

ANALYSIS OF POSITIVE SOLUTIONS FOR THE FRACTIONAL DERIVATIVE WITH DELAY AND INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. The goal of the manuscript is to investigate the existence of positive solutions of the nonlinear fractional differential equation with delay and integral boundary conditions. The system is governed by the Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$ and includes a nonlinear term influenced by both the current state and a solution-dependent delay. To establish the existence of a positive solution, we transform the fractional differential equation into an equivalent integral equation and apply the Guo-Krasnoselskii fixed-point theorem in a cone. Additionally, a practical example is provided to validate the theoretical results and demonstrate their real-world relevance.

1. INTRODUCTION

Fractional order systems, also known as systems with fractional derivative and integrals, have been studied extensively by several researchers in the field of engineering and science in the recent times. The fractional calculus system theory has the potential to facilitate the concise representation of an increasing number of the behaviours of physical systems. The applications of fractional differential equations in a variety of fields, such as physics [33], chemistry [19], biology [20], economics [25], control theory [23], signal and image processing [35], biophysics [30], blood flow phenomena [7], aerodynamics [6], and experimental data fitting [36], have contributed to the rise in popularity of these equations.

In recent studies, researchers have focused on proving the existence and uniqueness of fractional differential equations. These studies demonstrate that a given problem contains at least one solution and it can be unique [10]-[15]. Some works have explored non-linear fractional Cauchy problems, while others have studied impulsive fractional stochastic differential equations including infinite delay [16]. These problems appear in many real-world situations, such as financial markets [23], control systems [35], and biological processes [25]. In these cases, fixed-point theory has a crucial role for proving the existence of solution, helping to establish a strong mathematical foundation for fractional models [3].

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Another important area of research is integral boundary conditions, which appear in problems related to blood flow [25], chemical engineering [32], thermoelasticity, and population dynamics [6]. Unlike standard boundary conditions, integral boundary conditions allow the system to depend on values over an entire range than at specific points. This makes them useful in situations where past influences play a role in determining future states. Many researchers have investigated such conditions to improve mathematical models for biological and engineering systems. These studies show that using integral constraints makes fractional models can be more accurate in predicting real-world behaviour [22]-[9].

This paper focuses on studying fractional differential equations incorporating integral boundary conditions and proving the existence of positive solution using fixed-point theorems. Our work extends previous studies by considering state-dependent delays, which makes the model even more applicable to memory-dependent systems like disease spread, ecological populations, and mechanical systems.

Inspired from the previously stated work, this study discusses the existence of positive solution to the following non-linear FDE's including integral boundary conditions along with state dependent delay which is given as

$$\begin{cases} {}^R\mathfrak{D}^\alpha \varphi(x) + g(x, \varphi(x), \tau(\varphi(x))) = 0, & 2 < \alpha \leq 3, \quad x \in [0, 1] \\ \varphi(0) = \varphi'(0) = 0, \quad \varphi(1) = \gamma \int_0^1 \varphi(s) ds, \end{cases} \quad (1.1)$$

where ${}^R\mathfrak{D}^\alpha$ denotes Riemann - Liouville fractional derivative of order α with $2 < \alpha \leq 3$, $g : [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous non - linear function. Also, $\tau(\varphi(x))$ represents a state dependent delay function which depends on the current state $\varphi(x)$ and γ is a positive constant.

The paper is organized into several sections and each section contributes to the effectiveness of our research. Section 2 introduces the initial findings necessary for the subsequent sections. Section 3 examines the existence of a positive solution of system (1.1). Section 4 presents a practical example to validate the theoretical results and demonstrate their relevance in the real world.

2. PRELIMINARIES

This section introduces specific notations, definitions, and lemmas.

Definition 2.1. [8] For a given function g , the Riemann-Liouville fractional order derivative with $\alpha > 0$ can be expressed as

$$\mathfrak{D}^\alpha g(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^x (x-s)^{n-\alpha-1} g(s) ds,$$

where $n = [\alpha] + 1$ (the smallest integer greater than α), $\frac{d}{dt}$ is the ordinary differential operator and the fractional derivative is valid for $x > 0$, on the condition that the integral exists.

Definition 2.2. [8] For a given function g , the fractional order integral with $\alpha > 0$ is expressed as

$$I^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds,$$

where $\Gamma(\alpha)$ is the Gamma function and $x > s \geq 0$.

Lemma 2.3. [8] *Suppose for a given $\alpha > 0$, the fractional order differential equation,*

$${}^R\mathfrak{D}^\alpha \varphi(x) = 0$$

contains a unique solution given as

$$\varphi(x) = \sum_{j=0}^{[\alpha]} \frac{\varphi^{(j)}(0)}{j!} x^j.$$

Lemma 2.4. [8] *Suppose for a given $\alpha > 0$, the solution is given as*

$$I^{\alpha R} \mathfrak{D}^\alpha \varphi(x) = \varphi(x) - \sum_{j=0}^{[\alpha]} \frac{\varphi^{(j)}(0)}{j!} x^j.$$

Theorem 2.5. [8] *Suppose for given $2 < \alpha < 3$ and $\gamma \neq 0$. Then, for the given problem*

$$\begin{cases} {}^R\mathfrak{D}^\alpha \varphi(x) + \Theta(x) = 0, & 2 < \alpha \leq 3, \quad x \in [0, 1] \\ \varphi(0) = \varphi'(0) = 0, & \varphi(1) = \gamma \int_0^1 \varphi(s) ds, \end{cases} \quad (2.1)$$

contain its unique solution given as

$$\varphi(x) = \int_0^1 \mathcal{G}(x, s) \Theta(s) ds,$$

where $\mathcal{G}(x, s)$ represents the Green's function given as

$$\mathcal{G}(x, s) = \begin{cases} \frac{x^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\gamma+\gamma s) - (\alpha-\gamma)(x-s)^{\alpha-1}}{(\alpha-\gamma)\Gamma(\alpha)}, & 0 \leq s \leq x \leq 1, \\ \frac{x^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\gamma+\gamma s)}{(\alpha-\gamma)\Gamma(\alpha)}, & 0 \leq x \leq s \leq 1. \end{cases} \quad (2.2)$$

After analysing the Green's function $\mathcal{G}(x, s)$, it allows us to establish the following results as given below.

Lemma 2.6. [8] *Suppose $\mathcal{G}$ be a Green's function associated with the issue (2.1) which is defined by the expression (2.2), for all $\alpha \in (2, 3]$ and $\gamma \geq 0$, the following characteristics are satisfied*

- (1) $\mathcal{G}(0, s) = \mathcal{G}(x, 1) = 0$ for all $x, s \in [0, 1]$ and $\gamma \neq \alpha$. Which ensures the boundary conditions are satisfied at $x = 0$ and $s = 1$, with the condition $\gamma \neq \alpha$.
- (2) $\mathcal{G}(1, s) = 0$ or all $s \in (0, 1)$ if and only if $\gamma = 0$. Which states that the Green's function $\mathcal{G}(x, s)$ vanishes at $x = 1$ for any $s \in (0, 1)$ if γ is zero.
- (3) $(\alpha - \gamma)\mathcal{G}(1, s) > 0$ for all $s \in (0, 1)$ if and only if $\gamma \neq \alpha$. Which shows positivity of $\mathcal{G}(1, s)$ scaled by $(\alpha - \gamma)$, provided $\gamma \neq \alpha$.

- (4) $\mathcal{G}(x, 0) = 0$ for all $x \in [0, 1]$. Which indicates that $\mathcal{G}(x, s)$ vanishes at $s = 0$, fulfilling the initial condition.
- (5) $\mathcal{G}(x, s) \leq \frac{1}{(\alpha-\gamma)\Gamma(\alpha-1)}$ for all $x, s \in [0, 1]$ and $\gamma \in [0, \alpha)$. Which provides an upper bound for the Green's function, depending on α, γ , and the gamma function $\Gamma(\alpha - 1)$.
- (6) $\mathcal{G}(x, s)$ is a continuous function for all $x, s \in [0, 1]$ and $\gamma \neq \alpha$. Which establishes the continuity of the Green's function, which is crucial for the existence of solutions.

Thus, here we will establish two supplementary inequalities of the Green's function $\mathcal{G}$. These properties, along with earlier discussion, play a crucial role to ensure the existence of positive solution for the issue (2.1) which will be illustrated in the next section.

Lemma 2.7. For $2 < \alpha \leq 3$ and $0 < \gamma < \alpha$. Let $\mathcal{G}(x, s)$ be the Green's function associated to the issue (2.1), expressed by the equation (2.2). Then, the upcoming inequalities hold:

$$x^{\alpha-1}\mathcal{G}(1, s) \leq \mathcal{G}(x, s) \leq \frac{\alpha}{\gamma}\mathcal{G}(1, s), \quad \text{for all } x, s \in (0, 1). \quad (2.3)$$

Proof. Assume $0 < x \leq s < 1$. In this case:

$$h(x, s) \equiv \frac{\mathcal{G}(x, s)}{\mathcal{G}(1, s)} = \frac{x^{\alpha-1}(\alpha - \gamma(1 - s))}{\gamma s}, \quad \text{for all } 0 < x \leq s < 1.$$

Then, the following inequalities are verified:

$$x^{\alpha-1} < x^{\alpha-1} \left(1 + \frac{\alpha - \gamma}{\gamma s} \right) = h(x, s) \leq x^{\alpha-1} \frac{\alpha}{\gamma s} \leq x^{\alpha-2} \frac{\alpha}{\gamma} < \frac{\alpha}{\gamma}, \quad \text{for all } 0 < x \leq s < 1.$$

Conversely, if $0 < s \leq x < 1$, we obtain:

$$h(x, s) = \frac{x^{\alpha-1}(1 - s)^{\alpha-1}(\alpha - \gamma(1 - s)) - (\alpha - \gamma)(x - s)^{\alpha-1}}{\gamma s(1 - s)^{\alpha-1}}.$$

From $s \geq x$, we deduce:

$$h(x, s) \geq \frac{x^{\alpha-1}(1 - s)^{\alpha-1}[\alpha - \gamma(1 - s)] - (\alpha - \gamma)}{\gamma s(1 - s)^{\alpha-1}} = x^{\alpha-1}.$$

From the preceding case, one can verify that $h(x, s) \leq \frac{\alpha}{\gamma}$ whenever $0 < s \leq x < 1$. Therefore, from the lemma 2.6, (3), we observe that the inequality (2.3) are satisfied. $\square$

Corollary 2.8. Suppose $\mathcal{G}$ be the Green's function associated to the issue (2.1), which is expressed by the expression (2.2). Then, $\forall \alpha \in (2, 3]$ and $\gamma \geq 0$, the following property holds:

$$\mathcal{G}(x, s) > 0, \text{ for all } x, s \in (0, 1) \text{ and all } \gamma \in [0, \alpha).$$

3. EXISTENCE OF POSITIVE SOLUTIONS

This section deals with the solution which gives guarantee that the expression (1.1) contains a positive solution, and to prove this we use Guo-Krasnoselskii fixed point theorem, which is given in the subsequent theorem.

Theorem 3.1. [8] Suppose $\mathcal{E}$ be a complete norm linear space, and let $\mathcal{P} \subset \mathcal{E}$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of $\mathcal{E}$ with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $T : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{P}$ be a completely continuous mapping in such a way that

$$(1) \|T\varphi\| \geq \|\varphi\|, \varphi \in \mathcal{P} \cap \partial\Omega_1, \text{ and } \|T\varphi\| \leq \|\varphi\|, \varphi \in \mathcal{P} \cap \partial\Omega_2;$$

$$(2) \|T\varphi\| \leq \|\varphi\|, \varphi \in \mathcal{P} \cap \partial\Omega_1, \text{ and } \|T\varphi\| \geq \|\varphi\|, \varphi \in \mathcal{P} \cap \partial\Omega_2.$$

Then the mapping T contains at least one fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Now, define the mapping $T : C[0, 1] \rightarrow C[0, 1]$ as

$$T\varphi(x) := \int_0^1 \mathcal{G}(x, s)g(s, \varphi(s), \tau(\varphi(s)))ds \quad (3.1)$$

with $\mathcal{G}$ described in (2.2).

From the theorem (2.5), it is clear that the fixed points of operator T coincide with the solutions of issue (1.1).

Suppose $\mathcal{E} = C[0, 1]$ be a complete norm linear space endowed with the usual supremum norm $\|\cdot\|$, and suppose the subsequent assumption:

(H) The function $g : [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function.

Now, define the cone $\mathcal{P} \subset \mathcal{E}$ is given as

$$\mathcal{P} = \left\{ \varphi \in \mathcal{E}, \varphi(x) \geq 0 \quad \forall x \in [0, 1], \quad \varphi(x) \geq \frac{x^{\alpha-1}\gamma}{\alpha} \|\varphi\|, \quad \forall x \in \left[\frac{1}{2}, 1\right] \right\} \quad (3.2)$$

Set

$$g_0 = \lim_{\varphi \rightarrow 0^+} \left\{ \min_{x \in [1/2, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\} \quad \text{and} \quad g_\infty = \lim_{\varphi \rightarrow \infty} \left\{ \min_{x \in [1/2, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\}$$

and

$$g^0 = \lim_{\varphi \rightarrow 0^+} \left\{ \max_{x \in [0, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\} \quad \text{and} \quad g^\infty = \lim_{\varphi \rightarrow \infty} \left\{ \max_{x \in [0, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\}.$$

Now, using the above assumptions, we will define a theorem and prove that equation (1.1) contains a positive solution.

Theorem 3.2. Suppose the constraint (H) holds coupled with one of the two upcoming conditions:

(1) (sub-linear case). $g_0 = \infty$ and $g^\infty = 0$.

(2) (super-linear case). $g^0 = 0$, and $g_\infty = \infty$.

Then, $\forall \alpha \in (2, 3]$ and $\gamma \in (0, \alpha)$, the issue (1.1) contains a positive solution that belongs to the cone $\mathcal{P}$ explained in (3.2).

Proof. First, we will prove that $T : \mathcal{P} \rightarrow \mathcal{P}$ is completely continuous.

From the non-negativity and continuity for the function $\mathcal{G}$ and g on their domain of definition, we assume that if $\varphi \in \mathcal{P}$ then $T\varphi \in \mathcal{E}$ and $T\varphi(x) \geq 0$ for all $x \in [0, 1]$.

Let's see that $T(\mathcal{P}) \subset \mathcal{P}$. Take $\varphi \in \mathcal{P}$, then, for all $x \in [0, 1]$, and from Lemmas (2.6) and Lemma (2.7), the following inequalities are satisfied

$$\begin{aligned} T\varphi(x) &= \int_0^1 \mathcal{G}(x, s)g(s, \varphi(s), \tau(\varphi(s)))ds \\ &\geq x^{\alpha-1} \int_0^1 \mathcal{G}(1, s)g(s, \varphi(s), \tau(\varphi(s)))ds \\ &\geq \frac{x^{\alpha-1}\gamma}{\alpha} \int_0^1 \max_{x \in [0,1]} \{\mathcal{G}(x, s)\}g(s, \varphi(s), \tau(\varphi(s)))ds \\ &\geq \frac{x^{\alpha-1}\gamma}{\alpha} \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s)g(s, \varphi(s), \tau(\varphi(s)))ds \right\} \\ &= \frac{x^{\alpha-1}\gamma}{\alpha} \|T\varphi\|. \end{aligned}$$

In view of the continuity for the functions $\mathcal{G}$ and g , the operator $T : \mathcal{P} \rightarrow \mathcal{P}$ is continuous.

Let $\Omega \subset \mathcal{P}$ be bounded, which is to say there exists a positive constant $M > 0$ such that $\|\varphi\|_\infty \leq M$ for all $\varphi \in \Omega$.

Define now,

$$L = \max_{0 \leq x \leq 1, 0 \leq \varphi \leq M} |g(x, \varphi, \tau(\varphi))| + 1.$$

Then, for all $\varphi \in \Omega$, it is satisfied that

$$|T\varphi(x)| \leq L \int_0^1 \mathcal{G}(x, s)ds \leq \frac{L\alpha}{(\alpha - \gamma)\Gamma(\alpha)}, \quad \text{for all } x \in [0, 1], \quad (3.3)$$

Therefore, the set $T(\Omega)$ is bounded in $\mathcal{E}$.

For every given $\varphi \in \Omega$, we get

$$\begin{aligned} |(T\varphi)'(x)| &= \left| - \int_0^x \frac{(x-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s, \varphi(s))ds + \int_0^1 \frac{x^{\alpha-2}(1-s)^{\alpha-1}(\alpha-\gamma+\gamma s)}{(\alpha-\gamma)\Gamma(\alpha-1)} g(s, \varphi(s))ds \right| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^x (x-s)^{\alpha-2} |g(s, \varphi(s))|ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(\alpha - \gamma)\Gamma(\alpha - 1)} \int_0^1 x^{\alpha-2} (1-s)^{\alpha-1} (\alpha - \gamma + \gamma s) |g(s, \varphi(s))| ds \\
 & \leq \frac{L}{\Gamma(\alpha - 1)} \int_0^x (x-s)^{\alpha-2} ds + \frac{\alpha L}{(\alpha - \gamma)\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-1} ds \\
 & \leq \frac{L}{\Gamma(\alpha)} + \frac{\alpha L}{(\alpha - \gamma)\Gamma(\alpha)} := N.
 \end{aligned}$$

As consequence, for all $x_1, x_2 \in [0, 1], x_1 < x_2$, we have

$$|(T\varphi)(x_2) - (T\varphi)(x_1)| \leq \int_{x_1}^{x_2} |(T\varphi)'(s)| ds \leq N(x_2 - x_1)$$

Therefore, the set $T(\Omega)$ is equicontinuous in $\mathcal{E}$.

In conclusion, it follows that $\overline{T(\Omega)}$ is compact using Arzela-Ascoli theorem, i. e., $T : \mathcal{P} \rightarrow \mathcal{P}$ is a completely continuous operator.

Case (1). Sublinear case ($g_0 = \infty$ and $g^\infty = 0$).

As per the given information that $g_0 = \infty$, then $\exists$ a constant $\rho_1 > 0$ in such a way that $g(x, \varphi, \tau(\varphi)) \geq \delta_1 \varphi$, $\forall \quad 0 < \varphi \leq \rho_1$, where $\delta_1 > 0$ fulfills

$$\delta_1 \frac{\gamma}{\alpha} \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 s^{\alpha-1} \mathcal{G}(x, s) ds \right\} \geq 1 \quad (3.4)$$

Taking $\varphi \in \mathcal{P}$, in such a way that $\|\varphi\| = \rho_1$, as derived from the equation (3.4), we can conclude the subsequent inequalities

$$\begin{aligned}
 \|T\varphi\| &= \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) g(s, \varphi(s), \tau(\varphi(s))) ds \right\} \\
 &\geq \delta_1 \|\varphi\| \frac{\gamma}{\alpha} \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 s^{\alpha-1} \mathcal{G}(x, s) ds \right\} \geq \|\varphi\|.
 \end{aligned}$$

Since, we have a continuous function g defined on $[0, \infty)$. Therefore, we can define a function given as:

$$\tilde{g}(x, \varphi) = \max_{s \in [0,1]} \{g(x, s, \tau(s))\}$$

Clearly $\tilde{g}$ is non-decreasing on $[0, \infty)$. Furthermore, as $g^\infty = 0$. it is clear that (see (36)).

$$\lim_{\varphi \rightarrow \infty} \left\{ \max_{x \in [0,1]} \frac{\tilde{g}(x, \varphi)}{\varphi} \right\} = 0$$

Pick $\delta_2 > 0$ fulfilling the property:

$$\frac{\delta_2}{(\alpha - \gamma)\Gamma(\alpha - 1)} \leq 1 \quad (3.5)$$

Therefore, there exists a constant $\rho_2 > \rho_1 > 0$ such that $\tilde{g}(x, \varphi) \leq \delta_2 \varphi$ for all $\varphi \geq \rho_2$.

Let now $\varphi \in \mathcal{P}$ be such that $\|\varphi\| = \rho_2$, then, from the definition of $\tilde{g}$, equation (3.5) and property (5) in Lemma 2.2 we accomplish at the subsequent inequalities:

$$\begin{aligned} \|T\varphi\| &= \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) g(s, \varphi(s), \tau(\varphi(s))) ds \right\} \\ &\leq \delta_2 \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) \varphi(s) ds \right\} \\ &\leq \delta_2 \|\varphi\| \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) ds \right\} \\ &\leq \frac{\delta_2}{(\alpha - \gamma)\Gamma(\alpha - 1)} \|\varphi\| \leq \|\varphi\| \end{aligned}$$

Therefore, by the sub linear case of Guo-Krasnoselskii fixed point theorem, we determine that issue (1.1) contains at least one positive solution φ in such a way that

$$\rho_1 \leq \|\varphi\| \leq \rho_2$$

Case (2) Super linear case ($g^0 = 0$, and $g_\infty = \infty$).

Suppose $\delta_2 > 0$ is provided as in equation (3.5). Because $g^0 = 0$, then there exists a constant $r_1 > 0$ such that $g(x, \varphi, \tau(\varphi)) \leq \delta_2 \varphi$ for $0 \leq \varphi \leq r_1$.

Taking $\varphi \in \mathcal{P}$, in such a way that $\|\varphi\| = r_1$. Thus, we get

$$\begin{aligned} \|T\varphi\| &= \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) g(s, \varphi(s), \tau(\varphi(s))) ds \right\} \\ &\leq \delta_2 \|\varphi\| \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) ds \right\} \\ &\leq \frac{\delta_2}{(\alpha - \gamma)\Gamma(\alpha - 1)} \|\varphi\| \leq \|\varphi\| \end{aligned} \tag{3.6}$$

Now, take into account $\delta_3 > 0$ fulfilling

$$\delta_3 \frac{\gamma}{2^{\alpha-1}\alpha} \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 \mathcal{G}(x, s) ds \right\} \geq 1 \tag{3.7}$$

The fact that $g^\infty = \infty$ says that $\exists$ a constant $r_2 > r_1 > 0$ with $r_2 \alpha 2^{\alpha-1} > r_1 \gamma$ such that $g(x, \varphi, \tau(\varphi)) \geq \delta_3 \varphi$ for all $\varphi \geq r_2$.

Let now $\varphi \in \mathcal{P}$ be such that $\|\varphi\| = r_2 \frac{\alpha}{\gamma} 2^{\alpha-1}$. Note that from the definition of the cone $\mathcal{P}$, we have that $\varphi(x) \geq r_2$ for all $x \in [1/2, 1]$.

So, condition (2) gives us the following properties:

$$\begin{aligned}
 \|T\varphi\| &= \max_{x \in [0,1]} \left\{ \int_0^1 \mathcal{G}(x, s) g(s, \varphi(s), \tau(\varphi(s))) ds \right\} \\
 &\geq \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 \mathcal{G}(x, s) g(s, \varphi(s), \tau(\varphi(s))) ds \right\} \\
 &\geq \delta_3 \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 \mathcal{G}(x, s) \varphi(s) ds \right\} \\
 &\geq \delta_3 \frac{\gamma}{2^{\alpha-1} \alpha} \|\varphi\| \max_{x \in [0,1]} \left\{ \int_{\frac{1}{2}}^1 \mathcal{G}(x, s) ds \right\} \geq \|\varphi\|.
 \end{aligned}$$

Consequently, by the super linear case of Guo-Krasnoselskii fixed point theorem, we can determine that fractional differential equation (1.1) contains at least one positive solution. $\square$

Remark 3.3. It is evident that because $\mathcal{G}(0, s) = 0$. Therefore, to ensure the existence of r_2 in case (2) of the preceding theorem, it is required to decrease the interval from $[0, 1]$ to $[1/2, 1]$.

Thus, for the given real constant $0 < a \leq b \leq 1$, we redefine

$$g_0 = \lim_{\varphi \rightarrow 0^+} \left\{ \min_{x \in [a, b]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\} \quad \text{and} \quad g_\infty = \lim_{\varphi \rightarrow \infty} \left\{ \min_{x \in [a, b]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} \right\}.$$

which shows that Theorem (3.2) remains true in the cone

$$\mathcal{P} = \left\{ \varphi \in \mathcal{E}, \varphi(x) \geq 0 \quad \text{for all } x \in [0, 1], \quad \varphi(x) \geq \frac{x^{\alpha-1} \gamma}{\alpha} \|\varphi\|, \quad \text{for all } x \in [a, b] \right\}.$$

4. APPLICATION

The following fractional differential equation describes the mechanical behaviour of viscoelastic material that exhibits history dependent deformation.

$$\begin{cases} {}^R \mathfrak{D}^{2.5} \varphi(x) + g(x, \varphi(x), \tau(\varphi(x))) = 0, & x \in [0, 1] \\ \varphi(0) = \varphi'(0) = 0, & \varphi(1) = \gamma \int_0^1 \varphi(s) ds, \end{cases} \quad (4.1)$$

where

$$g(x, \varphi(x), \tau(\varphi(x))) = \sqrt{\varphi(x)} + \log(x\varphi^2(x)(x - \tau(\varphi(x)) + 2)),$$

and

$$\alpha = 2.5, \quad \tau(\varphi(x)) = \frac{1}{1+\varphi(x)},$$

where combination of square root term and logarithmic term represents a non linear stiffness response and integral boundary condition that includes a global coupling effect.

Here, it can be observed that for all $\varphi > 0$

$$\min_{x \in [1/2, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} = \frac{\sqrt{\varphi} + \log \left(\frac{\varphi^2}{2} \left(\frac{1}{2} - \frac{1}{1+\varphi(x)} \right) + 2 \right)}{\varphi}$$

$$\max_{x \in [0, 1]} \frac{g(x, \varphi, \tau(\varphi))}{\varphi} = \frac{\sqrt{\varphi} + \log \left(\varphi^2 \left(1 - \frac{1}{1+\varphi(x)} \right) + 2 \right)}{\varphi}$$

By direct calculation, we obtain $g_0 = \infty$ and $g^\infty = 0$. From the first part of Theorem (3.2), we find that the problem (4.1) has a positive solution. Similarly, we can take a case which satisfies the super linear case as well.

5. CONCLUSION

This study looks at solving a fractional differential equation with a state-dependent delay using Guo-Krasnoselskii's Fixed-Point Theorem. The delay in the system changes based on its current state, making the model more realistic by considering memory effects and past influences. Our results show that under certain conditions, there is always at least one positive solution, meaning the system behaves in a stable way. Also, we have one numerical example to check our findings. This makes the model useful for situations where the present state is influenced by the past values.

These findings are applicable in many real-life situations. For instance, in epidemics, the spread of a disease may depend on how many people are already infected. In ecology, the growth of a species may be affected by population levels. In finance, investors take time to react to market changes, and in neuroscience, reaction times depend on past experiences. Even though this study gives valuable insights, there are still open questions, such as expanding the model for more complex systems, adding random effects for unpredictable situations, and improving computational methods to handle large-scale problems faster. Solving these challenges will help make fractional calculus models even more useful in real-world applications.

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