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UNITS OF SOME ABELIAN IMAGINARY NUMBER FIELDS OF TYPE $(2,4)$

A. AZIZI, I. JERRARI, A. ZEKHNINI, AND M. TALBI

Abstract. Let K be an abelian imaginary number field of type $(2,4)$, i.e. the Galois goup $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. In this paper, we determine the fundamental system of units of some field K.

1. Notations

Let m be an integer and K be a number field. Throughout this paper, we adopt the following notations:

- E_K: the unit group of K;
- K^+: the maximal real subfield of K, if it is a CM-field;
- $Q_{K^+} = [E_K : W_K E_{K^+}]$ is Hasse's unit index, if K is a CM-field;
- $\mathfrak{p}_K$: the prime ideal of K;
- $\left(\frac{x,y}{\mathfrak{p}_K}\right)$ (resp. $\left(\frac{x}{\mathfrak{p}_K}\right)$): the Hilbert symbol (resp. the quadratic residue symbol) for the prime $\mathfrak{p}_K$ applied to (x,y) (resp. x);
- ξ_m: a primitive m-th root of unity;
- $\mathbb{Q}(\xi_m)$: the m-th cyclotomic number field.

2. Units of some abelian imaginary number fields of type $(2,4)$

By using the results of M. N. Gras [5], we will define a fundamental system of units (FSU) of a real cyclic quartic number field. Let L be a real cyclic quartic number field of Galois group $H = \langle \sigma_L \rangle$ and of quadratic subfield k, and let ε be the fundamental unit of k. Let f_L (resp. f_k) be the conductor of L (resp. k); then $f_L = f_k \cdot g$ where g is an integer, and the discriminant of L is equal to $\mathrm{disc}(L) = f_k \cdot f_L^2$. Since f_L is the conductor of L, then we have $L \subset \mathbb{Q}(\xi_{f_L})$. Let χ_L be a rational character of $\mathbb{Q}(\xi_{f_L})$, $E_{\chi_L} = \{\omega \in E_L : \omega^{1+\sigma_L^2} = \pm 1\}$, $|E_L|$ (resp. $|E_k|$, $|E_{\chi_L}|$) be the group of the absolute values of E_L (resp. E_k, E_{χ_L}), $|E^L| = |E_k| \oplus |E_{\chi_L}|$, $Q_L = [|E_L| : |E^L|]$ and ε_{χ_L} be a generator of E_{χ_L}. From [5], we have $Q_L = 1$ or 2, which gives two possible structures for the unit group of L:

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* Corresponding author.

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(1) If $Q_L = 1$, then we have $E_L = \langle -1, \varepsilon, \varepsilon_{\chi_L}, \varepsilon_{\chi_L}^{\sigma_L} \rangle$;
(2) If $Q_L = 2$, then we have $E_L = \langle -1, \xi_L, \xi_L^{\sigma_L}, \xi_L^{\sigma_L^2} \rangle$ with $\xi_L^2 = \pm\varepsilon\varepsilon_{\chi_L}^{1-\sigma_L}$, $\xi_L^{1+\sigma_L} = \pm\varepsilon_{\chi_L}$ and $\xi_L^{1+\sigma_L^2} = \pm\varepsilon$.

Note that ξ_L is called a Minkowski's unit.

Proposition 2.1. *Let $l \equiv 1 \pmod 4$ be a prime number. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit. Then the field $L = k(\sqrt{\varepsilon\sqrt{l}})$ admits a Minkowski's unit if and only if $l \equiv 1 \pmod 8$.*

Proof. It is known that $Q_L = 2$ if and only if L admits a Minkowski's unit, and from [3, Théorème II.2], the field L admits a Minkowski's unit if and only if the unit ε is norm in L.
Let $\mathfrak{p}$ be a prime ideal of k, so we have:

- If $\mathfrak{p}$ is not above l, then $v_{\mathfrak{p}}(\varepsilon\sqrt{l}) = 0$, so;

$$\begin{aligned}\left(\frac{\varepsilon, \varepsilon\sqrt{l}}{\mathfrak{p}}\right) &= \left(\frac{\varepsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(\varepsilon\sqrt{l})} \\ &= 1.\end{aligned}$$

- If $\mathfrak{p}$ lies above l, then $v_{\mathfrak{p}}(\varepsilon\sqrt{l}) = 1$. Put $\varepsilon = x + y\sqrt{l}$, then $x = 2^j c$ with c odd and $j \geq 1$, so;

$$\begin{aligned}\left(\frac{\varepsilon, \varepsilon\sqrt{l}}{\mathfrak{p}}\right) &= \left(\frac{\varepsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(\varepsilon\sqrt{l})} \\ &= \left(\frac{\varepsilon}{\mathfrak{p}}\right) \\ &= \left(\frac{x}{\mathfrak{p}}\right) \\ &= \left(\frac{2^j c}{\mathfrak{p}}\right) \\ &= \left(\frac{2^j c}{l}\right) \\ &= \left(\frac{2}{l}\right)^j \left(\frac{c}{l}\right) \\ &= \left(\frac{2}{l}\right)^j \left(\frac{l}{c}\right)\end{aligned}$$

Since $x^2 - ly^2 = 4^j c^2 - ly^2 = -1$, then $ly^2 = 4^j c^2 + 1 \equiv 1 \pmod c$. Thus,

$$\left(\frac{\varepsilon, \varepsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{2}{l}\right)^j$$

If $l \equiv 5 \pmod 8$ then $j = 1$ and $\left(\frac{2}{l}\right) = -1$, and if $l \equiv 1 \pmod 8$ then $j > 1$ and $\left(\frac{2}{l}\right) = 1$.

This means that the unit ε is not norm in L in the first case, whereas in the second case ε is. So the result. □

Corollary 2.2. *Let $l \equiv 1 \pmod 4$ be a prime number. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit, and set $L = k(\sqrt{\varepsilon\sqrt{l}})$. Then:*

(1) *If $l \equiv 5 \pmod 8$, we have $Q_L = 1$;*
(2) *If $l \equiv 1 \pmod 8$, we have $Q_L = 2$.*

Remark 2.3. In [2], the authors have shown 2 in another way.

Lemma 2.4. *Let $p \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$ be different primes. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit. Let $L = k(\sqrt{p\varepsilon\sqrt{l}})$, then $Q_L = 1$.*

Proof. It is well known that p divide the conductor f_L of field L and the norm of the unit ε is equal to -1, and so, from [5, Proposition 4], we have $Q_L = 1$. □

Let M_0 be a totally real number field, β be a positive number M_0, $M = M_0(\sqrt{-\beta})$ be a quadratic extension of M_0, absolutely abelian and finite. It is known that Hasse's unit index Q_{M^+} is equal to 1 or 2. Which means that a FSU of M_0 is also a FSU of M in the first case, whereas both systems different in the second case. From [1], we have the following results: Let j be an integer greater than or equal to 2 and ξ_{2^j} be a primitive 2^j-th root of unity, then there exist two real numbers μ_j and λ_j defined inductively:

$$\xi_{2^n} = \tfrac{1}{2}(\mu_n + \lambda_n\sqrt{-1}), \quad \text{where } \mu_n = \sqrt{2+\mu_{n-1}}, \quad \lambda_n = \sqrt{2-\lambda_{n-1}}$$
$$\mu_2 = 0, \lambda_2 = 2 \qquad \text{and } \mu_3 = \lambda_3 = \sqrt{2}.$$

Let j_0 be the greatest integer such that $\xi_{2^{j_0}}$ is contained in M.

Proposition 2.5 ([1])**.** *Let M_0 be a totally real number field, $M = M_0(\sqrt{-1})$ be a quadratic extension of M_0, absolutely abelian and finite, and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be a FSU of M_0 (whose the units are all positive). So we have:*

- *If there exists a unit λ of M_0 of the form $\varepsilon_1^{j_1}\varepsilon_2^{j_2}\ldots\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r$ (up to a permutation), where $j_k \in \{0,1\}$, such that $(2+\mu_{j_0})\lambda$ is a square in M_0, then $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r-1}, \sqrt{\xi_{2^{j_0}}\lambda}\}$ is a FSU of M.*
- *Otherwise, $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ is a FSU of M.*

Proposition 2.6 ([1])**.** *Let M_0 be a totally real number field, β be a positive number of M_0, square free and such that $\sqrt{\beta} \notin M_0$, $M = M_0(\sqrt{-\beta})$ be a quadratic extension of M_0, absolutely abelian and finite, and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be a FSU of M_0. Assume that the units ε_j are positive. So we have:*

- *If there exists a unit λ of M_0 of the form $\varepsilon_1^{j_1}\varepsilon_2^{j_2}\ldots\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r$ (up to a permutation), where $j_k \in \{0,1\}$, such that $\beta\lambda$ is a square in M_0, then $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r-1}, \sqrt{-\lambda}\}$ is a FSU of M.*
- *Otherwise, $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ is a FSU of M.*

Lemma 2.7. *Let $p \equiv 3 \pmod 4$ and $l \equiv 5 \pmod 8$ be different primes. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit. Let $F = k(\sqrt{p\varepsilon\sqrt{l}}, \sqrt{-1})$, then $Q_{F^+} = 1$.*

Proof. From Proposition 2.5, to show that the index $Q_{F^+} = 1$, it suffices to show that $(2+\mu_2)\lambda = 2\lambda$ is not a square in F^+. In fact, suppose that 2λ is a square in F^+, which equivalent to $(2)_{F^+} = \mathcal{H}^2$ where $\mathcal{H}$ is a ideal of F^+, which is not the case, since 2 cannot ramify in F^+. □

Lemma 2.8. *Let $p \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$ be different primes. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit. Let $F' = k(\sqrt{\varepsilon\sqrt{l}}, \sqrt{-p})$, then $Q_{F'^+} = 1$.*

Proof. From Proposition 2.6, to show that the index $Q_{F'^+} = 1$, it suffices to show that $p\lambda$ is not a square in F'^+. In fact, suppose that $p\lambda$ is a square in F'^+, which equivalent to $(p)_{F'^+} = \mathcal{H}^2$ where $\mathcal{H}$ is a ideal of F'^+, which is impossible, since p cannot ramify in F'^+. □

Corollary 2.2, and Lemmas 2.4, 2.7 and 2.8 allow us to deduce the following theorem:

Theorem 2.9. *Let $p \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$ be different primes. Put $k = \mathbb{Q}(\sqrt{l})$ and denote by ε its fundamental unit, and set $K = k(\sqrt{-p\varepsilon\sqrt{l}})$. Let $F = K(\sqrt{-1})$ and $F' = K(\sqrt{-p})$, then*

(1) *If $l \equiv 5 \pmod 8$, we have:*
- $E_F = \langle \sqrt{-1}, \varepsilon, \varepsilon_{\chi_{F^+}}, \varepsilon_{\chi_{F^+}}^{\sigma_{F^+}} \rangle$;
- $E_{F'} = \begin{cases} \langle -1, \varepsilon, \varepsilon_{\chi_{F'^+}}, \varepsilon_{\chi_{F'^+}}^{\sigma_{F'^+}} \rangle & \text{if } p \neq 3; \\ \langle \xi_6, \varepsilon, \varepsilon_{\chi_{F'^+}}, \varepsilon_{\chi_{F'^+}}^{\sigma_{F'^+}} \rangle & \text{if } p = 3. \end{cases}$

(2) *If $l \equiv 1 \pmod 8$, we have*

$$E_{F'} = \begin{cases} \langle -1, \xi_{F'^+}, \xi_{F'^+}^{\sigma_{F'^+}}, \xi_{F'^+}^{\sigma^2_{F'^+}} \rangle & \text{if } p \neq 3; \\ \langle \xi_6, \xi_{F'^+}, \xi_{F'^+}^{\sigma_{F'^+}}, \xi_{F'^+}^{\sigma^2_{F'^+}} \rangle & \text{if } p = 3. \end{cases}$$

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Abdelmalek Azizi: Mohammed First University, Department of Mathematics, Faculty of Sciences, Oujda, Morocco.
E-mail address: abdelmalekazizi@yahoo.fr

Idriss Jerrari: Mohammed First University, Department of Mathematics, Faculty of Sciences, Oujda, Morocco.
E-mail address: Idriss_math@hotmail.fr

Abdelkader Zekhnini: Mohammed First University, Department of Mathematics and Informatics, Pluridisciplinary Faculty of Nador, Morocco.
E-mail address: zekha1@yahoo.fr

Mohammed Talbi: Regional Center of Education and Training, Oujda, Morocco.
E-mail address: talbimm@yahoo.fr