta = 1 + \sum_{w=1}^{\infty}\frac{[\epsilon+p+mw(1-\Lambda)]\Gamma(\frac{mw+p}{\tau+1}+1)\Gamma(\frac{p}{\tau+1}+\varsigma+1)}{(\epsilon+p+mw)(\Gamma(\frac{p}{\tau+1}+1)\Gamma(\frac{mw+p}{\tau+1}+\varsigma+1)} a_{mw+p}z^{mw+1}.$$

Therefore we get the intended result by employing the same methods as in proving Theorem 4.2 (i).
(ii) From (4.22), we get

$$(\epsilon+p)\aleph^{\varsigma}_{\tau,p}\chi_m(z) = \epsilon\aleph^{\varsigma}_{\tau,p}\chi_m(z) + z\left(\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)\right)'. \tag{4.27}$$

Let

$$\hbar(z) = \frac{\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)}{z^p}, \ z \in \mathscr{D}. \tag{4.28}$$

Then, based on (4.28), (4.27) and (4.25), we observe that

$$(1-\lambda)\frac{\aleph^{\varsigma}_{\tau,p}\chi_m(z)}{z^p} + \lambda\frac{\aleph^{\varsigma}_{\tau,p}\mathscr{L}_{\varepsilon,p}(\chi_m)(z)}{z^p} = \hbar(z) + \frac{(1-\lambda)}{\epsilon+p}z\hbar'(z) \prec \Omega(A,B;z).$$

Thus, we obtain (4.26), which proves Theorem 4.4, by applying the same method as in the proof of Theorem 4.2 (ii). ∎

5. Conclusion

We investigated the set of analytic functions with $m-$fold$-p-$valent symmetry characteristics in the unit disk and defined a new fractional integral operator. Several geometric characteristics of this operator are demonstrated. Furthermore, using differential subordination, the behavior of the fractional integral operator in the unit disk is described.

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References

1. Abdulnaby, Z. E., & Ibrahim, R. W. (2020). On a subclass of analytic functions of fractal power with negative coefficients. Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science, 13(2), 387-398. https://doi.org/10.31926/but.mif.2020.13.62.2.2
2. Abdulnaby, Z. E., & Ibrahim, R. W. (2024). Applications of Differential Inequalities Employing A New Convoluted Operator Constructed by The Supertrigonometric Function, Iraqi Journal of Science, 65(6), 3302-3312. https://doi.10.24996/ijs.2024.65.6.29
3. Bernardi, S. D. (1969). Convex and starlike univalent functions. Transactions of the American Mathematical Society, 135, 429-446. https://doi.org/10.2307/1995025
4. Feng, J., & Li, X. (2014). Some properties on certain new subclasses of analytic functions. Gulf Journal of Mathematics, 2(2). https://doi.org/10.56947/gjom.v2i2.200
5. Gour, M. M., Sharma, D. K., Yadav, L. K., Sharma, G. S., & Purohit, S. D. (2024). Majorization property for specific classes of analytic functions. Gulf Journal of Mathematics, 16(2), 204-212. https://doi.org/10.56947/gjom.v16i2.1881
6. Ibrahim, R. W. (2011). On generalized Srivastava-Owa fractional operators in the unit disk. Advances in Difference Equations, 2011, 1-10. https://doi.org/10.1186/1687-1847-2011-55
7. Ibrahim, R. W., Kılıçman, A., & Abdulnaby, Z. E. (2017). Boundedness of fractional differential operator in complex spaces. Asian-European Journal of Mathematics, 10(4). https://doi.10.1142/S1793557117500759
8. Koepf, W. (1989). Coefficients of symmetric functions of bounded boundary rotation. Proceedings of the American Mathematical Society, 105(2), 324-329. https://doi.org/10.1090/S0002-9939-1989-0930244-7
9. Kumar, A., Mondal, S. R., & Das, S. (2022). Certain geometric properties of the Fox–Wright functions. Axioms, 11(11), 629. https://doi.org/10.3390/axioms11110629
10. Libera, R. J. (1965). Some classes of regular univalent functions. Proceedings of the American Mathematical Society, 16(4), 755-758. https://doi:10.1090/S0002-9939-1965-0178131-2
11. Masih, V. S., Ebadian, A., & Sokol, J. (2022). On strongly starlike functions related to the Bernoulli lemniscate. Tamkang Journal of Mathematics, 53(3), 187-199. https://doi.org/10.5556/j.tkjm.53.2022.3234
12. Mehrez, K. (2018). New integral representations for the Fox–Wright functions and its applications. Journal of Mathematical Analysis and Applications, 468(2), 650-673. https://doi.org/10.1016/j.jmaa.2018.08.053
13. Mehrez, S., Miraoui, M., & Agarwal, P. (2024). Expansion formulas for a class of function related to incomplete Fox–Wright function. Boletín de la Sociedad Matemática Mexicana, 30(1), 22. https://doi.org/10.1007/s40590-024-00596-6

14. Miller, S. S., & Mocanu, P. T. (1981). Differential subordinations and univalent functions. Michigan Mathematical Journal, 28(2), 157-172. https://doi:10.1307/mmj/1029002507
15. Miller, S. S, & Mocanu, P. T. (2000). Differential subordination: theory and applications. Series on monographs and textbooks in pure and applied mathematics. Marcel Dekker Incorporated, New York and Basel, vol 225. https://doi.org/10.1201/9781482289817
16. Mishra, A. K., & Choudhury, M. (1994). Coefficients of multivalent symmetric functions of bounded boundary rotation. Rendiconti del Circolo Matematico di Palermo Series 2, 43(3), 403-412. https://doi.org/10.1007/BF02844251
17. Owa, S., & Srivastava, H. M. (1987). Univalent and starlike generalized hypergeometric functions. Canadian Journal of Mathematics, 39(5), 1057-1077. https://doi.org/10.4153/CJM-1987-054-3
18. Owa, S., & Srivastava, H. M. (2004). Some subordination theorems involving a certain family of integral operators. Integral Transforms and Special Functions, 15(5), 445–454. https://doi.org/10.1080/10652460410001727563
19. Pommerenke, C. (1962). On the coefficients of close-to-convex functions. Michigan Mathematical Journal, 9(3), 259-269. https://doi:10.1307/mmj/1028998726
20. Srivastava, H. M. (2021). An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. Journal of Advanced Engineering and Computation, 5(3), 135-166. http://dx.doi.org/10.55579/jaec.202153.340

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