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# ON A TYPE OF REPRODUCING KERNEL HILBERT $C^*$-MODULES

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Abstract. Given a Hilbert $C^*$-module $M$ and a locally compact Hausdorff group $G$, we investigate a reproducing kernel semi-inner product $C^*$-module by supplying a positive semi-definite kernel on $G$ taking values in the space of adjointable operators on $M$. We examine the analogue of the evaluation functions and we establish several topological properties among which are boundedness, compactness and especially continuity.

## 1. Introduction

A basic concept in the theory of reproducing kernel Hilbert spaces (RKHS), as well as in many other branches of mathematics and machine learning, is that of "positive definite kernel", sometimes referred to as "positive semi-definite kernel". Aronszajn was among the first pathfinders to rigorously formulate the theory of reproducing kernels. In 1940s, he laid the conceptual foundations of RKHS by defining reproducing kernel and studying their fundamental properties [1, 2]. After, in 1967, Sarason established necessary and sufficient conditions for a kernel to be a reproducing kernel [17], thus providing a precise theoretical framework for the construction and characterisation of RKHS. Three years later, he explored properties of reproducing kernels with a continuous spectrum, opening up new perspectives in the analysis of RKHS. In 1964, the subjet gained interest of Schwartz in [18] where profound properties of RKHS are provided. More examples of reproducing kernel Hilbert space and its applications can be found in [16, 19].

Hilbert $C^*$-modules are natural generalization of Hilbert spaces where the inner product is a $C^*$-algebra valued sesquilinear map. They were first introduced by Kaplansky [10] in 1953, in proving that all derivation of an $AW^*$-algebra of type I is inner. His aim was to extend module theory beyond the classical framework of rings and vector spaces. One may find more details in [3, 11, 12]. The new trend is to recover for Hilbert modules the results obtained for the theory of Hilbert spaces.

In [15], Murphy introduced reproducing kernel Hilbert modules (RKHM) associated with $B(M)$-valued positive definite kernels which satisfied the Kolmogorov

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decompostion. Murphy explored the relationship between positive definite kernels and Hilbert $C^*$- modules, offering an in-depth analysis of properties and applications of these advanced mathematical concepts. A RKHM gives richer data than a RKHS.

In 2010, Szafraniec discussed the result of Murphy in [20] and underlined some limitations of [15]. In [13], Moslehian examined the concept of conditionally positive definite kernel in the framework of Hilbert $C^*$-modules. Meanwhile, he also studied the vector valued reproducing kernel Hilbert spaces [14]. Many examples of RKHM can be found in [8, 9]. Recently, in [4], the definition of RKHM from [8] was modified following some ideas of [20]. The authors also provided a construction of the exterior tensor product of RKHMs. Later, in 2023, Hashimoto et al. [6] investigated RKHM as a toolkit for kernels methods to examine deep learning. Readers are referred to [5] for more details on the applications of RKHM in machine learning.

In [7], Hashimoto et al. constructed a vector valued RKHS by defining a positive definite kernel with values in the $C^*$- algebra $B(H)$ where $H$ is a Hilbert space. In this paper, we consider the positive definite kernel as given in [14, 15]. We define the RKHM in the way to generalize the RKHS introduced in [7] by Hashimoto et al. Our goal is to go through the same walk with a positive definite kernel with values in $B(M)$, the space of adjointable operators on a Hilbert $C^*$-module $M$. We introduce the analogue of evaluation function and scrutnize some of their properties.

The paper is organized as follows. Section 2 sets the notations and definitions which will need in the sequel. Section 3 exposes our main results.

## 2. Preliminaries

Let $A$ be a $C^*$-algebra. We denote by $*$ the involution of $A$ and by $\|\cdot\|_A$ the norm of $A$.

**Definition 2.1.** An $A$-module $M$ is said to be a (right) pre-Hilbert $A$-module if $M$ is equipped with a sesquilinear map $\langle\cdot,\cdot\rangle_M : M \times M \to A$ such that

(i) $\langle x, x\rangle_M \geqslant 0$ for each $x \in M$,
(ii) $\langle x, x\rangle_M = 0$ implies that $x = 0$,
(iii) $\langle y, x\rangle_M = \langle x, y\rangle_M^*$ for all $x, y \in M$,
(iv) $\langle x, ya\rangle_M = \langle x, y\rangle_M a$ for all $x, y \in M$ and for all $a \in A$.

The sesquilinear map $\langle\cdot,\cdot\rangle_M$ is called an $A$-valued inner product. If the condition (ii) does not hold, then it is called semi-inner product and $M$ is said to be a semi-inner product $A$-module. The map $M \to \mathbb{R}_+$, $x \mapsto \|x\|_M = \|\langle x, x\rangle_M\|_A^{\frac{1}{2}}$ is a norm on $M$ and the following properties hold :

(i) $\|xa\|_M \leqslant \|x\|_M\|a\|_A$ for all $x \in M$, $a \in A$,
(ii) $\langle x, y\rangle_M\langle y, x\rangle_M \leqslant \|y\|_M\langle x, x\rangle_M$ for all $x, y \in M$,
(iii) $\|\langle x, y\rangle_M\|_A \leqslant \|x\|_M\|y\|_M$ for all $x, y \in M$.

If $M$ is complete under the norm $\|\cdot\|_M$, then $M$ is called a Hilbert $A$-module or Hilbert $C^*$-module over $A$.

**Example 2.2.** (i) Every $C^*$-algebra $A$ is a Hilbert $A$-module where the sesquilinear map is defined by $\langle a, b\rangle = a^*b$.

(ii) Let $H$ be an Hilbert space. Then, $H$ is a Hilbert $B(H)$-module where the sesquilinear map is defined by $\langle x, y\rangle = \langle x, y\rangle_H I_H$ where $I_H$ is the identity operator on $H$ and $\langle\cdot,\cdot\rangle_H$ is the inner product of $H$. The action of the $B(H)$-module $H$ is given by $T\cdot x = Tx$, $T \in B(H), x \in H$.

(iii) Let $X$ be a nonempty compact space. Let $C(X)$ be the algebra of continuous complex valued functions on $X$ and et $L^2(X)$ be space of square module integrable complex valued functions on $X$. It is well known that $C(X)$ is a $C^*$-algebra and $L^2(X)$ is a Hilbert space. Let us denote by $(\cdot,\cdot)_{L^2(X)}$ its inner product. The space $L^2(X)$ is a Hilbert $C(X)$-module where the action of $C(X)$ on $L^2(X)$ is the natural product of functions and the $C(X)$-valued inner product $\langle\cdot,\cdot\rangle_{L^2(X)}$ is defined by $\langle f, g\rangle_{L^2(X)} : X \to \mathbb{C}$, $s \mapsto (f, g)_{L^2(X)}$ for all $f, g \in L^2(X)$.

**Definition 2.3.** (See [11, page 8]) Let $(M, \langle\cdot,\cdot\rangle_M)$ and $(N, \langle\cdot,\cdot\rangle_N)$ be Hilbert $A$-modules. A map (or operator) $T : M \to N$ is said to be adjointable if there exists a map $T^* : N \to M$ such that $\langle Tx, y\rangle_N = \langle x, T^*y\rangle_M$ for all $x \in M$ and for all $y \in N$.

As explained in [11, page 8] or in [12, page 15], Definition 2.3 implies that any adjointable map is necessarily $\mathbb{C}$-linear, $A$-linear and bounded. The operator norm of $T$ is given by

$$\|T\| = \sup\{\|Tx\|_N : \|x\|_M \leqslant 1\}.$$

We designate by $B(M)$ the space adjointable operators on the Hilbert $A$-module $M$. Let $G$ be a locally compact, Hausdorff group. We denote by $B(M)^G$ the space of $B(M)$-valued functions on $G$.

## 3. Main results

**Definition 3.1.** A map $k : G \times G \to B(M)$ is called a positive semi-definite kernel if it satisfies the following conditions:

(i) $\forall x, y \in G$, $k(x, y) = k(y, x)^*$,

(ii) $\displaystyle\sum_{i,j=1}^{n} \langle u_i, k(x_i, x_j)u_j\rangle_M \geqslant 0$, $\forall n \in \mathbb{N}$, $\forall u_1, \cdots u_n \in M$, $\forall x_1, \cdots, x_n \in G$.

**Example 3.2** (see [14])**.** Consider a function $f : G \to M$. The map $k$ from $G\times G$ into $B(M)$ defined by $k(x, y)(u) = f(x)\langle f(y), u\rangle_M$ for $x, y \in G$ and $u \in M$, is a positive semi-definite kernel. In fact,

(i) For all $x, y \in G$ and $u, v \in M$, we have

$$\begin{aligned}
\langle k(x, y)u, v\rangle_M &= \langle f(x)\langle f(y), u\rangle_M, v\rangle_M \\
&= \langle f(x), v\rangle_M \langle f(y), u\rangle_M^* \\
&= \langle f(x), v\rangle_M \langle u, f(y)\rangle_M \\
&= \langle f(x), v\rangle_M \langle u, f(y)\langle f(x), v\rangle_M\rangle_M \\
&= \langle u, k(y, x)v\rangle_M.
\end{aligned}$$

Hence, $k(x,y)^* = k(y,x)$.

(ii) Let $u_1, \cdots, u_n \in M$; $x_1, \cdots, x_n \in G$ with $n \in \mathbb{N}$. We have

$$\begin{aligned}
\sum_{i,j=1}^{n} \langle u_i, k(x_i,x_j)u_j\rangle_M &= \sum_{i,j=1}^{n} \langle u_i, f(x_i)\langle f(x_j), u_j\rangle_M\rangle_M \\
&= \sum_{i,j=1}^{n} \langle u_i, f(x_i)\rangle_M \langle f(x_j), u_j\rangle_M \\
&= \sum_{i,j=1}^{n} \langle f(x_i), u_i\rangle_M^* \langle f(x_j), u_j\rangle_M \\
&= \left(\sum_{i=1}^{n} \langle f(x_i), u_i\rangle_M\right)^* \left(\sum_{j=1}^{n} \langle f(x_j), u_j\rangle_M\right) \geqslant 0.
\end{aligned}$$

Consider the map $\phi : G \to B(M)^G$, $x \mapsto \phi(x) = k(\cdot, x)$. Set

$$H_{k,0} = \left\{ \sum_{i=1}^{n} \phi(x_i)u_i : n \in \mathbb{N}, u_i \in M, x_i \in G \right\} \tag{3.1}$$

and consider the map $\langle \cdot, \cdot\rangle_{H_k} : H_{k,0} \times H_{k,0} \to A$ defined by

$$\left\langle \sum_{i=1}^{n} \phi(x_i)u_i, \sum_{j=1}^{m} \phi(y_j)v_j \right\rangle_{H_{k,0}} = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle u_i, k(x_i,y_j)v_j\rangle_M. \tag{3.2}$$

**Proposition 3.3.** *(i) The space $H_{k,0}$ is an $A$-submodule of $B(M)^G$.*
*(ii) The map $\langle \cdot, \cdot\rangle_{H_{k,0}}$ is an $A$-valued positive sesquilinear map on $H_{k,0}$.*

*Proof.* (i) The space $H_{k,0}$ is a $\mathbb{C}$-vector space. Let us show that $H_{k,0}$ is stable for the action of $A$ on $H_{k_0}$. Let $\mu = \sum_{i=1}^{n} \phi(x_i)u_i \in H_{k,0}$. We have $a\mu = a\sum_{i=1}^{n} \phi(x_i)u_i = \sum_{i=1}^{n} a\phi(x_i)u_i = \sum_{i=1}^{n} \phi(x_i)(au_i) \in H_{k,0}$ since $au_i \in M$. Hence, $H_{k,0}$ is an $A$-module included in $B(M)^G$.

(ii) On one hand, the map $\langle \cdot, \cdot\rangle_{H_k}$ is well defined. Indeed, if $\mu$ is an element of $H_{k,0}$ with two distinct representations

$$\mu = \sum_{i=1}^{n} \phi(x_i)u_i = \sum_{j=1}^{m} \phi(y_j)v_j,$$

then we have

$$\begin{aligned}
\sum_{i,j=1}^{n}\langle u_i, k(x_i,x_j)u_j\rangle_M &= \sum_{i=1}^{n}\left\langle u_i, \sum_{j=1}^{n} k(x_i,x_j)u_j\right\rangle_M = \sum_{i=1}^{n}\langle u_i, \mu(x_j)\rangle_M \\
&= \sum_{i=1}^{n}\left\langle u_i, \sum_{l=1}^{m} k(x_j,y_l)v_l\right\rangle_M \\
&= \sum_{l=1}^{m}\left\langle \sum_{i=1}^{n} k(y_l,x_i)u_i, v_l\right\rangle_M \\
&= \sum_{l=1}^{m}\left\langle \sum_{j=1}^{m} k(y_l,y_j)v_j, v_l\right\rangle_M \\
&= \sum_{l,j=1}^{m}\langle k(y_l,y_j)v_j, v_l\rangle_M = \sum_{l,j=1}^{m}\langle v_j, k(y_j,y_l)v_l\rangle_M.
\end{aligned}$$

So, the definition of the map $\langle\cdot,\cdot\rangle_{H_{k,0}}$ is independent of the representation of $\mu$. Let $\mu=\sum_{i=1}^{n}\phi(x_i)u_i$, $\nu=\sum_{j=1}^{m}\phi(y_j)v_j$, $\omega=\sum_{l=1}^{p}\phi(z_k)w_k\in H_{k,0}$ and $a,b\in A$.

(1) We have

$$\begin{aligned}
\langle\mu,\mu\rangle_{H_{k,0}} &= \left\langle \sum_{i=1}^{n}\phi(x_i)u_i, \sum_{j=1}^{n}\phi(x_j)u_j\right\rangle_{H_{k,0}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle u_i, k(x_i,x_j)u_j\rangle_M \geqslant 0 \text{ by hypothesis.}
\end{aligned}$$

(2) We have

$$\begin{aligned}
\langle\mu,\nu\rangle^*_{H_{k,0}} &= \left\langle \sum_{i=1}^{n}\phi(x_i)u_i, \sum_{j=1}^{n}\phi(y_j)v_j\right\rangle^*_{H_{k,0}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle u_i, k(x_i,x_j)v_j\rangle^*_M \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle v_j, k(x_j,x_i)u_i\rangle_M = \langle\nu,\mu\rangle_{H_{k,0}}.
\end{aligned}$$

(3) We have

$$
\begin{aligned}
\langle \mu, a\nu\rangle_{H_{k,0}} &= \left\langle \sum_{i=1}^{n} \phi(x_i)u_i, a\sum_{j=1}^{m} \phi(y_j)v_j \right\rangle_{H_{k,0}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle u_i, k(x_i,y_j)(av_j)\rangle_M \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle u_i, k(x_i,y_j)v_j\rangle_M a \\
&= \langle \mu, \nu\rangle_{H_{k,0}} a.
\end{aligned}
$$

(4) Let us set $(t_q, r_q) = (y_q, v_q)$ if $q = 1, \cdots, m$ and $(t_q, r_q) = (z_l, w_l)$ if $q - m = l$, $l = 1, \cdots, p$. We have $\nu + \omega = \sum_{j=1}^{m} \phi(y_j)v_j + \sum_{l=1}^{p} \phi(z_k)w_k = \sum_{q=1}^{m+p} \phi(t_q)r_q$, and

$$
\begin{aligned}
\langle \mu, \nu\rangle_{H_{k,0}} + \langle \mu, \omega\rangle_{H_{k,0}} &= \left\langle \sum_{i=1}^{n} \phi(x_i)u_i, \sum_{j=1}^{m} \phi(y_j)v_j \right\rangle_{H_{k,0}} + \left\langle \sum_{i=1}^{n} \phi(x_i)u_i, \sum_{l=1}^{p} \phi(z_l w_l \right\rangle_{H_{k,0}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle u_i, k(x_i,y_j)v_j\rangle_M + \sum_{i=1}^{n}\sum_{l=1}^{p} \langle u_i, k(x_i,z_l)w_l\rangle_M \\
&= \sum_{i=1}^{n}\sum_{p=1}^{m+p} \langle u_i, k(x_i,t_q)r_q\rangle_M \\
&= \left\langle \sum_{i=1}^{n} \phi(x_i)u_i, \sum_{q=1}^{m+p} \phi(t_q)r_q \right\rangle_{H_{k,0}} \\
&= \langle \mu, \nu+\omega\rangle_{H_{k,0}}.
\end{aligned}
$$

Then, it follows that $\langle\cdot,\cdot\rangle_{H_{k,0}}$ is an $A$-valued positive sesquilinear map on $H_{k,0}$. It defines a semi-inner product on $H_{k,0}$ turning it into a semi-inner product $C^*$-module. □

Set

$$
N_k = \left\{ \mu = \sum_{i=1}^{n} \phi(x_i)u_i \in H_{k,0} : n \in \mathbb{N}, x_i \in G, u_i \in M \text{ and } \langle \mu, \mu\rangle_{H_{k,0}} = 0 \right\}.
$$

**Proposition 3.4.** *The space $N_k$ is an $A$-submodule of $H_{k,0}$.*

*Proof.* The proof can be checked by using the Cauchy-Schwarz inequality. □

Let us designate by $Cl(N_k)$ the closure of $N_k$ with respect to the semi-norm derived from the semi-inner product of $H_{k,0}$ and $\tilde{H}_{k,0} = H_{k,0}/Cl(N_k) = \{\tilde{\mu} = \mu + Cl(N_k), \mu \in H_{k,0}\}$ the quotient $A$-module. The map $\langle\cdot,\cdot\rangle : \tilde{H}_{k,0} \times \tilde{H}_{k,0} \to A$ defined by

$$\langle \tilde{\mu}, \tilde{\nu} \rangle = \langle \mu, \nu \rangle_{H_{k,0}}$$

is an $A$-valued inner product on $\tilde{H}_{k,0}$. The following consequence holds.

**Corollary 3.5.** *The space $\tilde{H}_{k,0}$ is a pre-Hilbert $A$-module equipped with $\langle \cdot, \cdot \rangle$.*

**Proposition 3.6.** *The map $\langle \cdot, \cdot \rangle_{H_{k,0}}$ has the following reproducing property :*

$$\langle \phi(x)w, \mu \rangle_{H_{k,0}} = \langle w, \mu(x) \rangle_M, \quad \forall \mu \in H_{k,0}, \forall x \in G, \forall w \in M. \tag{3.3}$$

*Proof.* Let $\mu = \sum_{i=1}^{n} \phi(x_i)u_i \in H_{k,0}$, $x_i \in G$, $u_i \in M$, $x \in G$ and $w \in M$. We have

$$\begin{aligned} \langle \phi(x)w, \mu \rangle_{H_{k,0}} &= \left\langle \phi(x)w, \sum_{i=1}^{n} \phi(x_i)u_i \right\rangle_{H_{k,0}} \\ &= \left\langle w, \sum_{i=1}^{n} k(x, x_i)u_i \right\rangle_M \\ &= \langle w, \mu(x) \rangle_M. \end{aligned}$$

□

Denote by $H_k$ the completion of $H_{k,0}$ with respect to the semi-inner product $\langle \cdot, \cdot \rangle_{H_{k,0}}$ and designate by $\langle \cdot, \cdot \rangle_{H_k}$ the extended semi-inner product to $H_k$. Naturaly, $\| \cdot \|_{H_k}$ denotes the corresponding semi-norm. We have the following statement.

**Proposition 3.7.** *If the kernel $k$ is continuous then the map $G \times M \to H_{k,0}$, $(x, w) \mapsto \phi(x)w = k(\cdot, x)w$ is also continuous.*

*Proof.* Let $(x_0, w_0) \in G \times M$. For $(x, w)$ in a neighborhood of $(x_0, w_0)$, we get

$$\begin{aligned} \|\phi(x)w - \phi(x_0)w_0\|_{H_k}^2 &= \|\langle \phi(x)w - \phi(x_0)w_0, \phi(x)w - \phi(x_0)w_0 \rangle_{H_k}\|_A \\ &= \|\langle w, k(x,x)w \rangle_M - \langle w, k(x, x_0)w_0 \rangle_M \\ &\quad - \langle w_0, k(x_0, x)w \rangle_M + \langle w_0, k(x_0, x_0)w_0 \rangle_M\|_A \\ &\leqslant \|\langle w, k(x,x)w - k(x, x_0)w_0 \rangle_M\|_A \\ &\quad + \|\langle w_0, k(x_0, x_0)w_0 - k(x_0, x)w \rangle_M\|_A \\ &\leqslant \|w\|_M^2 \|k(x,x) - k(x, x_0)\| + \|w\|_M \|k(x, x_0)(w - w_0)\|_M \\ &\quad + \|w_0\|_M \|k(x_0, x)(w - w_0)\|_M \\ &\quad + \|w_0\|_M^2 \|k(x_0, x) - k(x_0, x_0)\| \\ &\leqslant \|w\|_M^2 \|k(x,x) - k(x, x_0)\| + \|w\|_M \|w - w_0\|_M \|k(x, x_0)\| \\ &\quad + \|w_0\|_M \|w - w_0\|_M \|k(x_0, x)\| \\ &\quad + \|w_0\|_M^2 \|k(x_0, x) - k(x_0, x_0)\|. \end{aligned}$$

The conclusion follows from the continuity of $k$. □

The following statement holds.

**Proposition 3.8.** *Let $(f_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements of $H_k$ with limit $f \in H_k$. Then, for all $x \in G$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ in $M$. Furthermore, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $G$.*

*Proof.* For $x \in G$, $w \in M$, we have

$$\|\phi(x)w\|^2_{H_k} = \|\langle \phi(x)w, \phi(x)w\rangle_{H_k}\|_A = \|\langle w, k(x,x)w\rangle_M\|_A \leqslant \|w\|^2_M \|k(x,x)\|.$$

Hence,

$$\|\phi(x)w\|_{H_k} \leqslant \left(\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}}\right) \|w\|_M.$$

Assume that $\|f_n - f\|_{H_k} \to 0$, as $n \to +\infty$. Let $x \in G$. We have from (3.3)

$$\begin{aligned} \|f_n(x) - f(x)\|^2_M &= \|\langle f_n(x) - f(x), f_n(x) - f(x)\rangle_M\|_A \\ &= \|\langle \phi(x)(f_n(x) - f(x)), f_n - f\rangle_{H_k}\|_A \\ &\leqslant \|\phi(x)(f_n(x) - f(x))\|_{H_k} \|f_n - f\|_{H_k} \\ &\leqslant \left(\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}}\right) \|f_n(x) - f(x)\|_M \|f_n - f\|_{H_k}. \end{aligned}$$

Therefore,

$$\|f_n(x) - f(x)\|_M \left(\|f_n(x) - f(x)\|_M - \left(\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}}\right) \|f_n - f\|_{H_k}\right) \leqslant 0.$$

The latter inequality implies $\|f_n(x) - f(x)\|_M \leqslant \left(\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}}\right) \|f_n - f\|_{H_k}$.
Thus, $\|f_n(x) - f(x)\|_M \to 0$ as $n \to +\infty$ since $\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}} < \infty$ by the definition of $k$.
On the other hand, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $G$ since

$$\sup_{x\in G} \|f_n(x) - f(x)\|_M \leqslant \left(\sup_{s\in G} \|k(s,s)\|^{\frac{1}{2}}\right) \|f_n - f\|_{H_k} \to 0 \text{ as } n \to +\infty.$$

□

We have the following definition of the evaluation map.

**Definition 3.9.** Let $w_0$ be a non null vector of $M$. For $x \in G$, the evaluation map at $x$ is the map $\varphi_x : H_k \to A$, $\mu \mapsto \langle \phi(x)w_0, \mu\rangle_{H_k}$.

**Proposition 3.10.** *For $x \in G$, the evaluation map $\varphi_x$ is an adjointable operator (hence it is a bounded $\mathbb{C}$-linear $A$-linear) with $\|\varphi_x\| = \|\phi(x)w_0\|_{H_k}$.*

*Proof.* Let $x \in G$, $\mu \in H_k$ and $a \in A$. We have

$$\begin{aligned} \langle \varphi_x(\mu), a\rangle_A &= (\varphi_x(\mu))^* a \\ &= \langle \phi(x)w_0, \mu\rangle^*_{H_k} a \\ &= \langle \mu, \phi(x)w_0\rangle_{H_k} a \\ &= \langle \mu, a\phi(x)w_0\rangle_{H_k}. \end{aligned}$$

Thus, $\varphi_x$ is adjointable with adjoint map $\varphi^*_x : A \to H_k$ defined by $\varphi^*_x(a) = a\phi(x)w_0$. Moreover, for $x \in G$ and $\mu \in H_k$, we have $\|\varphi_x(\mu)\|_A = \|\langle \phi(x)w_0, \mu\rangle_{H_k}\|_A \leqslant$

$\|\phi(x)w_0\|_{H_k}\|\mu\|_{H_k}$. Then, $\|\varphi_x\| = \sup\{\|\varphi_x(\mu)\|_A : \|\mu\|_{H_k} \leqslant 1\} \leqslant \|\phi(x)w_0\|_{H_k}$. Furthermore, put $\eta = \dfrac{\phi(x)w_0}{\|\phi(x)w_0\|_{H_k}}$. We obtain

$$\begin{aligned}
1 &= \|\eta\|_{H_k}^2 \\
&= \|\langle \eta, \eta\rangle\|_A = \left\|\left\langle \frac{\phi(x)w_0}{\|\phi(x)w_0\|_{H_k}}, \eta\right\rangle\right\|_A \\
&= \frac{1}{\|\phi(x)w_0\|_{H_k}}\|\langle \phi(x)w_0, \eta\rangle\|_A \\
&= \frac{1}{\|\phi(x)w_0\|_{H_k}}\|\varphi_x(\eta)\|_A \\
&\leqslant \frac{1}{\|\phi(x)w_0\|_{H_k}}\|\varphi_x\|\|\eta\|_{H_k} \\
&\leqslant \frac{1}{\|\phi(x)w_0\|_{H_k}}\|\varphi_x\|.
\end{aligned}$$

Hence, $\|\phi(x)w_0\|_{H_k} \leqslant \|\varphi_x\|$. Therefore, $\|\varphi_x\| = \|\phi(x)w_0\|_{H_k}$. □

**Proposition 3.11.** *If the kernel $k$ is continuous, then the map $\varphi : G \to A^{H_k}$, $x \mapsto \varphi_x$ is also continuous.*

*Proof.* The continuity of $\varphi$ results from Proposition 3.7 together with the following computation : for $x_0 \in G$ and $x$ in a neighborhood of $x_0$,

$$\begin{aligned}
\|\varphi(x) - \varphi(x_0)\| &= \|\varphi_x - \varphi_{x_0}\| \\
&= \sup\{\|\varphi_x(\mu) - \varphi_{x_0}(\mu)\|_A : \|\mu\|_{H_k} \leqslant 1\} \\
&= \sup\{\|\langle \phi(x)w_0, \mu\rangle - \langle \phi(x_0)w_0, \mu\rangle\|_A : \|\mu\|_{H_k} \leqslant 1\} \\
&= \sup\{\|\langle (\phi(x) - \phi(x_0))w_0, \mu\rangle\|_A : \|\mu\|_{H_k} \leqslant 1\} \\
&\leqslant \sup\{\|(\phi(x) - \phi(x_0))w_0\|_{H_k}\|\mu\|_{H_k} : \|\mu\|_{H_k} \leqslant 1\} \\
&\leqslant \|(\phi(x) - \phi(x_0))w_0\|_{H_k}.
\end{aligned}$$

□

## 4. Conclusion

In this paper, we introduced a type of reproducing kernel Hilbert $C^*$-module and checked some of its fundamental properties. Next, we intend to investigate reproducing kernel quasi-inner product $C^*$-modules, thereby extending the research conducted in the present article.

## Statements and Declarations

On behalf of all authors, the corresponding author states that there is no financial or proprietary interests in any material discussed in this article.

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