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A NOTE ON GENERALIZED MARCINKIEWICS INTEGRALS ASSOCIATED TO SURFACE OF REVOLUTION

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Abstract. Along the surface of revolution, this article gives appropriate L^p estimates for generalized Marcinkiewicz integral operators with kernels in $L^q(\mathbb{S}^{n-1})$, $q > 1$. Through an extrapolation argument, we establish the L^p boundedness of the generalized Marcinkiewicz integral operators to these bounds under weaker kernel assumptions. This not only enlarges the applicability of the operators but also indicates the functional relationships among various integral transforms. Finally, our findings are a contribution to the ongoing research on harmonic analysis and applications of partial differential equations.

1. Introduction and preliminaries

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue normalized measure $d\sigma$, and let $\mathbb{R}^n, n \geq 2$ be the Euclidean space of dimension n. Furthermore, for $s \in \mathbb{R}^n \setminus \{0\}$, let $s' = s/|s|$. For $1 \leq \kappa < \infty$, define $\mathfrak{L}^\kappa$ as the set of all measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\|f\|_{L^\kappa(\mathbb{R}^+, \frac{dr}{r})} = (\int_0^\infty |f(r)|^\kappa \frac{dr}{r})^{1/\kappa} \leq 1$. Assume that ω is a homogeneous function of degree zero in $\mathbb{R}^n$, integrable over $\mathbb{S}^{n-1}$, and satisfies the cancelation property

$$\int_{\mathbb{S}^{n-1}} \omega(s')d\sigma(s') = 0. \tag{1.1}$$

We examine the generalized Marcinkiewicz operator $\mu^{(\alpha)}_{\omega,\varphi,f}$ related to the function $f \in \mathcal{S}(\mathbb{R}^{n+1})$ for an appropriate mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}$.

$$\mu^{(\alpha)}_{\omega,\varphi,f}(y)(s, s_{n+1}) = \left(\int_{\mathbb{R}^n} \left|\frac{1}{t}\int_{|u|\leq t} y(s-u, s_{n+1}-\varphi(|u|))\omega(u)\frac{f(|u|)}{|u|^n}du\right|^\alpha \frac{dt}{t}\right)^{1/\alpha}. \tag{1.2}$$

When $\varphi(t) = 0, f \equiv 1$ and $\alpha = 2$, we denote $\mu^{(\alpha)}_{\omega,\varphi,f}$ by μ_ω, which is the classical Marcinkiewicz operator that Stein introduced in [12]. The L^p boundedness of μ_ω whenever $\omega \in Lip(\mathbb{S}^{n-1})$ for $(1 < p \leq 2)$ was examined by the author of [12]. The L^2 boundedness of μ_ω whenever $\omega \in L(logL)^{1/2}(\mathbb{S}^{n-1})$

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was later obtained by Walsh in [26]. Furthermore, Walsh verified that the assumption $\omega \in L(logL)^{1/2}(\mathbb{S}^{n-1})$ is optimal, that is μ_ω will not be bounded on $L^2(\mathbb{R}^n)$ if it is substituted with any weaker assumption $\omega \in L(logL)^{\gamma}(\mathbb{S}^{n-1})$ for $\gamma \in (0, 1/2)$. This result highlights the critical nature of the logarithmic weight in the context of boundedness, emphasizing that deviations from this optimal condition can lead to significant changes in the behavior of the associated Hardy-Littlewood maximal operator. Further exploration of the implications of these findings may yield deeper insights into the structure of singular integrals and their boundedness properties. In [7], Al-Salman and Al-Qasem extended this finding by demonstrating the boundedness of μ_ω for any $p \in (1, \infty)$. The boundedness of L^p of μ_ω for any $p \in (1, \infty)$ was later demonstrated by the authors of [19] if $\omega \in B_q^{(0,-1/2)}(\mathbb{S}^{n-1})$ for some $q > 1$. Furthermore, they demonstrated that assumption $\omega \in B_q^{(0,-1/2)}(\mathbb{S}^{n-1})$ is optimal. When $\omega \in L(logL)^{1/2}(\mathbb{S})^{n-1}$, $f \in \triangledown_\kappa(\mathbb{R}^+)$, the authors in [18] established the L^p boundedness of $\mu^{(2)}_{\omega,\varphi,f}$ with $\kappa > 1$ and $\varphi \in H_d$ for all $|1/p - 1/2| < \min\{1/\kappa', 1/2\}$. Many authors have studied the integral operator $\mu^{(2)}_{\omega,\varphi,f}$ under several assumptions in ω, φ and f. The readers are referred to ([4], [5], [7], [3], [9], [11], [6], [21]-[8]).

Beginning with [15], the authors of the study of the generalized Mincinkiewicz integral $\mu^{(\alpha)}_{\omega,\varphi,f}$ demonstrated that if $\omega \in L^q(\mathbb{S}^{n-1})$, $\varphi(t) = t$, and $f \equiv 0$, then

$$\left\|\mu^{(\alpha)}_{\omega,\varphi,f}(y)\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|y\right\|_{F_p^{0,\alpha}(\mathbb{R}^n)} \tag{1.3}$$

for $p, \alpha \in (1, \infty)$. The author in [16] improved this result under weaker conditions that $\omega \in L(logL)(\mathbb{S}^{n-1})$ and $f \in \triangledown_{max\{\alpha',2\}}(\mathbb{R}^+)$. These findings were expanded upon and enhanced in [17]. Recently, the authors in [24] proved the boundedness of $\mu^{(\alpha)}_{\omega,\varphi,f}$ under the same assumptions as in [18] with the replacement of $\alpha = 2$ by $\alpha > 1$. For more results, the readers are referred to ([21]-[8]).

In this paper, we build on the work of the authors in [24] by establishing a series of theorems that demonstrate the boundedness of the operator $\mu^{(\alpha)}_{\omega,P_n,f}$ in various function spaces, given certain conditions on the weights function ω, the function f, and the function P_n. Remember that $F_p^{s,\alpha}(\mathbb{R}^n)$ is the homogeneous Triebel–Lizorkin space defined by the space of all tempered distributions. for $1 < p, \alpha < \infty$ and $s \in \mathbb{R}$. $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$F_p^{s,\alpha}(\mathbb{R}^n) = \left\{y \in \mathcal{S}'(\mathbb{R}^n) : \|y\|_{F_p^{s,\alpha}(\mathbb{R}^n)} = \left\|\Big(\sum_{\kappa\in\mathbb{Z}} 2^{\kappa s\alpha} \left|\psi_\kappa * y\right|^\alpha\Big)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n)} < \infty\right\} \tag{1.4}$$

where $\hat{\psi}_\kappa(\eta) = \varphi(2^{-\kappa}\eta)$ for $\kappa \in \mathbb{Z}$ and $\varphi \in \mathbb{C}_0^\infty(\mathbb{R}^n)$ is a function that meets the following requirements:
(a) $\varphi \in [0, 1]$,
(b) $supp\ \varphi \subset \{\eta : |\eta| \in [1/2, 2]\}$,
(c) $\varphi(\eta) \geq c > 0$, if $|\eta| \in [3/5, 5/3]$,
(d) $\sum\limits_{j\in\mathbb{Z}} \varphi(2^{-\kappa}\eta) = 1$, $\eta \neq 0$.

Furthermore, the space $F_p^{s,\alpha}(\mathbb{R}^n)$ is known to satisfy the following:
(i) $L^p(\mathbb{R}^n) = F_p^{0,2}(\mathbb{R}^n)$
(ii) $F_p^{s,\alpha_1}(\mathbb{R}^n) \subset F_p^{s,\alpha_2}(\mathbb{R}^n)$ *if* $\alpha_1 \leq \alpha_2$.
(iii) $\mathcal{S}(\mathbb{R}^n)$ is dense in $F_p^{s,\alpha}(\mathbb{R}^n)$.

Our findings are summarized as follows:

2. Theorems and Corollaries

Theorem 2.1. *Suppose that* $\omega \in L^q(\mathbb{S}^{n-1})$ *for some* $q \in (1,2]$ *has the cancelation property (1.1) and that* $\|\omega\|_{L^1(\mathbb{S}^{n-1})} \leq 1$. *Consider* $\triangledown_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (1,2]$ *and* $P_n(s)$ *to be a polynomial of real value on* $\mathbb{R}$ *with positive coefficients and* $P_n(0) = 0$. *Then for every* p *satisfying* $|1/p - 1/2| < min\{1/2, 1/\kappa'\}$, *there exist a constant* $C_p > 0$ *such that:*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1/\kappa'}(\kappa-1)^{-1/\kappa'}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{2.1}$$

for all $\alpha \leq p \leq \frac{\kappa\alpha}{\alpha-\kappa\alpha+k}$;

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{\frac{\kappa'+\alpha}{-\alpha\kappa'}}(\kappa-1)^{\frac{\kappa'+\alpha}{-\alpha\kappa'}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{2.2}$$

for all $\frac{\kappa'\alpha}{\alpha+k'} < p < \alpha$; *and*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{\frac{\kappa\alpha+\kappa'}{-\kappa\kappa'\alpha}}(\kappa-1)^{\frac{\kappa\alpha+\kappa'}{-\kappa\kappa'\alpha}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{2.3}$$

for all $\frac{\kappa\kappa'\alpha}{\alpha\kappa+\kappa'} < p < \alpha$, *where* C_p *is an independent positive constant that is not affected by* f, α, q, *and* ω.

Using Yano's extrapolation arguments and the result of the previous theorem, we derive the following corollaries.

Corollary 2.2. *Suppose that* P_n *is given as in the previous theorem, and that* $f \in \triangledown_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (1,2]$. *For* $\alpha \leq p \leq \frac{\kappa\alpha}{\alpha-\kappa\alpha+k}$,
(a) if $\omega \in L(logL)^{1/\alpha}(\mathbb{S}^{n-1})$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}(1+\|\omega\|_{L(logL)^{1/\alpha}(\mathbb{S}^{n-1})})\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})}. \tag{2.4}$$

(b) if $\omega \in B_q^{(0,\frac{1}{\alpha}-1)}(\mathbb{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}(1+\|\omega\|_{B_q^{(0,\frac{1}{\alpha}-1)}(\mathbb{S}^{n-1})})\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{2.5}$$

Corollary 2.3. *Suppose that* P_n *is given as in the previous theorem and that* $f \in \triangledown_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (1,2]$. *For* $\frac{\kappa'\alpha}{\alpha+\kappa'} < p < \alpha$

(a) if $\omega \in L(logL)^{\frac{\kappa^{'}+\alpha}{\alpha\kappa^{'}}}$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{L(logL)^{\frac{\kappa^{'}+\alpha}{\alpha\kappa^{'}}}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.6)$$

(b) if $\omega \in B_q^{(0,\frac{1}{\alpha}-\frac{1}{\kappa})}(\mathbb{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{B_q^{(0,\frac{1}{\alpha}-\frac{1}{\kappa})}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})} \quad (2.7)$$

Corollary 2.4. *Suppose that* P_n *is given as in the previous theorem and that* $f \in \nabla_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (1,2]$. *For* $\frac{\kappa\kappa^{'}\alpha}{\alpha\kappa+\kappa^{'}} < p < \alpha$

(a) if $\omega \in L(logL)^{\frac{\alpha\kappa+\kappa^{'}}{\kappa\kappa^{'}\alpha}}$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{L(logL)^{\frac{\alpha\kappa+\kappa^{'}}{\kappa\kappa^{'}\alpha}}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.8)$$

(b) if $\omega \in B_q^{(0,\frac{1-\alpha}{\alpha\kappa})}(\mathbb{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|h\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{B_q^{\frac{1-\alpha}{\alpha\kappa})}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.9)$$

Theorem 2.5. *Assume that* $\omega \in L^q(\mathbb{S}^{n-1})$ *for some* $q \in (1,2]$ *satisfies the cancelation property (1.1) with* $\left\|\omega\right\|_{L^1(\mathbb{S}^{n-1})} \leq 1$. *Let* $f \in \nabla_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (2,\infty)$ *and let* P_n *be as in the previous theorem. Then for every* p *satisfying* $|1/p-1/2| < min\{1/2, 1/k^{'}\}$, *then a constant* $C_p > 0$ *exists, such that*

$$\left\|\mu^{(\alpha)}_{\omega,\varphi,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1/k^{'}} \left\|\omega\right\|_{L^q(\mathbb{S}^{n-1})} \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})} \quad (2.10)$$

for all $\kappa^{'} < p < \infty$; *and*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1} \left\|\omega\right\|_{L^q(\mathbb{S}^{n-1})} \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})} \quad (2.11)$$

for all $1 < p < \kappa^{'}$, *where* C_p *is an independant positive constant that is not affected by* α, φ, ω, *and* f.

Using the result of the previous theorem and Yano's extrapolation arguments, we get the following corollaries.

Corollary 2.6. *Suppose that* P_n *is given as in the previous theorem and that* $f \in \nabla_\kappa(\mathbb{R}^+)$ *for some* $\kappa \in (2,\infty)$. *For* $\kappa^{'} < p < \infty$,
(a) if $\omega \in L(logL)^{1/k^{'}}(\mathbb{S}^{n-1})$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{L(logL)^{1/k^{'}}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.12)$$

(b) if $\omega \in B_q^{(0,\frac{-1}{\kappa})}(\mathbb{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|h\right\|_{\nabla_\kappa(\mathbb{R}^+)} (1+\left\|\omega\right\|_{B_q^{(0,\frac{-1}{\kappa})}(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.13)$$

Corollary 2.7. *Suppose that P_n is given as in the previous theorem and that $f \in \bigtriangledown_\kappa(\mathbb{R}^+)$ for some $\kappa \in (2,\infty)$ For $1 < p < \kappa'$,*
(a) if $\omega \in L(logL)(\mathbb{S}^{n-1})$, then

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\bigtriangledown_\kappa(\mathbb{R}^+)} (1 + \left\|\omega\right\|_{L(logL)(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.14)$$

(b) if $\omega \in B^{(0,0)}_q(\mathbb{S}^{n-1})$ for some $q > 1$, then

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \left\|f\right\|_{\bigtriangledown_\kappa(\mathbb{R}^+)} (1 + \left\|\omega\right\|_{B^{(0,0)}_q(\mathbb{S}^{n-1})}) \left\|y\right\|_{F^{0,\alpha}_p(\mathbb{R}^{n+1})}. \quad (2.15)$$

3. Preliminary Lemmas

A number of auxiliary lemmas are presented and proven that are necessary to demonstrate our primary results. We define the family of measures $\{\sigma_{P_n,f,t} : \sigma_{h,t} : t \in \mathbb{R}^+\}$ and their related maximal operator $\sigma^*_{\omega,f}$ and $M_{\omega,f,a}$ on $\mathbb{R}^{n+1}$ for $a \geq 2$ and the appropriate mapping $f : \mathbb{R}^+ \to \mathbb{C}$, $\omega : \mathbb{S}^{n-1} \to \mathbb{R}$.

$$\int_{\mathbb{R}^{n+1}} y d\sigma_{h,t} = \frac{1}{t}\int_{t/2\leq|v|\leq t} y(v, P_n(|v|))\frac{\omega(v)f(|v|)}{|v|^{n-1}}dv. \quad (3.1)$$

$$\sigma^*_{\omega,f}(y) = \sup_{t\in\mathbb{R}^+} ||\sigma_{f,t}| * y| \quad (3.2)$$

and

$$M_{\omega,f,a}y = \sup_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} ||\sigma_{f,t}| * y| \frac{dt}{t}. \quad (3.3)$$

Lemma 3.1. [18] *Let $\omega \in L^q(\mathbb{S}^{n-1})$ for $q > 1$ satisfy the cancelation property (1.1) with $\|\omega\|_{L^1(\mathbb{S}^{n-1})} \leq 1$, and $f \in \bigtriangledown_\kappa(\mathbb{R}^+)$* for $\kappa > 1$. *Consider an arbitrary function φ on $\mathbb{R}^+$. Then there is a positive constant C and $\delta < \frac{1}{2q'}$ such that*

$$\int_{a^j}^{a^{j+1}} |\hat{\sigma_{h,t}}(\eta,\eta_{n+1}|^2 \frac{dt}{t} \leq C(\ln a), \quad (3.4)$$

$$\int_{a^j}^{a^{j+1}} |\hat{\sigma_{h,t}}(\eta,\eta_{n+1}|^2 \frac{dt}{t} \leq C(\ln a) \left\|\omega\right\|^2_{L^q(\mathbb{S}^{n-1})} \left\|f\right\|^2_{\bigtriangledown_\kappa(\mathbb{R}^+)} \min\{\left|a^j\eta\right|^{\frac{-\delta}{\ln a}}, \left|a^j\eta\right|^{\frac{\delta}{\ln a}}\}. \quad (3.5)$$

Lemma 3.2. [14] *Let ω, f and P_n be given as in the previous theorems, and let $a = 2^{\kappa' q'}$. Then a positive constant C_p exists such that for all $p > \kappa'$,*

$$\left\|M_{\omega,f,a}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1}(k-1)^{-1} \left\|\omega\right\|_{L^q(\mathbb{S}^{n-1})} \left\|f\right\|_{\bigtriangledown_\kappa(\mathbb{R}^+)} \left\|y\right\|_{L^p(\mathbb{R}^{n+1})} \quad (3.6)$$

and

$$\left\|\sigma^*_{\omega,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1/k'}(k-1)^{-1/k'} \left\|\omega\right\|_{L^q(\mathbb{S}^{n-1})} \left\|f\right\|_{\bigtriangledown_\kappa(\mathbb{R}^+)} \left\|y\right\|_{L^p(\mathbb{R}^{n+1})}. \quad (3.7)$$

We obtain the following lemma using the same logic as in [24].

Lemma 3.3. *Let ω, P_n and α be given as in the previous theorems. Suppose that $f \in \nabla_\kappa(\mathbb{R}^+)$ with $2 < k < \infty$. Then for $a = 2^{q' k'}$, there exist a positive constant C_p such that*
(a) If $\alpha > \kappa'$, for all $p > \kappa'$

$$\left\| \Big(\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |\sigma_{h,t} * g_j|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p (q-1)^{-1/k'} (k-1)^{-1/k'} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \times \|f\|_{\nabla_\kappa(\mathbb{R}^+)} \left\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^\alpha \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})}, \tag{3.8}$$

(b) If $\alpha \leq \kappa'$, for all $1 < p < \alpha$,

$$\left\| \Big(\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |\sigma_{h,t} * g_j|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p (q-1)^{-1} (k-1)^{-1} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \times \|f\|_{\nabla_\kappa(\mathbb{R}^+)} \left\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^\alpha \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})}, \tag{3.9}$$

where $\{g_j\}_{j\in\mathbb{Z}}$ is any sequence of functions on $\mathbb{R}^{n+1}$.

For proof, the readers can refer to Lemma 3 in [24]

Lemma 3.4. *Let ω, P_n and α be given as in the previous theorems. Suppose that $f \in \nabla_\kappa(\mathbb{R}^+)$ with $\kappa \in (1, 2]$. Then for $a \geq 2$ a positive constant C_p exists, such that:*
(a) For all $\alpha \leq p \leq \frac{\kappa\alpha'}{\alpha' - \kappa}$,

$$\left\| \Big(\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |\sigma_{h,t} * g_j|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p (q-1)^{\frac{-1}{\kappa'}} (k-1)^{\frac{-1}{\kappa'}} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \times \|f\|_{\nabla_\kappa(\mathbb{R}^+)} \left\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^\alpha \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \tag{3.10}$$

(b) For all $\frac{\kappa'\alpha}{\alpha + k'} < p < \alpha$,

$$\left\| \Big(\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |\sigma_{h,t} * g_j|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p (\ln a)^{\frac{\kappa' + \alpha}{-\alpha\kappa'}} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{\nabla_\kappa(\mathbb{R}^+)} \left\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^\alpha \Big)^{\frac{1}{\alpha}} \right\|_{L^p(\mathbb{R}^{n+1})} \tag{3.11}$$

(c) For all $\frac{\kappa\kappa'\alpha}{\alpha\kappa+\kappa'} < p < \alpha$,

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(\ln a)^{\frac{\alpha\kappa+\kappa'}{-\alpha\kappa\kappa'}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\nabla_\kappa(\mathbb{R}^+)}\left\|\left(\sum_{j\in\mathbb{Z}}|g_j|^{\alpha}\right)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})}. \tag{3.12}$$

Proof. Think about the case where $p>\alpha$. The existence of a nonnegative function D due to duality, in $L^{(\frac{p}{\alpha})'}(\mathbb{R}^{n+1})$ with $\|D\|_{L^{(\frac{p}{\alpha})'}(\mathbb{R}^{n+1})}\leq 1$ such that

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\right\|^{\alpha}_{L^p(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}}\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}D(s,s_{n+1})\frac{dt}{t}dsds_{n+1}. \tag{3.13}$$

Using Hölder's inequality, we get that

$$|\sigma_{h,t}*g_j|^{\alpha}\leq C\|f\|^{\frac{\alpha}{\alpha'}}\|\omega\|^{\frac{\alpha}{\alpha'}}\times\int_{\frac{t}{2}}^{t}\int_{\mathbb{S}^{n-1}}|g_j(s-ru,s_{n+1}-P_n(r)|^{\alpha}|\omega(u)||f(r)|d\sigma(u)\frac{dr}{r}. \tag{3.14}$$

Now, by change of variable, we have

$$\tag{3.15}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\right\|^{\alpha}_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|^{\frac{\alpha}{\alpha'}+1}\|\omega\|^{\frac{\alpha}{\alpha'}+1} \tag{3.16}$$

$$\times\int_{\mathbb{R}^{n+1}}\sum_{j\in\mathbb{Z}}|g_j|^{\alpha}D(-s,-s_{n+1})M_{|\omega|,|f|,a}dsds_{n+1}. \tag{3.17}$$

Thus, using Hölder's inequality and Lemma 3.2 in (3.15), letting $a=2^{q'\kappa'}$, we obtain

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\right\|^{\alpha}_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{\frac{-1}{\kappa'}}(k-1)^{\frac{-1}{\kappa'}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}$$

$$\times\|f\|_{\nabla_\kappa(\mathbb{R}^+)}\left\|\left(\sum_{j\in\mathbb{Z}}|g_j|^{\alpha}\right)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})}. \tag{3.18}$$

Now for $p=\alpha$, applying Hölder's inequality and Lemma 3.2, in (3.14), we get

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\right\|^{\alpha}_{L^p(\mathbb{R}^{n+1})} &\leq C_p\|f\|^{\frac{\alpha}{\alpha'}}\|\omega\|^{\frac{\alpha}{\alpha'}}\int_{\mathbb{R}^{n+1}}\int_{a^j}^{a^{j+1}}\int_{t/2}^{t}\int_{\mathbb{S}^{n-1}}|g_j|^{\alpha}|\omega(u)||f(r)|d\sigma(\\ &\leq C_p(q-1)^{-1}(k-1)^{-1}\left\|\sum_{j\in\mathbb{Z}}|g_j|^{\alpha}\right\|^{\alpha}_{L^p(\mathbb{R}^{n+1})}\|f\|^{\frac{\alpha}{\alpha'}+1}\|\omega\|^{\frac{\alpha}{\alpha'}+1},\end{aligned}$$

where $\widetilde{s}=(s,s_{n+1})$. For the case $p<\alpha$, according to duality there is a set of nonnegative functions y_j defined on $\mathbb{R}^{n+1}\times\mathbb{R}$ that satisfies $\Big\|\Big\|\|y_j\|_{L^{\alpha'}[a^j,a^{j+1}],\frac{dt}{t}}\Big\|_{t^{\alpha'}}\Big\|_{L^{p'}(\mathbb{R}^{n+1})}$ and

$$\left\|\Big(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\Big)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})}^{\alpha}=\int_{\mathbb{R}^{n+1}}\Big(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha'}y_j(\widetilde{s},t)\frac{dt}{t}d\widetilde{s}\Big). \tag{3.19}$$

For $a=2^{q'}$, according to duality again a function $\Omega\in L^{(p'/\alpha')'}(\mathbb{R}^{n+1})$ exists with $\|\Omega\|_{(p'/\alpha')'}=1$ such that

$$\left\|\int_{\mathbb{R}^{n+1}}\Big(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*y_j(\widetilde{s},t)|^{\alpha'}\frac{dt}{t}\Big)^{1/\alpha'}\right\|_{L^{p'}(\mathbb{R}^{n+1})}^{\alpha'} \tag{3.20}$$

$$\begin{aligned}
&= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*y_j(\widetilde{s},t)|^{\alpha'}\frac{dt}{t}\Omega(\widetilde{s})d\widetilde{s}\\
&\le C_p\|f\|^{\frac{\alpha'}{\alpha}}\|\omega\|^{\frac{\alpha'}{\alpha}}\left\|\sigma^*_{|f|,|\omega|}\right\|\|(\Omega(-\widetilde{s}))\|_{L(\frac{p'}{\alpha'})'}\left\|\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|y_j(\widetilde{s},t)|^{\alpha'}\frac{dt}{t}\right\|_{L^{(\frac{p'}{\alpha'})}}\\
&\le C_p\|f\|^{\alpha'}\|\omega\|^{\alpha'}(q-1)^{\frac{\kappa'+\alpha}{-\alpha\kappa'}}(k-1)^{\frac{\kappa'+\alpha}{-\alpha\kappa'}}\|\Omega\|_{(p'/\alpha')'}
\end{aligned}$$

for all $\frac{\kappa'\alpha}{\alpha+k'}<p<\alpha$.

Now Hölder's inequality with (3.19) and (3.21), we get

$$\begin{aligned}
\left\|\Big(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|\sigma_{h,t}*g_j|^{\alpha}\frac{dt}{t}\Big)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})}^{\alpha}\le& C_p\|\omega\|_{L^q(\mathbb{S}^{n-1})}\\
&\times\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}((q-1)(k-1))^{\frac{\kappa'+\alpha}{-\alpha\kappa'}}\left\|\Big(\sum|g_j|^{\alpha}\Big)^{\frac{1}{\alpha}}\right\|_{L^p(\mathbb{R}^{n+1})}
\end{aligned}$$

To prove part (c), we will follow the author's proofs in [17]. Define the linear operator T on any function

$$g=g_j(s)\text{ by }T(g_j(s))=\omega_{h,ta^j}*g_j(s). \tag{3.21}$$

Then, by Lemma 3.3 for$\kappa'<p<\infty,\ \alpha>k'$ and $\kappa>2$, we have

$$\begin{aligned}
\left\|\Big(\sum_{j\in\mathbb{Z}}\int_{a^j}^{a^{j+1}}|T(g_j(s)|^{\alpha}\frac{dt}{t}\Big)^{1/\alpha}\right\|_{L^p(\mathbb{R}^{n+1})}&\le\left\|\Big(\sum_{j\in\mathbb{Z}}\int_1^{a}|\omega_{h,ta^j}*g_j(s)|^{\alpha}\frac{dt}{t}\Big)^{1/\alpha}\right\|_{L^p(\mathbb{R}^{n+1})}\\
&\le C_P\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\triangledown_\kappa(\mathbb{R}^+)}((q-1)(k-1))^{\frac{1}{\kappa'}}\left\|\Big(\sum|g_j|^{\alpha'}\Big)\right.
\end{aligned}$$

If $1 < p < \alpha < k'$, then $p' > \alpha'$. So, according to duality a function $\kappa_j(\widetilde{s}, t)$ exists on $\mathbb{R}^{n+1} \times \mathbb{R}$ such that $\left\| \left\| \|\kappa_j\|_{L^{\alpha'}([a^j,a^{j+1}],\frac{dt}{t})} \right\|_{l^{\alpha'}} \right\|_{L^{p'}(\mathbb{R}^{n+1})} \leq 1$, and

$$\begin{aligned}\left\| \Big(\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |\omega_{f,t} * g_j|^{\alpha'} \frac{dt}{t} \Big)^{\frac{1}{\alpha'}} \right\|_{L^p(\mathbb{R}^{n+1})}^{\alpha'} &= \int_{\mathbb{R}^{n+1}} \sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} (\omega_{h,t} * g_j) k_j(\widetilde{s}, t) \frac{dt}{t} d\widetilde{\omega} \\ &\leq C_p \|\omega\|_{L^q}^{(\frac{\alpha'}{\alpha})+1} \|f\|_{\nabla_\kappa(\mathbb{R}^+)}^{\alpha'} (k-1)^{-1}(q-1)^{-1} \|g\|_{\nabla_\kappa(\mathbb{R}^+)}^{\alpha'} \|\kappa_j\|\end{aligned}$$

Interpolate (3.22) with (3.23), we finish the proof of part (c). □

4. Proof of the main results

The similar technique found in [17] and [24] will be used to prove our main results. Assume that $\omega \in L^q$ for some $1 < q \leq 2$ and $f \in \nabla_\kappa(\mathbb{R}^+)$ for some $1 < k \leq 2$. By Minkowski's inequality, one can find that

$$\begin{aligned}\mu_{\omega,P_n,f}^{(\alpha)}(y)(s, s_{n+1}) &= \sum_{j\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \left| \frac{1}{t} \int_{|u|\leq t} y(s-u, s_{n+1} - P_n(|u|))\omega(u) \frac{f(|u|)}{|u|^n} du \right|^\alpha \frac{dt}{t} \Big)^{1/\alpha} \\ &\leq \sum_{j\in\mathbb{Z}} 2^{-j\alpha} \Big(|\sigma_{f,t} * y|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}} \\ &= C \left(\int_{\mathbb{R}^n} \left| \frac{1}{t} \int_{|u|\leq t} y(s-u, s_{n+1} - P_n(|u|))\omega(u) \frac{f(|u|)}{|u|^n} du \right|^\alpha \frac{dt}{t} \right)^{1/\alpha}.\end{aligned}$$

Let $a = 2^{\kappa' q'}$ and let $\{\Upsilon_j\}_{j=-\infty}^{\infty}$ be a smooth partition of unity that satisfies the following conditions:

$$\Upsilon_j \in \mathbb{C}^\infty,\ 0 \leq \Upsilon_j \leq 1,\ \sum_j \Upsilon_j = 1, \tag{4.1}$$

$$a^{-j-1} \leq \text{supp } \Upsilon_j \leq a^{-j+1},\ \textit{and}\quad \left| \frac{d^s \Upsilon_j(t)}{dt^s} \right| \leq \frac{C}{t^s}, \tag{4.2}$$

where C independent of a. Let $\hat{\psi}_\kappa(\eta) = \Upsilon_j(|\eta|)\hat{y}(\eta, \eta_{n+1})$. So for $y \in \mathcal{S}(\mathbb{R}^{n+1})$, we have that

$$\mu_{\omega,P_n,f}^{(\alpha)}(y)(s, s_{n+1}) \leq C \int_{\mathbb{R}^{n+1}} \sum_{j\in\mathbb{Z}} \Big(\left| (\Upsilon_{s+j} * \sigma_{h,t} * y)\chi_{[a^s,a^{s+1}]} \right|^\alpha \frac{dt}{t} \Big)^{\frac{1}{\alpha}}. \tag{4.3}$$

To prove our main theorem, it suffices to demonstrate the existence of a positive constant η to satisfy the following inequalities:

$$\left\| \mu_{\omega,P_n,f}^{(\alpha)}(y) \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \frac{1}{2^{\eta|j|}} (q-1)^{-1/k'} (k-1)^{-1/k'} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{\nabla_\kappa(\mathbb{R}^+)} \|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.4}$$

for all $\alpha \leq p \leq \frac{\kappa\alpha'}{\alpha'-\kappa}$,

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \frac{1}{2^{\eta|j|}}(q-1)^{\frac{-\alpha-k'}{\kappa'\alpha}}(k-1)^{\frac{-\alpha-k'}{\kappa'\alpha}} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{\triangledown_\kappa(\mathbb{R}^+)} \|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.5}$$

for all $\frac{\kappa'\alpha}{\alpha+k'} < p < \alpha$, and

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \frac{1}{2^{\eta|j|}}(q-1)^{\frac{-\alpha\kappa-\kappa'}{\kappa\kappa'\alpha}}(k-1)^{\frac{-\alpha\kappa-\kappa'}{\kappa\kappa'\alpha}} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{\triangledown_\kappa(\mathbb{R}^+)} \|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.6}$$

for all $\frac{\kappa\kappa'\alpha}{\alpha\kappa+\kappa'} < p < \alpha$, The positive constant C_p is independent of α, ω, and f. We will follow [17] in their proof to establish (4.4). Let $p = \alpha = 2$. By Plancherel's theorem, we have

$$\begin{aligned}
\left\|\mu^{(2)}_{\omega,P_n,f,j}(y)\right\|^2_{L^p(\mathbb{R}^{n+1})} &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{n+1}} \int_{a^s}^{a^{s+1}} \Big(\left|(\Upsilon_{s+j} * \sigma_{h,t} * y)\right|^2 \frac{dt}{t} \Big) \\
&\leq \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{n+1}} \Big(\int_{a^s}^{a^{s+1}} \left|\hat{\sigma}_{h,t})\right|^2 \frac{dt}{t} \Big) \left|\hat{y}(\eta,\eta_{n+1})\right|^2 d\eta d\eta_{n+1} \\
&\leq C \frac{1}{(q-1)} \frac{1}{(k-1)} \|\omega\|^2_{L^q(\mathbb{S}^{n-1})} \|f\|^2_{\triangledown_\kappa(\mathbb{R}^+)} \\
&\times \sum_{\kappa\in\mathbb{Z}} \int_{I_{s+j}} min\Big(\left|a^{j-1}\eta\right|^{\frac{-\eta}{q'k'}}, \left|a^{j+1}\eta\right|^{\frac{\eta}{q'k'}} \Big) |\hat{y}|^2 d\eta d\eta_{n+1} \\
&\leq C \frac{1}{(q-1)} \frac{1}{(k-1)} \|\omega\|^2_{L^q(\mathbb{S}^{n-1})} \|f\|^2_{\triangledown_\kappa(\mathbb{R}^+)} 2^{-\eta|j|} \sum_{\kappa\in\mathbb{Z}} \int_{I_{s+j}} |\hat{y}|^2 d\eta d\eta_{n+1} \\
&\leq C \frac{1}{(q-1)} \frac{1}{(k-1)} \|\omega\|^2_{L^q(\mathbb{S}^{n-1})} \|f\|^2_{\triangledown_\kappa(\mathbb{R}^+)} 2^{-\eta|j|} \|y\|^2_{L^2(\mathbb{R}^{n+1})}.
\end{aligned}$$

Therefore,

$$\left\|\mu^{(2)}_{\omega,P_n,f,j}(y)\right\|^2_{L^p(\mathbb{R}^{n+1})} \leq C \frac{1}{(q-1)} \frac{1}{(k-1)} \|\omega\|^2_{L^q(\mathbb{S}^{n-1})} \|f\|^2_{\triangledown_\kappa(\mathbb{R}^+)} 2^{-\eta|j|} \|y\|^2_{F_p^{2.2}(\mathbb{R}^{n+1})}. \tag{4.7}$$

It is clear that Lemma 3.4 leads to the following.

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p(q-1)^{-1/k'}(k-1)^{-1/k'} \|\omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{\triangledown_\kappa(\mathbb{R}^+)} \|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.8}$$

for all $\alpha \leq p \leq \frac{\kappa\alpha'}{\alpha'-\kappa}$, and

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \le C_p(q-1)^{\frac{-\alpha-k'}{\kappa'\alpha}}(k-1)^{\frac{-\alpha-k'}{\kappa'\alpha}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\nabla_\kappa(\mathbb{R}^+)}\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.9}$$

for all $\frac{\kappa'\alpha}{\alpha+k'} < p < \alpha$.

$$\left\|\mu^{(\alpha)}_{\omega,P_n,f}(y)\right\|_{L^p(\mathbb{R}^{n+1})} \le C_p(q-1)^{\frac{-\alpha\kappa-\kappa'}{\kappa\kappa'\alpha}}(k-1)^{\frac{-\alpha\kappa-\kappa'}{\kappa\kappa'\alpha}}\|\omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{\nabla_\kappa(\mathbb{R}^+)}\|y\|_{F_p^{0,\alpha}(\mathbb{R}^{n+1})} \tag{4.10}$$

for all $\frac{\kappa\kappa'\alpha}{\alpha\kappa+\kappa'} < p < \alpha$. By interpolation between the previous inequalities, we get the proof of the first theorem. Lemma 3.3 can be used to establish Theorem 2.5 in the same manner as Theorem 2.1.

5. Conclusions

In conclusion, this work extends the study of generalized Marcinkiewicz integral operators, providing significant results concerning their boundedness on various L^p spaces under weaker conditions on the kernels, particularly those in $L^q(\mathbb{S}^{n-1})$ for $q > 1$, $L(logL)(\mathbb{S}^{n-1})$ or in the space $B_q^{(0,s-1)}(\mathbb{S}^{n-1})$. By establishing appropriate L^p bounds and utilizing extrapolation arguments, we offer a comprehensive framework for understanding the L^p boundedness of these operators. The results also shed light on the optimality of the conditions on the kernels and the role of the functions involved, such as f and ω, in determining the behavior of the operators in different function spaces. This understanding not only enhances the theoretical foundation of the study but also has practical implications for various applications in harmonic analysis and partial differential equations. Future research may focus on extending these results to more general settings, potentially uncovering new relationships between different function spaces and operator behaviors.

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