

IMAGINARY BIQUADRATIC PÓLYA FIELDS OF THE FORM $\mathbb{Q}(\sqrt{d}, \sqrt{-2})$

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ABSTRACT. A number field k , with ring of integers $\mathcal{O}_k$, is called a Pólya field if the module of integer-valued polynomials over $\mathcal{O}_k$ has a regular basis. In this paper, we characterize all imaginary biquadratic Pólya fields of the form $\mathbb{Q}(\sqrt{d}, \sqrt{-2})$ where d is a square-free integer.

1. INTRODUCTION

Let k be a number field and denote by $\mathcal{O}_k$ its ring of integers. Consider the ring of *integer-valued polynomials* on $\mathcal{O}_k$, that is

$$\text{Int}(\mathcal{O}_k) = \{P \in k[X] \mid P(\mathcal{O}_k) \subset \mathcal{O}_k\}.$$

The ring $\text{Int}(\mathcal{O}_k)$ is a free $\mathcal{O}_k$ -module (see [16]). According to Pólya ([11]), a basis $(e_n)_{n \in \mathbb{N}}$ of $\text{Int}(\mathcal{O}_k)$ is said to be a *regular basis* if the degree of each polynomial e_n is n . The field k is then said to be a *Pólya field* if $\text{Int}(\mathcal{O}_k)$ has a regular basis.

For each $n \in \mathbb{N}$, the leading coefficients of the polynomials in $\text{Int}(\mathcal{O}_k)$ of degree n together with zero form a fractional ideal of $\mathcal{O}_k$, denoted by $\mathfrak{I}_n(k)$ and called the *characteristic ideal of k of index n* ([16]). Then k is a Pólya field if and only if the ideals $\mathfrak{I}_n(k)$ are principal ([16]). A. Ostrowski gave in [7] an other characterization of Pólya fields (see Proposition 2.2 below). Finally, Cahen and Chabert, in [16], characterized Pólya fields k using primes p that divide their discriminants (see Proposition 2.5 below).

H. Zantema [13] has completely characterized quadratic Pólya fields. In 2011, A. Leriche, in [6], gave some results on the Pólya fields. She studied fields which are product of two linearly disjoint Pólya fields, after she concluded some biquadratic Pólya fields, and in 2013, she characterized [5] cubic, quartic and sextic Pólya fields. Recently, M. Taous in [8, Chap. 13] characterized all biquadratic Pólya fields of the form $\mathbb{Q}(\sqrt{d}, \sqrt{-1})$ where d is a square-free integer.

After these Pólya fields, it seems natural to characterize all biquadratic Pólya fields which are or are not product of two quadratic Pólya fields, it is still an open question.

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In this paper, we determine the Pólya group of the field $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$ and we deduce all the imaginary biquadratic Pólya fields of the form $\mathbb{Q}(\sqrt{d}, \sqrt{-2})$ where d is a square-free integer.

2. PRELIMINARIES

Let k be a number field and $\mathcal{O}_k$ be its ring of integers. Recall that the class group of k is the quotient $C_k = I_k/P_k$ of the group I_k of nonzero fractional ideals of $\mathcal{O}_k$ by the subgroup P_k of nonzero principal ideals. For each integer $q \geq 2$, we denote by $\prod_q(k)$ the product of all prime ideals of $\mathcal{O}_k$ with norm $q\mathbb{Z}$, if these ideals do not exist we set $\prod_q(k) = \mathcal{O}_k$.

In what follows, denote by $N_{L/k}$ the norm of some extension L/k , by E_k (resp. E_L) the unit group of k (resp. L) and by $[H]$ the ideal class of an ideal H . Finally, denote by ϵ_m the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{m})$, for some positive square-free integer m , and by $N(\epsilon_m)$ its absolute norm.

Definition 2.1. The subgroup $P(k)$ of C_k generated by the classes of the ideals $\prod_q(k)$ is called the *Pólya group* of k . In fact, $P(k)$ is also the subgroup of C_k generated by the classes of the characteristic ideals of K .

Proposition 2.2 ([7]). *If k is a Galois extension of $\mathbb{Q}$, then k is a Pólya field if and only if the Pólya group $P(k)$ is trivial.*

Proposition 2.3 ([13]). *A quadratic field $\mathbb{Q}(\sqrt{d})$ for square-free integer d is a Pólya field if and only if d is of one of the following:*

1. $d = -1, d = 2, d = -2$,
2. $d = -p$, where $p \equiv 3 \pmod{4}$ is prime,
3. $d = p$, where p is an odd prime,
4. $d = 2p$, where p is prime and
 - either $p \equiv 3 \pmod{4}$
 - or $p \equiv 1 \pmod{4}$ and $N(\epsilon_d) = 1$,
5. $d = pq$, where p and q are primes and
 - either $p, q \equiv 3 \pmod{4}$
 - or $p, q \equiv 1 \pmod{4}$ and $N(\epsilon_d) = 1$.

Let L/k be a finite extension of number fields, and let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_k$. As $\mathcal{O}_L$ is a Dedekind ring, then

$$\mathcal{P}\mathcal{O}_L = \mathcal{B}_1^{e_1}\mathcal{B}_2^{e_2}\cdots\mathcal{B}_g^{e_g},$$

where $\mathcal{B}_i$ are prime ideals of $\mathcal{O}_L$ said lying above a $\mathcal{P}$. The integer e_i is called the *ramification index* of $\mathcal{B}_i$ over $\mathcal{P}$, it is often denoted by $e(\mathcal{B}_i/\mathcal{P})$. If L/k is a Galois extension of degree n , then $e_1 = e_2 = \cdots = e_g := e_{\mathcal{P}}(L/k)$. In this case we say that $e_{\mathcal{P}}(L/k)$ is the *ramification index* of $\mathcal{P}$ in L/k . The quotient $f_{\mathcal{P}}(L/k) = \frac{n}{ge_{\mathcal{P}}(L/k)}$ is called the *inertia degree* of $\mathcal{P}$ in L/k .

Definition 2.4. Let L be a finite Galois extension of a number field k and $\mathcal{B}$ be a prime ideal of $\mathcal{O}_L$ lying above a prime ideal $\mathcal{P}$ of $\mathcal{O}_k$. We say that $\mathcal{P}$ is *ramified* in L if $e_{\mathcal{P}}(L/k) > 1$. The extension L/k is said *unramified* if its ramification degree is $e_{\mathcal{P}}(L/k) = 1$ for all prime ideals $\mathcal{P}$, and is said *ramified* otherwise.

Since, in a Dedekind ring, an ideal I is maximal if and only if it is a prime ideal, then we can deduce the following proposition.

Proposition 2.5 ([16]). *Let D_k be the discriminant of k . If $k/\mathbb{Q}$ is Galois extension, then*

$$P(k) = \langle [\prod_{p|f} (k)] \mid p \text{ divides } D_k \text{ and } f \text{ is the inertia degree of } p \text{ in } k/\mathbb{Q} \rangle.$$

Definition 2.6. Let L/k be a finite extension. An ideal class of k is said to *capitulate* in L if it is in the kernel of the homomorphism $j : C_k \rightarrow C_L$, induced by extension of ideals from k to L , and if, in addition, $j(P(k)) = \{1\}$, then we said that L/k is a *Pólya extension*.

Theorem 2.7 ([10]). *Let L/k be an unramified cyclic extension, then the number of classes that capitulate in L/k is: $[L : k][E_k : N_{L/k}(E_L)]$.*

Theorem 2.8 (Chap. 13 of [8]). *Let L/k be an unramified cyclic extension, then $j(P(k)) = P(L)$. Consequently L/k is a Pólya extension if and only if L is a Pólya field.*

Corollary 2.9. *Let L/k be an unramified cyclic extension and $|P(k)| > [L : k][E_k : N_{L/k}(E_L)]$, then L is never a Pólya field.*

Proof. It is a consequence of Theorems 2.7 and 2.8. □

Let G be a group, and let $(A, \times)$ be a G -module. We consider $Z^1(G, A)$ the group of 1-cocycles, i.e., the applications $a : G \rightarrow A$ such that for all $\sigma, \tau \in G$, $a(\sigma\tau) = a(\sigma) \times \sigma \cdot a(\tau)$. Let $B^1(G, A)$ be the subgroup of 1-coboundaries, i.e., the applications $a : G \rightarrow A$ for which there exists $x \in A$ such that $a(\sigma) = x^{-1}\sigma \cdot x$, for all $\sigma \in G$. The quotient group $H^1(G, A) = Z^1(G, A)/B^1(G, A)$ is called the *first cohomology group* of G with coefficients in A .

K. Iwasawa showed in [15] that if L/k is an unramified Galois extension, then $H^1(G, E_L) \simeq \ker j$, with $G = \text{Gal}(L/k)$. If $k/\mathbb{Q}$ is a Galois extension, we denote by G_k the Galois group of $k/\mathbb{Q}$. The following proposition gives a cohomological approach of a Pólya group.

Proposition 2.10 ([13]). *Let $k/\mathbb{Q}$ be a Galois extension and D_k be its discriminant. Denote by e_p the ramification index of a prime number p in k . Then the following sequence is exact*

$$1 \rightarrow H^1(G_k, E_k) \rightarrow \bigoplus_{p|D_k} \mathbb{Z}/e_p\mathbb{Z} \rightarrow P(k) \rightarrow 1.$$

In particular $|H^1(G_k, E_k)||P(k)| = \prod_{p|D_k} e_p$.

Corollary 2.11 (Chap. 13 of [8]). *Let L/k be a finite unramified Galois extension and let θ be the restriction of j to $P(k)$, then*

$$\frac{|H^1(G_L, E_L)|}{|H^1(G_k, E_k)|} = |\ker \theta| = \frac{|P(k)|}{|P(L)|}.$$

3. THE PÓLYA GROUP OF $\mathbb{Q}(\sqrt{d}, \sqrt{-2})$

Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$ where $d > 2$ is a square-free integer. We denote by $Q = [E_k : \langle -1, \epsilon_d \rangle]$ the Hasse's unit index of k . It is well known that $Q = 1$ or 2 (see [12] §21, Satz 15). On the other hand, according to [2] or [1], we have the following result.

Lemma 3.1. *Let $d > 2$ be a square-free integer and $\epsilon_d = s + t\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$.*

- i. *If $N(\epsilon_d) = -1$, then $E_k = \langle -1, \epsilon_d \rangle$, i.e. $Q = 1$.*
- ii. *If $N(\epsilon_d) = 1$, then $E_k = \langle -1, \sqrt{-\epsilon_d} \rangle$ if only if $s \pm 1$ is a square in $\mathbb{N}$ (i.e., if only if $2\epsilon_d$ is a square in $\mathbb{Q}(\sqrt{d})$), i.e. $Q = 2$. Else, $E_k = \langle -1, \epsilon_d \rangle$, i.e. $Q = 1$.*

Lemma 3.2 ([3], Lemma 6). *Let $q \equiv -1 \pmod{4}$ be a prime and $\epsilon_q = x + y\sqrt{q}$ be the fundamental unit of $\mathbb{Q}(\sqrt{q})$. Then x is an even integer, $x \pm 1$ is a square in $\mathbb{N}$ and $2\epsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$.*

Lemma 3.3 ([3], Lemma 7). *Let p be an odd prime and $\epsilon_{2p} = x + y\sqrt{2p}$. If $N(\epsilon_{2p}) = 1$, then $x \pm 1$ is a square in $\mathbb{N}$ and $2\epsilon_{2p}$ is a square in $\mathbb{Q}(\sqrt{2p})$.*

Lemma 3.4 ([17]). *Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$ for some square-free integer d , and D_k be its discriminant. If p is a prime number dividing D_k , then $p = 2$ or p divides d .*

Lemma 3.5 ([13]). *Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$ where $d > 2$ is a square-free integer and let ϵ_d be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Then*

$$H^1(G_k, E_k) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } Q = N(\epsilon_d) = 1; \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$$

Let $d = 2^s d'$ with $d' = p_1 p_2 \cdots p_n$, where $s = 0$ or $s = 1$ and p_i are different primes. If we denote by G_m an elementary abelian 2-group of order 2^m with some integer m , then the following corollary holds.

Corollary 3.6. *Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$, where $d > 2$ is a square-free integer, Q be the Hasse's unit index of k and ϵ_d be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Denote by e_2 the ramification index of 2 in k .*

- 1. *If $e_2 = 2$, then $P(k) \simeq \begin{cases} G_{n-2}, & \text{if } Q = N(\epsilon_d) = 1; \\ G_{n-1}, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$*
- 2. *If $e_2 = 4$, then $P(k) \simeq \begin{cases} G_{n-1}, & \text{if } Q = N(\epsilon_d) = 1; \\ G_n, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$*

Proof. Put $k_0 = \mathbb{Q}(\sqrt{-2})$ and denote by h_0 its class number. Let p be a prime dividing the discriminant of k and let e_p be its ramification index in k , then $p = 2$ or $p = p_i$, $e_{p_i} = 2$ and $e_2 = 2$ or 4 .

First case: for $d = d' \equiv 1 \pmod{4}$ or $d = 2d'$ with $d' \equiv -1 \pmod{4}$, we have $e_2 = 2$. Then

- $2\mathcal{O}_k = \mathcal{H}$ (resp. $f = 2$) or $\mathcal{H}_1^2 \mathcal{H}_2^2$ (resp. $f = 1$), hence $(\prod_2(k))^2 = \mathcal{H}^2 = 2\mathcal{O}_k$ or $(\prod_2(k))^2 = \mathcal{H}_1^2 \mathcal{H}_2^2 = 2\mathcal{O}_k$.

- if $p_i \equiv 1 \pmod{8}$ or $p_i \equiv 3 \pmod{8}$, so there exist π and π' in k_0 such that

$p^{h_0}\mathcal{O}_{k_0} = \pi\pi'$. Thus $f = f_p(k/\mathbb{Q}) = 1$ and there exist $\mathcal{B}_1, \mathcal{B}_2$ two prime ideals of $\mathcal{O}_k$ lying above p satisfying $B_1^2 = \pi\mathcal{O}_k$ and $B_2^2 = \pi'\mathcal{O}_k$, hence $(\prod_{p_i^2}(k))^2 = \pi\pi'\mathcal{O}_k = p\mathcal{O}_k$.

- if $p_i \equiv 5 \pmod{8}$ or $p_i \equiv 7 \pmod{8}$, so p_i remains inert in k_0 , then there exists π in k_0 such that $p_i^{h_0}\mathcal{O}_{k_0} = \pi$. Thus $(\prod_{p_i^{f_p(k/\mathbb{Q})}}(k))^2 = \pi\mathcal{O}_k$.

Consequently, $(\prod_{p^{f_p(k/\mathbb{Q})}}(k))^2$ is a principal ideal in k , i.e. $P(k)$ is an elementary abelian 2-group. The order of $P(k)$ is given by the Proposition 2.10 and Lemma 3.5 as follows:

$$|P(k)| = \begin{cases} 2^{n-2}, & \text{if } Q = N(\epsilon_d) = 1; \\ 2^{n-1}, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$$

Second case: for $d = d' \equiv 3 \pmod{4}$ or $d = 2d' \equiv 2 \pmod{4}$ (i.e. $d' \equiv 1 \pmod{4}$), then $e_2 = 4$.

As $e_2 = 4$, so there exist an ideal $\mathcal{H}_0$ in k_0 and an ideal $\mathcal{H}$ in k such that $2\mathcal{O}_{k_0} = \mathcal{H}_0^2$, $2\mathcal{O}_k = \mathcal{H}_0^4$ and $\mathcal{H}_0\mathcal{O}_k = \mathcal{H}^2$, moreover, $f = 1$. Thus

$$\left(\prod_2(k)\right)^2 = \mathcal{H}^2 = \mathcal{H}_0\mathcal{O}_k.$$

Since the 2-class group of k_0 is trivial, so $(\prod_2(k))^2$ is principal.

On the other hand, we know that for all prime p dividing the discriminant of k we have $e_p = 2$, hence proceeding as above we prove $(\prod_p^f(k))^2 = p\mathcal{O}_k$ is principal. Therefore, by Proposition 2.10 and Lemma 3.5 we get

$$|P(k)| = \begin{cases} 2^{n-1}, & \text{if } Q = N(\epsilon_d) = 1; \\ 2^n, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$$

□

Theorem 3.7. Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$, where $d > 2$ is a square-free integer, Q be the Hasse's unit index of k and ϵ_d be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Then k is a Pólya field if only if d satisfies one of the following conditions:

1. $d = p \equiv 1 \pmod{4}$ is a prime number.
2. $d = 2q$, where $q \equiv 3 \pmod{4}$ is a prime number.
3. $d = pq$, where p and q are distinct primes and
 - either $p \equiv q \equiv 1 \pmod{4}$, (p or $q \equiv 5 \pmod{8}$) and $N(\epsilon_d) = 1$,
 - or $p \equiv q \equiv 3 \pmod{4}$ and $Q = 1$.
4. $d = 2pq$, where $p \equiv -q \equiv 1 \pmod{4}$ are distinct primes and $Q = 1$

Proof. We have two cases to distinguish.

First case: d is odd. Then the Corollary 3.6 shows that k is a Pólya field if only if

$$d \equiv 1 \pmod{4} \text{ and } \begin{cases} n = 2, & \text{if } Q = N(\epsilon_d) = 1; \\ n = 1, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$$

or $d \equiv 3 \pmod{4}$ and $n = 1$, if $Q = N(\epsilon_d) = 1$.

we find that

$$\begin{cases} d = pq \equiv 1 \pmod{4}, & \text{if } Q = N(\epsilon_d) = 1; \\ d = p \equiv 1 \pmod{4}, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases}$$

or $d = q \equiv 3 \pmod{4}$, if $Q = N(\epsilon_d) = 1$;

The last case can not occur because by Lemmas 3.1 and 3.2 we have $N(\epsilon_d) = 1$ and $Q = 2$. On the other hand, if $d = p \equiv 1 \pmod{4}$, then Lemma 3.1 yields that $Q = 1$. Thus k is a Pólya field if only if

$$d = pq \equiv 1 \pmod{4} \text{ and } Q = N(\epsilon_d) = 1 \text{ or } d = p \equiv 1 \pmod{4}.$$

The first case is equivalent to

$$\begin{aligned} d = pq, \quad & \text{where } p \equiv q \equiv 1 \pmod{4} \text{ and } Q = N(\epsilon_d) = 1 \text{ or} \\ d = pq, \quad & \text{where } p \equiv q \equiv 3 \pmod{4} \text{ and } Q = 1. \end{aligned}$$

For the case $d = pq$, where $p \equiv q \equiv 1 \pmod{4}$ and $Q = N(\epsilon_d) = 1$, $Q = 1$ if only if $s \pm 1$ is not a square in $\mathbb{N}$ (Lemma 3.1), where $\epsilon_d = s + t\sqrt{d}$. this in turn is equivalent to $p \equiv 5 \pmod{8}$ or $q \equiv 5 \pmod{8}$. Which completes the proof of the assertions 1. and 3.

Second case: $d = 2d'$ is even. Then the Corollary 3.6 shows that k is a Pólya field if only if

$$\begin{aligned} d' \equiv 3 \pmod{4} \text{ and } \begin{cases} n = 2, & \text{if } Q = N(\epsilon_d) = 1; \\ n = 1, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases} \\ \text{or } d' \equiv 1 \pmod{4} \text{ and } n = 1, \text{ if } Q = N(\epsilon_d) = 1. \end{aligned}$$

we find that

$$\begin{aligned} \begin{cases} d' = pq \equiv 3 \pmod{4}, & \text{if } Q = N(\epsilon_d) = 1; \\ d' = q \equiv 3 \pmod{4}, & \text{if } Q = 2 \text{ or } N(\epsilon_d) = -1. \end{cases} \\ \text{or } d' = p \equiv 1 \pmod{4}, \text{ if } Q = N(\epsilon_d) = 1; \end{aligned}$$

The last case can not occur because by Lemmas 3.1 and 3.3, if $d = 2p$ and $N(\epsilon_d) = 1$, where p is an odd prime, then $Q = 2$. Thus k is a Pólya field if only if

$$d' = pq \equiv 3 \pmod{4} \text{ and } Q = N(\epsilon_d) = 1 \text{ or } d' = q \equiv 3 \pmod{4}.$$

This in turn is equivalent to

$$\begin{aligned} d = 2pq, \quad & \text{where } p \equiv -q \equiv 1 \pmod{4}, \text{ and } Q = 1 \text{ or} \\ d = 2q, \quad & \text{where } q \equiv 3 \pmod{4}. \end{aligned}$$

Which completes the proof of the assertions 2. and 4. □

Remark 3.8. Let $d = 2pq$, where $p \equiv -q \equiv 1 \pmod{4}$ are distinct primes and $Q = 1$. From [6], we know that $k_1 = \mathbb{Q}(\sqrt{-pq})$ and $k_2 = \mathbb{Q}(\sqrt{2pq})$ are not Pólya, but $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$ is. This answers affirmatively a question raised in [5, Remark 5.11.] on the existence of biquadratic Pólya fields not obtainable as a composition of two Pólya subfields. In fact we get infinitely many Pólya fields with only one Pólya subfield, i.e. $\mathbb{Q}(\sqrt{-2})$.

Corollary 3.9. *Let $k = \mathbb{Q}(\sqrt{d}, \sqrt{-2})$, where $d = 2^s p_1 p_2 \cdots p_n$ is a square-free integer, Q be the Hasse's unit index of k and ϵ_d be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Denote by e_2 the ramification index of 2 in k , and let L be a quadratic unramified extension of k and Galois over $\mathbb{Q}$. Then L is never a Pólya field in the following cases:*

1. $e_2 = 2$ and
 - either $5 < n$ and $Q = N(\epsilon_d) = 1$,
 - or $4 < n$ and ($Q = 2$ or $N(\epsilon_d) = -1$).
2. $e_2 = 4$ and
 - either $4 < n$ and $Q = N(\epsilon_d) = 1$,
 - or $3 < n$ and ($Q = 2$ or $N(\epsilon_d) = -1$).

Proof. As $E_k^2 \subset N_{L/k}(E_L) \subset E_k$, so $[E_k : N_{L/k}(E_L)] \leq [E_k : E_k^2] = 4$. Then the results are assured by Corollaries 2.9 and 3.6. $\square$

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