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# ON THE FIBONACCI NUMBERS AND THEIR SUMS WHICH ARE CLOSE TO A POWER OF 3

ANOUAR GAHA[1*] and SOUFIANE MEZROUI[2]

Abstract. Let $(L_n)_{n\geq 0}$ be the Lucas sequence defined by $L_{n+2} = L_{n+1} + L_n$, for all $n \geq 0$, with initial values $L_0 = 2$ and $L_1 = 1$. In this paper, we find all the Fibonacci numbers $2F_n$ and sums of two Fibonacci numbers which are close to a power of 3. As a corollary, we determine all Lucas numbers close to a power of 3. To prove our results, we will use Baker's theory lower bound for linear forms in logarithms of algebraic numbers, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

## 1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence which is the binary recurrence relation given by $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Let also $(L_n)_{n\geq 0}$ be the Lucas sequence that follows the same recursive pattern as the Fibonacci numbers with initial conditions $L_0 = 2$ and $L_1 = 1$. The elements of these sequences are referred to as Fibonacci numbers and Lucas numbers, respectively. The problem of determining all integer solutions of Diophantine equations when Fibonacci numbers has piqued the curiosity of mathematicians and there is a broad literature on this subject. For instance, Bugeaud, Mignotte, and Siksek [2] solved that the only perfect powers among the Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$. More precisely, the only powers of 2 in the Fibonacci sequence are 1, 2 and 8.

We say that an integer $n$ is said to be close to a positive integer $m$, if it satisfies

$$|n - m| < \sqrt{m}.$$

In 2014, Chern and Cui [3] found all Fibonacci numbers $F_n$ that are close to a power of 2, in the sense that they solved the Diophantine inequality

$$|F_n - 2^m| < 2^{m/2}$$

in positive integers $n$, $m$. This result has been generalized in several ways. In particular, recently Hasanalizade [5] determined the Diophantine inequality

$$|F_n + F_m - 2^a| < 2^{a/2}$$

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* Corresponding author.

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in positive integers $(n, m, a)$ with $n \geq m$, and as a Corollary they also determined

$$|L_n - 2^a| < 2^{a/2}.$$

Here, $L_n$ denotes the $n$-th Lucas number, and the trick is that $L_n = F_{n-1} + F_{n+1}$. Previously, Nikiema and Odjoumani [7] give effective bounds for the Diophantine equation $N_m = U_n$ in positive unknowns $m$ and $n$. They solved explicitly this equation when Fibonacci, Pell and Balancing sequences cases. However, Ziegler [8] studied that Diophantine equation $F_n + F_m = y^a$ has a solution in positive integers $(n, m, a)$ with $n > m > 0$, unless $y = 2, 3, 4, 6$ or 10.

In the present paper, we study the solutions $(2F_n, 3^m)$ of the Diophantine inequality

$$|2F_n - 3^m| < 3^{m/2} \tag{1.1}$$

in positive integers $n$ and $m$ with $n \geq m$. Following that, we search for the sum of two Fibonacci numbers which are close to a power of 3, *i.e.*, we also study the Diophantine inequality

$$|F_n + F_m - 3^a| < 3^{a/2} \tag{1.2}$$

in positive integers $(n, m, a)$ with $n \geq m$. More specifically, we prove the following main theorems.

**Theorem 1.1.** *There are only* 5 *Fibonacci numbers which are close to a power of* 3*. Namely, the solutions* $(2F_n, 3^m)$ *of the inequality* (1.1) *in positive integers* $n$ *and* $m$ *with* $n \geq m$*, are given by*

$$(2F_n, 3^m) \in \{(2, 3), (4, 3), (10, 9), (26, 27), (754, 729)\}.$$

**Theorem 1.2.** *There are exactly* 37 *solutions of Diophantine inequality* (1.2) *in positive integers* $(n, m, a)$ *with* $n \geq m$*. All the solutions satisfy* $n \leq 29$*,* $m \leq 22$*, and* $a \leq 12$*. The list of solutions is given in the appendix (Table 2).*

**Corollary 1.3.** *There are only* 7 *Lucas numbers which are close to a power of* 3*. Namely, the solutions of the inequality*

$$|L_n - 3^a| < 3^{a/2}$$

*in positive integers* $(n, a)$ *with* $n \geq a$*, are given by*

$$(n, a) \in \{(2, 1), (3, 1), (4, 2), (5, 2), (7, 3), (9, 4), (16, 7)\}.$$

In order to prove Theorem 1.1 and 1.2, we shall need to use a few applications of a Baker's theory lower bound for nonzero linear forms in logarithms of algebraic numbers due to Matveev [6] and Dujella-Pethő [4] to get a large upper bound for $n$. However, we try to apply a version of the Baker-Davenport reduction method we reduce this large bound.

## 2. Preliminary results

First, we recall that the Binet formula for the general term of the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for all $n \geq 0$, where

$$\alpha := \frac{1+\sqrt{5}}{2} \text{ and } \beta := \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic polynomial $x^2 - x - 1 = 0$ of $(F_n)_{n\geq 0}$. The relation between the Fibonacci numbers and $\alpha$, are given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{2.1}$$

holds for all $n \geq 1$. From the equation (5) of Ziegler [8], we get

$$0.38\alpha^n \leq F_n \leq 0.48\alpha^n \tag{2.2}$$

for every positive integer $n > 1$.

**Definition 2.1** (Mahler measure)**.** Let $\eta$ be an algebraic number, we define its measure by the following identity

$$M(\eta) = |a_d| \prod_{i=1}^{d} \max\{1, |\eta_i|\},$$

with $\eta_1, \eta_2, \ldots, \eta_d$ are the roots of $f(x) = a_d \prod_{i=1}^{d}(x - \eta_i)$ is the minimal polynomial of $\eta$.

**Definition 2.2** (Absolute logarithmic height)**.** Let $\eta$ be an algebraic number of degree $d$ on $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d \prod_{i=1}^{d} (x - \eta_i),$$

where the $a_1, a_2, \ldots, a_d$ are relatively prime integers with $a_d > 0$ and the $\eta_1, \eta_2, \ldots, \eta_d$ are conjugates of $\eta$. The usual absolute logarithmic height of $\eta$ is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_d + \sum_{i=1}^{d} \log\left(\max\{1, |\eta_i|\}\right)\right) = \frac{1}{d}\log M(\eta).$$

In particular, if $\eta = \frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and $\gcd(p, q) = 1$, then $h(\eta) = \log\left(\max\{|p|, q\}\right)$.

The following properties of $h(\eta)$, which will be used in the next section. Let $\eta$, $\kappa$ be algebraic numbers and $t \in \mathbb{Z}$. Then,

- $h(\eta \pm \kappa) \leq h(\eta) + h(\kappa) + \log 2$;
- $h(\eta\kappa^{\pm 1}) \leq h(\eta) + h(\kappa)$;
- $h(\eta^t) = |t|h(\eta)$.

Using the above notation, we shall need a lower bound for linear forms in logarithms due to Matveev [6] or Bugeaud, Mignotte, and Siksek [2, Theorem 9.4] proved the following theorem.

**Theorem 2.3** (Matveev's theorem). *Assume that $\eta_1, \eta_2, \ldots, \eta_l$ are positive real algebraic numbers in a real algebraic number field $\mathbb{L}$ of degree $D$. Let $b_1, b_2, \ldots, b_l \in \mathbb{Z}$, and*

$$\Gamma := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 = \left( \prod_{i=1}^{l} \eta_i^{b_i} \right) - 1$$

*is not zero. Then*

$$|\Gamma| > \exp\left(-1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_l\right)$$

*where $A_1, A_2, \ldots, A_l$ are real numbers,*

$$B \geq \max\{|b_1|, \ldots, |b_l|\},$$

*and*

$$A_i \geq \max\{Dh(\eta_i), |\log(\eta_i)|, 0.16\} \text{ for all } i = 1, \ldots, l.$$

To prove our main results, we need a variant of the Baker-Davenport [1, Lemma], which is due to Dujella and Pethő [4]. To do this, we use some results from the theory of continued fractions, Diophantine approximation, and we also use a few applications of a Baker's theory lower bound for not zero linear forms in logarithms of algebraic numbers. For this reason, we state a result to Matveev [6] for the general lower bound on linear forms in logarithms, and we apply a version of the Baker-Davenport reduction method to reduce the upper bound which is generally very large.

Here, for a real number $x$, we write $\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}$ the distance of $x$ to the nearest integer.

We shall also need the following Lemma due to Dujella and Pethő [4].

**Lemma 2.4** (Dujella and Pethő). *Let $M$ be a positive integer and suppose that $p/q$ be a convergent of the continued fraction of the irrational number $\gamma$ such that $q$ exceeds $6M$, and let $A$, $B$, $\mu$ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |\lambda\gamma - \nu + \mu| < AB^{-\omega},$$

*in positive integers $\lambda$, $\nu$ and $\omega$ with*

$$\lambda \leq M \text{ and } \omega \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The next following result of Legendre's criterion, are well-known from the theory of Diophantine approximation, is used for reduce the upper bounds on variables that are very large.

**Lemma 2.5** (Legendre's criterion). *Let $\rho$ be an irrational number, $\frac{r}{s} = \frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ are all the convergents of the continued fraction of $\rho$ and $K$ be a positive integer. Let $N$ be a nonnegative integer such that $q_N > K$. Then putting $a_K := \max\{a_k : k = 0, 1, \ldots, N\}$, the inequality*

$$\left|\rho - \frac{r}{s}\right| > \frac{1}{(a_K + 2)s^2},$$

*is valid for all pairs* $(r, s)$ *of positive integers with* $0 < s < K$.

**Lemma 2.6.** *For any not zero real number* $x$*, we have the following:*

(1) $0 < x < |e^x - 1|$.
(2) *If* $x < 0$ *and* $|e^x - 1| < \frac{1}{2}$*, then* $|x| < 2|e^x - 1|$.

## 3. The Proof of Theorem 1.1

*Proof.* Assume throughout that equation (1.1) holds. Firstly, we discuss the two following cases.

- `Case 1`. In the range $1 \le m \le 10$, then we have list of the first 10 intervals to $I_m := (3^m - 3^{m/2}, 3^m + 3^{m/2})$ are given in the appendix (Table 1), and we find all the Fibonacci numbers $2F_n$ in them. For this, we obtain that the only Fibonacci numbers $2F_n$ which are close to $3^m$ with $1 \le m \le 10$ are 2, 4, 10, 26 and 754.
- `Case 2`. When $m > 10$, then by Binet formula of $(F_n)_{n\ge 0}$, we get

$$\left|2F_n - \frac{2}{\sqrt{5}}\alpha^n\right| = \frac{2}{\sqrt{5}}|\beta|^n = \frac{2}{\sqrt{5}\alpha^n}, \quad \text{because} \quad \beta := -\frac{1}{\alpha}.$$

Combining it with (1.1), we obtain

$$\left|3^m - \frac{2}{\sqrt{5}}\alpha^n\right| < 3^{m/2} + \frac{2}{\sqrt{5}\alpha^n},$$

Dividing both sides of the above inequality by $3^m$, we get

$$\left|1 - 2\cdot 3^{-m}\cdot\alpha^n\cdot\sqrt{5}^{-1}\right| < 3^{-m/2} + \frac{2}{3^m\alpha^n\sqrt{5}} < \frac{3^{-\frac{m}{2}+1}}{2}. \tag{3.1}$$

In order to apply Matveev's result Theorem 2.3, we take

$$(\eta_1, b_1) := (2, 1), \quad (\eta_2, b_2) := (3, -m), \quad (\eta_3, b_3) := (\alpha, n), \text{ and } (\eta_4, b_4) := (\sqrt{5}, -1).$$

Notice that $\mathbb{L} := \mathbb{Q}(\sqrt{5})$ contains $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ are the four positive real numbers and has $D := [\mathbb{L} : \mathbb{Q}] = 2$. For this end, we can take $h(\eta_1) = \log 2 = 0.6931\ldots$, $h(\eta_2) = \log 3 = 1.0986\ldots$, $h(\eta_3) = \frac{1}{2}\log\alpha = 0.2406\ldots$, and $h(\eta_4) = \log\sqrt{5} = 0.8047\ldots$. Therefore, we also take $A_1 := 1.4 > Dh(\eta_1)$, $A_2 := 2.2 > Dh(\eta_2)$, $A_3 := 0.5 > Dh(\eta_3)$, and $A_4 := 1.7 > Dh(\eta_4)$. So, according to inequalities (1.1) and (2.1), we obtain

$$3^m - 3^{m/2} < 2F_n \le 2\alpha^{n-1},$$

which holds for $m \le n$. Thus, we deduce that $B := \max\{|1|, |-m|, |n|, |-1|\} = n$. Suppose now that $\Gamma_1 := 2\cdot 3^{-m}\cdot\alpha^n\cdot\sqrt{5}^{-1} - 1$. We justify that $\Gamma_1 \ne 0$. Indeed, from the fact that if $\Gamma_1 = 0$, we would get the relation $\alpha^n = \frac{3^m\sqrt{5}}{2}$, so $\alpha^{2n} = \frac{5\cdot 3^{2m}}{4} \in \mathbb{Q}$, which is not possible since $\alpha^{2n} \notin \mathbb{Q}$. By Matveev's result Theorem 2.3, we obtain

$$|\Gamma_1| > \exp\left(-C(1+\log n)\cdot 1.4\cdot 2.2\cdot 0.5\cdot 1.7\right)$$

where

$$C := 1.4\cdot 30^7\cdot 4^{5.5}\cdot(1+\log 2) < 1.07\times 10^{14}.$$

From (3.2), we get

$$\log|\Gamma_1| < \left(-\frac{m}{2}+1\right)\log 3 - \log 2.$$

Hence, we have

$$\frac{m}{2} - 1 < 2.55 \times 10^{14}(1+\log n) \tag{3.2}$$

By inequalities (1.1) and (2.1), we obtain

$$\begin{aligned} 2\alpha^{n-2} \leq 2F_n < 3^m + 3^{\frac{m}{2}} = 3^{m+2}(3^{-2}+3^{-\frac{m}{2}-2}) &< 3^{m+2}(2^{-2}+2^{-\frac{m}{2}-2}) \\ &< 3^{m+2}\cdot\alpha^{-2}, \end{aligned}$$

which gives

$$n < \frac{(m+2)\log 3 - \log 2}{\log\alpha}. \tag{3.3}$$

Combining it with inequality (3.2), and by an easy calculation with Maple, we get

$$m < 2.01 \times 10^{16} \text{ and } n < 4.6 \times 10^{16}.$$

Secondly, we are going to reduce the upper bounds of $m$ and $n$. When $m > 10$, we have no the Fibonacci number $2F_n$ equals $3^m$. Hence, we discuss this following cases.

- If $2F_n > 3^m$. Notice that

$$3^m \leq 2F_n - 1 < \frac{2}{\sqrt{5}}\alpha^n,$$

we get

$$-m\log 3 + n\log\alpha + \log 2 - \log\sqrt{5} > 0.$$

Since $e^\Lambda - 1 > \Lambda$ for all $\Lambda > 0$, by using inequalities (3.1) and (3.3), we obtain

$$0 < -m\log 3 + n\log\alpha + \log 2 - \log\sqrt{5} < \frac{3^{-\frac{m}{2}+1}}{2} < \frac{3^{2-\frac{n\log\alpha+\log 2}{2\log 3}}}{2} < \frac{7}{2}\cdot\left(\frac{5}{4}\right)^{-n}.$$

Dividing both sides of the above last inequality by $\log 3$, we get

$$0 < n\frac{\log\alpha}{\log 3} - m + \frac{\log 2 - \log\sqrt{5}}{\log 3} < \frac{7}{2\log 3}\cdot\left(\frac{5}{4}\right)^{-n}. \tag{3.4}$$

Taking,

$$\gamma := \frac{\log\alpha}{\log 3} \notin \mathbb{Q}, \quad \mu := \frac{\log 2 - \log\sqrt{5}}{\log 3}, \quad A := \frac{7}{2\log 3}, \quad B := \frac{5}{4}, \text{ and } \omega := n.$$

Let us now $q_k > 6M$ be the denominator of the $k$th convergent of the continued fraction of $\gamma$. Since $n < 4.6\times 10^{16}$, we can take $M := 4.6\times 10^{16}$. However, applying Lemma 2.4 to inequality (3.4), we get

$$q_k := q_{40} = 1116972345258589541 > 6M,$$

and so

$$\varepsilon := \|\mu q_{40}\| - M\|\gamma q_{40}\| = 0.06854\cdots.$$

We conclude that there is no solution to inequality (3.4) for

$$n = \omega \geq \frac{\log\left(Aq_{40}/\varepsilon\right)}{\log B} = 203.43941\cdots$$

and therefore in the range

$$\left[\left\lfloor \frac{\log\left(Aq_{40}/\varepsilon\right)}{\log B}\right\rfloor + 1, M\right] \supset [204, 4.6 \times 10^{16}].$$

So, $n < 204$.

- If $2F_n < 3^m$. Denote that for $\Lambda < 0$, we get

$$0 < -\Lambda < e^{-\Lambda} - 1 = e^{-\Lambda}|e^{\Lambda} - 1|.$$

Here, we can choose

$$\Lambda := -m\log 3 + n\log\alpha + \log 2 - \log\sqrt{5} < 0.$$

and

$$|e^{\Lambda} - 1| < \frac{3}{2}\cdot\sqrt{3}^{-m} < \frac{7}{2}\cdot\left(\frac{5}{4}\right)^{-n} < \frac{1}{2}.$$

Since $\Lambda < 0$, it follows that $e^{\Lambda} \in (\frac{1}{2}, 1)$, so that $e^{-\Lambda} < 2$. Next, we get

$$0 < m\frac{\log 3}{\log\alpha} - n + \frac{\log\sqrt{5} - \log 2}{\log\alpha} < \frac{3}{\log\alpha}\cdot\sqrt{3}^{-m} < \frac{7}{\log\alpha}\cdot\left(\frac{5}{4}\right)^{-n}. \tag{3.5}$$

Again, by applying Lemma 2.4 to inequality (3.5), we deduce that there is no solution to this inequality for

$$m \geq 206.54506\cdots.$$

Through a similar argument, we obtain $m < 207$ and $n < 476$.

Lastly, by an easy calculation in Maple. Suppose now that $a := \frac{\log(2F_n)}{\log 3}$, where $a > 4$, $n \geq m > 10$ and

$$3^{a+1} - 3^{(a+1)/2} = 3^a(3 - 3^{(-a+1)/2}) > 3^a = 2F_n.$$

Hence, for this case $2F_n > 3^m$, we get

$$\frac{\log(2F_n)}{\log 3} < m < \frac{\log(2F_n)}{\log 3} + 1.$$

Thus, we only need to check whether $2F_n$ is in the interval $(3^m, 3^m + 3^{m/2})$ with

$$m = \left\lfloor \frac{\log(2F_n)}{\log 3}\right\rfloor + 1.$$

Through a calculation with the help of Maple, we deduce that there is no such $n$. For the case $2F_n < 3^m$, through a similar calculation with the help to Maple, we conclude that no such $n$ exists. This finishes the proof. □

## 4. The Proof of Theorem 1.2

### 4.1. **Upper bound for $n$.**

*Proof.* Assume throughout that inequality (1.2) holds. First of all, we try to apply a lower bound for linear forms in logarithms due to Matveev and Dujella-Pethő to get a large upper bound for $n$. If $m = 1$, then $(n, 1, a)$ is a solution of inequality (1.2). Since $F_1 = F_2 = 1$, then $(n, 2, a)$ is also a solution to inequality (1.2). So, from now on, we assume that $m \geq 2$. By Theorem 1.1, then all solutions of inequality (1.2) when $n = m$ are listed in the appendix (Table 2). In fact, since $F_{n+1} = F_n + F_{n-1}$, we can assume $n > m + 1$ and in particular $n - m \geq 2$.

**Proposition 4.1.** *There are exactly* 37 *solutions of inequality* (1.2) *in positive integers* $(n, m, a)$ *with* $m \leq n \leq 300$. *All the solutions satisfy* $n \leq 29$, $m \leq 22$, *and* $a \leq 12$. *A list of solutions is given in the appendix (Table 2).*

*Proof.* The solutions were obtained by a brute force search with a computational number theory system PARI/GP. □

Because of Proposition 4.1 for the rest of the paper, we assume that $n > 300$, and we will show that there exist no solutions for $n > 300$.

Observe that the inequalities (2.1) and (2.2) with $n > m > 1$, which gives the inequality

$$0.38\alpha^n < F_n < F_n + F_m < 0.48\alpha^n + 0.48\alpha^{n-1} < 0.78\alpha^n. \tag{4.1}$$

Let us now obtain a relation between $n$ and $a$ which for $a \geq 2$. Combining (1.2) with the right-hand of last inequality, we get

$$3^{a-1} < 3^a - 3^{a/2} < F_n + F_m < 0.78\alpha^n < \alpha^n.$$

On the other hand, if combining (1.2) with the left-hand of inequality (4.1), we obtain

$$0.38\alpha^n < F_n + F_m < 3^a + 3^{a/2} < \frac{3^{a+1}}{2}.$$

Thus,

$$n\frac{\log \alpha}{\log 3} + \frac{\log 0.76}{\log 3} - 1 < a < n\frac{\log \alpha}{\log 3} + 1 \tag{4.2}$$

with $\frac{\log \alpha}{\log 3} = 0.43801\cdots$. Therefore, we get $a < n$.

From the Binet formula, we have

$$\frac{\alpha^n + \alpha^m}{\sqrt{5}} - F_n - F_m = \frac{\beta^n + \beta^m}{\sqrt{5}} \tag{4.3}$$

Taking the absolute values in the last inequality and combining it with (1.2), we obtain

$$\left|\frac{\alpha^n(1 + \alpha^{m-n})}{\sqrt{5}} - 3^a\right| < \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} + 3^{a/2} < \frac{1}{3} + 3^{a/2} < \frac{3^{\frac{a}{2}+1}}{2}$$

which holds for all $n \geq 5$ and $m \geq 1$. By dividing both sides of the above inequality by $3^a$, we get

$$\left|3^{-a} \cdot \alpha^n \cdot \frac{1+\alpha^{m-n}}{\sqrt{5}} - 1\right| < \frac{3^{-\frac{a}{2}+1}}{2}. \tag{4.4}$$

In the first application of Matveev's result Theorem 2.3, we can choose the parameters $l := 3$ and

$$(\eta_1, b_1) := (3, -a), \quad (\eta_2, b_2) := (\alpha, n), \quad (\eta_3, b_3) := \left(\frac{1+\alpha^{m-n}}{\sqrt{5}}, 1\right).$$

As before, $\mathbb{L} := \mathbb{Q}(\sqrt{5})$ contains the three numbers $\eta_1, \eta_2, \eta_3$ are positive real, so we can take again $D := [\mathbb{L} : \mathbb{Q}] = 2$. Put $\Gamma_1 := 3^{-a}\alpha^n(1+\alpha^{m-n})\sqrt{5}^{-1} - 1 \neq 0$, note that otherwise, we would have the relation

$$3^a\sqrt{5} = \alpha^n + \alpha^m. \tag{4.5}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we have

$$-3^a\sqrt{5} = \beta^n + \beta^m. \tag{4.6}$$

However, combining the relations (4.5) and (4.6), we obtain

$$\alpha^n < \alpha^n + \alpha^m = |\beta^n + \beta^m| \leq |\beta|^n + |\beta|^m < 1$$

which is not possible for $n \geq 1$. Thus, $\Gamma_1 \neq 0$.

Since $h(\eta_1) = \log 3 = 1.0986\ldots$ and $h(\eta_2) = \frac{1}{2}\log\alpha = 0.2406\ldots$, we can take $A_1 := 2.2$ and $A_2 := 0.5$. Let us now estimate $h(\eta_3)$. We begin by observing that

$$\eta_3 = (1+\alpha^{m-n})\sqrt{5}^{-1} < 2\sqrt{5}^{-1} \text{ and } \eta_3^{-1} = \sqrt{5}(1+\alpha^{m-n})^{-1} < \sqrt{5}$$

so, it follows that $|\log\eta_3| < 1$. Next, we use the properties of the logarithmic height, we obtain

$$h(\eta_3) \leq \log\sqrt{5} + \frac{1}{2}|m-n|\log\alpha + \log 2 = \log 2\sqrt{5} + \frac{1}{2}(n-m)\log\alpha.$$

Therefore, we also have

$$A_3 := 3 + (n-m)\log\alpha > \max\{2h(\eta_3), |\log\eta_3|, 0.16\}.$$

Since $a < n$, we infer that $B := \max\{|-a|, |n|, |1|\} = n$, so we can choose $B := n$. Now, we try to apply Matveev's result Theorem 2.3 again, we obtain

$$\begin{aligned}|\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log n) \cdot 2.2 \cdot 0.5 \cdot (3+(n-m)\log\alpha)\right)\\ &> \exp\left(-2.14 \times 10^{12}\log n \cdot (3+(n-m)\log\alpha)\right)\end{aligned}$$

where we also used the inequality $1 + \log n < 2\log n$ for all $n \geq 3$.

By comparing the above inequality with (4.4), we have that

$$\left(\frac{a}{2} - 1\right)\log 3 + \log 2 < 2.14 \times 10^{12}\log n \cdot (3+(n-m)\log\alpha). \tag{4.7}$$

Let us consider a second linear form in logarithms by rewriting (4.3) as follows

$$\frac{\alpha^n}{\sqrt{5}} - (F_n + F_m) = \frac{\beta^n}{\sqrt{5}} - F_m$$

Again, by combining the above relation with (1.2), we obtain

$$\left|\frac{\alpha^n}{\sqrt{5}}-3^a\right|<3^{a/2}+\frac{|\beta|^n}{\sqrt{5}}+F_m<3^{a/2}+\frac{1}{2}+\alpha^m,\quad \text{because}\quad |\beta|^n<\frac{1}{2}\ \text{for all } n\geq 2.$$

By dividing both sides of the above inequality by $\alpha^n/\sqrt{5}$, and taking into account that $\alpha>\sqrt{2}$ and $n>m$, we get

$$\begin{aligned}|1-3^a\cdot\alpha^{-n}\cdot\sqrt{5}|<\frac{3^{\frac{a}{2}}\sqrt{5}}{\alpha^n}+\frac{\sqrt{5}}{2\alpha^n}+\frac{\sqrt{5}}{\alpha^{n-m}}<\frac{2^a\sqrt{5}}{\alpha^n}+\frac{\sqrt{5}}{2\alpha^n}+\frac{\sqrt{5}}{\alpha^{n-m}}\\ <3\sqrt{5}\max\{\alpha^{a-n},\alpha^{m-n}\}.\end{aligned}\tag{4.8}$$

In a first application of Matveev's result Theorem 2.3, we take again $l:=3$ and

$$(\eta_1,b_1):=(3,a),\quad (\eta_2,b_2):=(\alpha,-n),\quad (\eta_3,b_3):=(\sqrt{5},1).$$

Note that $\mathbb{L}:=\mathbb{Q}(\sqrt{5})$ contains the three numbers $\eta_1,\eta_2,\eta_3$ are positive real and has $D:=[\mathbb{L}:\mathbb{Q}]=2$. Setting $\Gamma_2:=1-3^a\alpha^{-n}\sqrt{5}$. We justify that $\Gamma_2\neq 0$. Indeed, if $\Gamma_2=0$, then we have the relation $3^a=\alpha^n\sqrt{5}$, so $\alpha^{2n}=5\cdot 3^{2a}\in\mathbb{Z}$, which is impossible because $\alpha^{2n}\notin\mathbb{Z}$. Thus, $\Gamma_2\neq 0$. Moreover, since $h(\eta_1)=\log 3=1.0986\ldots$, $h(\eta_2)=\frac{1}{2}\log\alpha=0.2406\ldots$, and $h(\eta_3)=\log\sqrt{5}=0.8047\ldots$, it follows that we have also taken $A_1:=2.2$, $A_2:=0.5$, and $A_3:=1.7$. Since $a<n$, we deduce that $B:=n$.

As before, applying Matveev's result Theorem 2.3 again, we get

$$\begin{aligned}|\Gamma_2|&>\exp\left(-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log n)\cdot 2.2\cdot 0.5\cdot 1.7\right)\\ &>\exp\left(-3.63\times 10^{12}\log n\right)\end{aligned}$$

with $1+\log n<2\log n$ for all $n\geq 3$. Comparing the resulting inequality with (4.8), we obtain

$$\min\{(n-a)\log\alpha,(n-m)\log\alpha\}<3.7\times 10^{12}\log n.$$

Now, we distinguish the arguments splits into two cases.

- `Case 1.` If $\min\{(n-a)\log\alpha,(n-m)\log\alpha\}=(n-a)\log\alpha$, then

  $$(n-a)\log\alpha<3.7\times 10^{12}\log n.$$

  Note that the right-hand inequality of (4.2) gives

  $$n-a>n\left(1-\frac{\log\alpha}{\log 3}\right)-1.$$

  Combining the above last inequalities and by a calculation in Maple, we get

  $$a<2.03\times 10^{14}\ \text{and}\ n<4.6\times 10^{14}.$$

- `Case 2.` If $\min\{(n-a)\log\alpha,(n-m)\log\alpha\}=(n-m)\log\alpha$, then

  $$(n-m)\log\alpha<3.7\times 10^{12}\log n.$$

  From (4.7), we get

$$\left(\frac{a}{2}-1\right)\log 3+\log 2<2.14\times 10^{12}\log n\cdot\left(3+3.7\times 10^{12}\log n\right)<8\times 10^{24}\log^2 n.$$

Combining the above with the left-hand inequality (4.2) and by a calculation in Maple, we obtain

$$a < 6.6 \times 10^{28} \text{ and } n < 1.5 \times 10^{29}.$$

Thus, in all the two cases, we obtain

$$a < 6.6 \times 10^{28} \text{ and } n < 1.5 \times 10^{29}.$$

Now, we need to reduce the above bound for $n$.

4.2. **Reducing of the bound on $n$.** Let us now assume that $n - a > 300$ and $n - m > 300$. We go back to inequality (4.8). Setting

$$\Lambda_1 := a \log 3 - n \log \alpha + \log \sqrt{5}.$$

Since we assume that $\min\{n - a, n - m\} > 300$, we have $|e^{\Lambda_1} - 1| < 1/2$. Hence, by Lemma 2.6, we get $|\Lambda_1| < 2|e^{\Lambda_1} - 1|$. If we replace $\Lambda_1$ by its formula and by (4.8), we obtain

$$|a \log 3 - n \log \alpha + \log \sqrt{5}| < \frac{6\sqrt{5}}{\alpha^{\min\{n-a,n-m\}}}.$$

By dividing both sides of the above inequality by $\log \alpha$, we can conclude

$$0 < \left| a\frac{\log 3}{\log \alpha} - n + \frac{\log \sqrt{5}}{\log \alpha} \right| < 28\alpha^{-\omega}$$

with $\omega := \min\{n - a, n - m\}$. Applying Lemma 2.4 again, we also put

$$\gamma := \frac{\log 3}{\log \alpha}, \quad \mu := \frac{\log \sqrt{5}}{\log \alpha}, \quad A := 28, \text{ and } B := \alpha.$$

It is clear that $\gamma \notin \mathbb{Q}$. We also choose $M := 6.6 \times 10^{28}$ which is an upper bound for $a$. We find that $q_k$ be the denominator of the $k$th convergent of the continued fraction of $\gamma$ is such that

$$q_k := q_{61} = 10519533034350002865545777070 71 > 6M$$

By using this we get that $\varepsilon := \|\mu q_{61}\| - M\|\gamma q_{61}\| = 0.15456\cdots$. Hence,

$$n - m < \frac{\log(28q_{61}/0.15456)}{\log \alpha} < 155 \text{ or } n - a < \frac{\log(28q_{61}/0.15456)}{\log \alpha} < 155.$$

Thus, we obtain that $n - m < 155$ or $n < 278$. The latter case contradicts our assumption that $n > 300$. Inserting the upper bound for $n - m$ into inequality (4.7), we have that $a < 1.12 \times 10^{16}$.

We shall use the inequality (4.4) to improve the bound for $n$. We also set

$$\Lambda_2 := a \log 3 - n \log \alpha + \log \varphi(n - m),$$

where $\varphi$ is the function defined by the formula $\varphi(u) := \sqrt{5}(1 + \alpha^{-u})^{-1}$. We may assume that $a > 12$. Hence, from (4.7) it follows that

$$\left|1 - e^{\Lambda_2}\right| < \frac{1}{2 \cdot 3^{\frac{a}{2}-1}}. \tag{4.9}$$

Note that $\Lambda_2 \neq 0$. Thus, we distinguish the following cases. If $\Lambda_2 > 0$, then

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 \leq \frac{1}{2 \cdot 3^{\frac{a}{2}-1}}.$$

If $\Lambda_2 < 0$. Then, by (4.9) we get that $\left|1 - e^{\Lambda_2}\right| < 1/2$ and so $e^{|\Lambda_2|} < 2$. Since $\Lambda_2 < 0$, we obtain

$$0 < |\Lambda_2| \leq e^{|\Lambda_2|} - 1 = e^{|\Lambda_2|} \left|e^{\Lambda_2} - 1\right| < \frac{1}{3^{\frac{a}{2}-1}}.$$

In any case, we get

$$0 < |\Lambda_2| \leq \frac{1}{3^{\frac{a}{2}-1}}.$$

Again, replacing $\Lambda_2$ in the above inequality and dividing both sides by $\log \alpha$, we obtain

$$0 < \left| a\frac{\log 3}{\log \alpha} - n + \frac{\log \varphi(n-m)}{\log \alpha} \right| < \frac{3}{\log \alpha} \cdot \sqrt{3}^{-a}. \tag{4.10}$$

Now we can take $M := 1.12 \times 10^{16}$ which is an upper bound on $a$ and applying Lemma 2.4, we conclude that

$$\gamma := \frac{\log 3}{\log \alpha} \notin \mathbb{Q}, \quad \mu := \frac{\log \varphi(n-m)}{\log \alpha}, \quad A := \frac{3}{\log \alpha}, \quad B := \sqrt{3}, \text{ and } \omega := a$$

for all choices $n - m \in \{1, 2, \ldots, 155\}$ except when $n - m = 2$. Moreover, a computer search with Maple, we find that if $(n, m, a)$ is a possible solution of the equation (1.2) where $n - m \neq 2$, then

$$a = \omega \geq \frac{\log\left(Aq_{39}/0.00127\right)}{\log \sqrt{3}} = 89.62298\cdots$$

with $q_{39} = 489253858114607333 > 6M$. Therefore, $a \leq 89$ and $n \leq 206$. But this is false because we have assumed $n > 300$.

Finally, we deal with the special case $n - m = 2$. We cannot study these case as before because when applying Lemma 2.4 on inequality (4.10), then the corresponding parameter $\mu$ appearing in Lemma 2.4 is either

$$\frac{\log \varphi(n-m)}{\log \alpha} = 1.$$

But, we have that the corresponding value of $\varepsilon$ from Lemma 2.4 is always negative and therefore the reduction method is not useful for reducing the bound to $n$ in these case. For this case, by inequality (4.10), we have that

$$0 < |a\gamma - n + 1| < \frac{3}{\log \alpha} \cdot 3^{-\frac{1}{2}\left(n\frac{\log \alpha}{\log 3} + \frac{\log 0.76}{\log 3} - 1\right)} \tag{4.11}$$

We recall that $a < 1.2 \times 10^{16}$. Let $[a_0, a_1, a_2, a_3, a_4, a_5, \ldots] = [2, 3, 1, 1, 6, 1, \ldots]$ be the continued fraction of $\gamma$. A quick inspection using Maple reveals that

$$85357944215968245 = q_{38} < 1.12 \times 10^{16} < q_{39} = 489253858114607333.$$

Moreover, $a_K := \max\{a_k : k = 0, 1, \ldots, 39\} = a_{76} = 871$. Thus, by Lemma 2.5 we get

$$|a\gamma - (n-1)| > \frac{1}{(a_K + 2)a}. \tag{4.12}$$

Comparing estimates (4.11) and (4.12), we obtain that $n < 193$. This contradicts our assumption that $n < 300$. This complete the analysis of the case $n - m = 1$ and therefore the proof of Theorem 1.2. □

## 5. The Code

In this section, for the sake of completeness, we present the PARI/GP [9] code used to confirm Proposition 4.1.

```
{
for(n = 1, 300, for(m = 1, 300, for(a = 1, 300, if(n >= m,
Fn = fibonacci(n);
Fm = fibonacci(m);
b = Fn + Fm - 3^a;
if(abs(b) < 3^(a/2),
print([n, m, a, b, 3^(a/2)])))))))
}
```

## 6. Appendix

Table 1. The Fibonacci numbers $2F_n$ to the inequality (1.1) with $1 \leq m \leq 10$ are given by

| $m$ | Integers in $I_m$ | The Fibonacci numbers $2F_n$ in $I_m$ |
|---|---|---|
| 1 | $2, 3, 4$ | $2, 4$ |
| 2 | $7, \ldots, 11$ | 10 |
| 3 | $22, \ldots, 32$ | 26 |
| 4 | $73, \ldots, 89$ | |
| 5 | $228, \ldots, 258$ | |
| 6 | $703, \ldots, 755$ | 754 |
| 7 | $2141, \ldots, 2233$ | |
| 8 | $6481, \ldots, 6641$ | |
| 9 | $19543, \ldots, 19823$ | |
| 10 | $58807, \ldots, 59291$ | |

Table 2. The solutions of Diophantine inequality (1.2) are given by

| | |
|---|---|
| $\lvert F_k + F_1 - 3^1\rvert < \sqrt{3}, \quad k = 1, 2, 3, 4$ | $\lvert F_{10} + F_l - 3^4\rvert < 3^2, \quad l = 8, 9$ |
| $\lvert F_k + F_2 - 3^1\rvert < \sqrt{3}, \quad k = 2, 3, 4$ | $\lvert F_{12} + F_{11} - 3^5\rvert < 3^{5/2}$ |
| $\lvert F_3 + F_3 - 3^1\rvert < \sqrt{3}$ | $\lvert F_{13} + F_l - 3^5\rvert < 3^{5/2}, \quad l = 1, \ldots, 8$ |
| $\lvert F_5 + F_l - 3^2\rvert < 3, \quad l = 3, 4, 5$ | $\lvert F_{14} + F_{14} - 3^6\rvert < 3^3$ |
| $\lvert F_6 + F_l - 3^2\rvert < 3, \quad l = 1, 2, 3, 4$ | $\lvert F_{15} + F_{12} - 3^6\rvert < 3^3$ |
| $\lvert F_7 + F_7 - 3^3\rvert < 3^{3/2}$ | $\lvert F_{17} + F_{15} - 3^7\rvert < 3^{7/2}$ |
| $\lvert F_8 + F_l - 3^3\rvert < 3^{3/2}, \quad l = 1, \ldots, 6$ | $\lvert F_{29} + F_{22} - 3^{12}\rvert < 3^6$ |

## 7. Conclusion

In this paper, we found all the Fibonacci numbers $2F_n$, the solutions $(2F_n, 3^m)$ of the inequality (1.1) in positive integers $n, m$ with $n \geq m$ and sums of two Fibonacci numbers which are close to a power of 3. Moreover, we determine all Lucas numbers close to a power of 3.

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[1] Mathematics and Intelligent Systems (MASI), National School of Applied Sciences of Tangier (ENSAT), Abdelmalek Essaadi University
*Email address*: anouar.gaha@etu.uae.ac.ma

[2] Mathematics and Intelligent Systems Team (MASI), Higher School of Technology of Tetouan (ESTTe), Abdelmalek Essaadi University
*Email address*: smezroui@uae.ac.ma