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NONPARAMETRIC RELATIVE ERROR REGRESSION ESTIMATOR UNDER η-WEAK DEPENDENCE WITH GEOMETRIC RATE

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Abstract. In this paper, we present a non-parametric regression approach to study the relationship between two variables. The relative error estimator serves as a more robust alternative to classical regression, particularly under the influence of outliers. First, we present the relative error regression estimator. Next, we then investigate both pointwise and uniform almost complete convergence, along with their rates, under the η-weak dependence condition. Finally, the numerical simulations demonstrate the effectiveness and accuracy of the proposed estimator.

1. Introduction

The regression model is among the leading models extensively studied models in both parametric and nonparametric statistics. [5] and [9] discussed the parametric estimation within the context of regression models. Nevertheless, the nonparametric regression is a widely applicable analysis method used in various fields, including finance, engineering, social sciences, medicine and marketing. The primary objective of nonparametric estimation of the regression is to construct an estimate for a function $r(.)$ that represents the relationship between the response variable Y and the covariate X, as described

$$Y = r\left(X\right) + \varepsilon, \tag{1.1}$$

where ε represents a stochastic error term with zero mean, independent of X and $r(.)$ is the regression function commonly estimated through the minimization of the expected value of the quadratic loss function, defined as

$$E\left[\left(Y - r(x)\right)^2 \mid X = x\right]. \tag{1.2}$$

However, this squared loss function is sensitive to outliers, which can produce inaccurate results, as it assumes all variables have equal weight and that residual variance is the same for all observations. A more robust alternative, especially in

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the presence of outliers, is minimizing the following mean squared relative error, for $Y > 0$

$$E\left[\left(\frac{Y - r(x)}{Y}\right)^2 \mid X = x\right]. \tag{1.3}$$

Clearly, the minimization problem's solution is

$$r(x) = \frac{E\left(Y^{-1} \mid X = x\right)}{E\left(Y^{-2} \mid X = x\right)}, \tag{1.4}$$

when the first two moments of the inverse of Y, conditioned on X, are finite. [15] have demonstrated that this solution satisfies

$$r(x) = \frac{E\left(Y^{-1} \mid X = x\right)}{E\left(Y^{-2} \mid X = x\right)} \leq E(Y \mid X = x). \tag{1.5}$$

The history of non-parametric relative regression error estimation evolved through several key contributions. [10] first explored the asymptotic performance of a consistent estimate for this model employing the kernel approach, establishing its quadratic convergence for independent and identically distributed observations. In 2010, [3] introduced the least absolute relative error estimation for multiplicative regression models, proving the consistency and asymptotic normality of the estimator and offering an inference approach through random weighting. These studies and others focus on independent data. However, in practice, data are often dependent, particularly in time series, which has led to extensive research on kernel estimation to address such dependencies. Recently, [12] advanced the field by studying nonparametric relative regression for associated variables, they proved its strong consistency and the asymptotic normality. However, [6] introduced a more general notion of imposing restrictions on dependence beyond α-mixing and association. In 2001, they studied the almost sure consistency of the kernel density estimator, see [7]. Later, the consistency of the kernel regression estimate is obtained by [2]. [13] established the asymptotic normality of recursive kernel estimation of the density and [14] provide the strong consistency of recursive kernel estimators of both density and regression under η-weak dependence.

Our contribution establishes the asymptotic properties of the relative error regression estimator under weak dependence, as defined by [6], using the Bernstein-type inequality established by [8], which yields the same rate as in the independent data case. To our knowledge, these results have not been investigated in statistical literature. Therefore, this work aims to address this gap. More precisely, in Section 2, we begin by recalling the definitions of weak dependence and the relative error regression estimator. Then, we prove its pointwise and uniform almost complete convergence, in Section 3. Finally, Section 4 displays computational findings to demonstrate the accuracy and performance of the proposed estimate. The Appendix includes some technical lemmas.

2. Definition and estimation

Consider $k \in \mathbb{N}$, $u, v \in \mathbb{N}^*$, $(s_1, ..., s_u) \in \mathbb{Z}^u$ and $(t_1, ..., t_v) \in \mathbb{Z}^v$ when

$$s_1 \leq ... \leq s_u \leq t_1 \leq ... \leq t_v. \tag{2.1}$$

We briefly recall that a sequence of real random variables $(X_t)_{t\in\mathbb{Z}}$ is recognized as η-weakly dependent dependent if there exists a decreasing sequence $(\eta_k)_{k\in\mathbb{N}}$ tending to 0 at infinity and if for all bounded functions $\varphi_1 : \mathbb{R}^u \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^v \to \mathbb{R}$ satisfying

$$|Cov\left(\varphi_1\left(X_{s_1}, ..., X_{s_u}\right), \varphi_2\left(X_{t_1}, ..., X_{t_v}\right)\right)| \leq \left[u \left\|\varphi_2\right\|_\infty Lip\left(\varphi_1\right) + v \left\|\varphi_1\right\|_\infty Lip\left(\varphi_2\right)\right] \eta_k \tag{2.2}$$

with $Lip\left(\varphi_l\right) = \sup\limits_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\|x-y\|_1}$, $\|x\|_1 = \sum\limits_{i=1}^{u} |x_i|$ and $Lip\left(\varphi_l\right) < \infty$, for $l = 1, 2$.
Let us explore some well-known examples of such sequences that are η-weakly dependent, including Bernoulli shift processes (see [1]), Chaotic Volterra models (see [8]), the bilinear process and stable Markov chains (see [4]). For further details, the reader is referred to [4] and [13].
Let $\{Z_i = (X_i, Y_i),\ 1 \leq i \leq n\}$ be n stationary random processes, identically distributed as the random pair $Z = (X, Y)$, taking values in $\mathbb{R} \times \mathbb{R}_+^*$. Throughout this paper, we assume that the marginal and joint densities f_X and $f_{(X_1,X_2)}$ of (X_1, X_2) exist.
Note that Nadaraya-Watson estimate of the classical regression is defined by

$$\bar{r}\left(x\right) = \frac{\sum\limits_{i=1}^{n} Y_i K\left(\frac{x-X_i}{h_n}\right)}{\sum\limits_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right)}, \tag{2.3}$$

with K is a kernel and h_n is a sequence of positive real numbers. Inspired by this estimator, [10] estimated the regression function $r(.)$ under the relative loss as

$$\widehat{r}\left(x\right) = \frac{\sum\limits_{i=1}^{n} Y_i^{-1} K\left(\frac{x-X_i}{h_n}\right)}{\sum\limits_{i=1}^{n} Y_i^{-2} K\left(\frac{x-X_i}{h_n}\right)}, \tag{2.4}$$

where K and h_n are given in (2.3).

3. Main results

3.1. **Pointwise almost complete convergence.** For investigating the pointwise almost complete convergence of $\widehat{r}$ for a fixed point x in $\mathbb{R}$, we first need to make the following assumptions.

- (H_1) (X_i, Y_i) is a stationary sequence of η-weak dependent real valued random variables and $\eta(r) = O\left(\exp\left(-ar\right)\right), a > 0$.
- (H_2) For $l = 1, 2$, the functions $r_l\left(x\right) = E\left(Y^{-l} \mid X = x\right)$ and f_X are continuous at x.
- (H_2') For $l = 1, 2$, the functions $r_l\left(x\right)$ and f_X are m times continuously differentiable around x.

(H_3) The bandwidth h_n satisfies

$$\lim_{n\to\infty} h_n = 0, \quad \lim_{n\to\infty} \frac{\ln n}{nh_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} nh_n^{6+\xi} = +\infty \text{ for some } \xi > 0.$$

(H_4) K is a Lipschitz density with compact support $[-b, b]$.
(H_5) $\int |z|^j K(z)\, dz = 0 \ \forall j = 1, .., m-1$.
(H_6) For $l = 1, 2$, $E\left(|Y|^{-l}\right) < \infty$, or $E\left(e^{|Y|^{-l}}\right) < \infty$.
(H_7) $\exists C > 0$, $E\left(Y^{-4} \mid X = x\right) < C$.
(H_8) The function $f_{(X_1,X_2)}$ is continuous around (x, x).

The above hypotheses are commonly made the nonparametric estimation framework. The assumption (H_1) means that (X_i, Y_i) are weakly dependent with a geometric rate, which is a general choice for the dependence context, rather than mixing and association dependence. It has numerous practical applications, as it is satisfied by many stochastic processes, including certain time series models. The regularity conditions (H_2), (H_2') and (H_8) help us evaluate the bias and the variance terms, enabling us to obtain our results. Additionally, these conditions ensure smooth and stable relationships between variables, which is crucial for making reliable predictions in real-world applications like regression and, consequently, for ensuring the robustness of our estimate. Finally, the technical conditions (H_3)–(H_7) play a vital role in supporting the proofs of our results. These assumptions are essential for ensuring that the nonparametric estimation process is both statistically sound and practically meaningful. By guaranteeing the smoothness, stability, and reasonable behavior of the variables involved, these conditions help ensure that the estimates we obtain are reliable, robust, and well-behaved.
We are now ready to present our first result.

Theorem 3.1. *Assume that hypotheses (H_1), (H_3), (H_4) and (H_6)–(H_8) are satisfied.*
(i) Under the additional hypothesis (H_2), we obtain

$$|\widehat{r}(x) - r(x)| = o_{a.co}(1). \tag{3.1}$$

(ii) If, in addition, (H_2') and (H_5) are satisfied, we get

$$|\widehat{r}(x) - r(x)| = O\left(h_n^m\right) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n}}\right). \tag{3.2}$$

Proof. Let us set

$$\widehat{r}(x) = \frac{\widehat{r}_1(x)}{\widehat{r}_2(x)}, \tag{3.3}$$

where for $l = 1, 2$

$$\widehat{r}_l(x) = \frac{1}{nh_n} \sum_{i=1}^{n} Y_i^{-l} K\left(\frac{x - X_i}{h_n}\right), \tag{3.4}$$

which will have a significant role in the demonstration due to the below decomposition.

$$\begin{aligned}\widehat{r}(x) - r(x) &= \frac{1}{\widehat{r}_2(x)}\left[\widehat{r}_1(x) - E(\widehat{r}_1(x)) + E(\widehat{r}_1(x)) - g_1(x)\right] \\ &+ \frac{r(x)}{\widehat{r}_2(x)}\left[g_2(x) - E(\widehat{r}_2(x)) + E(\widehat{r}_2(x)) - \widehat{r}_2(x)\right], \qquad (3.5)\end{aligned}$$

where $g_l(x) := r_l(x) f(x)$. □

Lemma 3.2. *Suppose that the conditions (H_3) and (H_4) are fulfilled.*
(i) Under the additional hypothesis (H_2), we obtain

$$E[\widehat{r}_l(x)] - g_l(x) = o(1). \qquad (3.6)$$

(ii) If, in addition, (H_2') and (H_5) are satisfied, we have

$$E[\widehat{r}_l(x)] - g_l(x) = O(h_n^m). \qquad (3.7)$$

Proof. We have

$$\begin{aligned}E[\widehat{r}_l(x)] &= \frac{1}{h_n}E\left[K\left(\frac{x-X_1}{h_n}\right)E\left(Y_1^{-l} \mid X_1\right)\right] \\ &= \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{x-u}{h_n}\right)\underbrace{r_l(u) f_X(u)}_{g_l(u)}du \\ &= \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{x-u}{h_n}\right) g_l(u)\,du, \qquad (3.8)\end{aligned}$$

(i) Setting $z = x - u$, under (H_2) Bochner's theorem gives

$$\begin{aligned}E[\widehat{r}_l(x)] &= \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{z}{h_n}\right) g_l(x-z)\,dz \\ &\to g_l(x)\int_{\mathbb{R}} K(z)\,dz = g_l(x), \qquad (3.9)\end{aligned}$$

because K is density.

(ii) Now, we set $z = \dfrac{x-u}{h_n}$ in (3.8) and since K is a density, we obtain

$$E[\widehat{r}_l(x)] - g_l(x) = \int_{-b}^{b} K(z)\left(g_l(x - h_n z) - g_l(x)\right)dz. \qquad (3.10)$$

Hypotheses (H_2') and (H_3)–(H_5) combined with Taylor expansion around x permit us to write

$$\left|E[\widehat{r}_l(x)] - g_l(x)\right| \leq \frac{h_n^m}{m!}\int_{-b}^{b} |z|^m K(z)\left|g_l^{(m)}(\alpha_n)\right| dz, \qquad (3.11)$$

with α_n is between $x - h_n z$ and x. Moreover, since $h_n \to 0$ and $g_l^{(m)}$ is continuous at the point x, we get for any $\epsilon > 0$

$$\exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ \ |\alpha_n - x| \leq |x - h_n z - x| \leq |z| h_n \leq b h_n \implies |g_l^{(m)}(\alpha_n) - g_l^{(m)}(x)| < \epsilon,$$

thus,

$$\begin{aligned} |E[\widehat{r}_l(x)] - g_l(x)| &\leq \frac{h_n^m}{m!} \int_{-b}^{b} |z|^m K(z) \underbrace{\left| g_l^{(m)}(\alpha_n) - g_l^{(m)}(x) \right|}_{<\varepsilon} dz + \frac{h_n^m}{m!} |g_l^{(m)}(x)| \int_{-b}^{b} |z|^m K(z)\, dz \\ &\leq \left[\frac{b^m \left(|g_l^{(m)}(x)| + \varepsilon \right)}{m!} \right] h_n^m = O(h_n^m), \end{aligned} \tag{3.12}$$

thanks to hypothesis (H_4). □

Lemma 3.3. *Under hypotheses* (H_1), (H_2) *(or* (H_2')*)*, (H_3), (H_4) *and* (H_6)*–*(H_8)*, we have*

$$\widehat{r}_l(x) - E[\widehat{r}_l(x)] = O_{a.co}\left(\sqrt{\frac{\ln n}{n h_n}} \right). \tag{3.13}$$

Proof. To achieve the proof, we base it on the following decomposition

$$\widehat{r}_l(x) - E[\widehat{r}_l(x)] = \widehat{r}_l(x) - \widetilde{r}_l(x) + \widetilde{r}_l(x) - E[\widetilde{r}_l(x)] + E[\widetilde{r}_l(x)] - E[\widehat{r}_l(x)],$$

where $\widetilde{r}_l(x)$ represents the truncated version of $\widehat{r}_l(x)$, defined by

$$\widetilde{r}_l(x) = \frac{1}{n h_n} \sum_{i=1}^{n} Y_i^{-l} 1_{\{|Y_i|^{-l} \leq b_n\}} K\left(\frac{x - X_i}{h_n} \right). \tag{3.14}$$

So, we need to show taking into account the weak dependence assumption of the observations, if necessary, that for $l = 0, 1$

$$|\widehat{r}_l(x) - \widetilde{r}_l(x)| = O_{a.co}\left(\sqrt{\frac{\ln n}{n h_n}} \right), \tag{3.15}$$

$$|E[\widehat{r}_l(x)] - E[\widetilde{r}_l(x)]| = O\left(\sqrt{\frac{\ln n}{n h_n}} \right) \tag{3.16}$$

and

$$|\widetilde{r}_l(x) - E[\widetilde{r}_l(x)]| = O_{a.co}\left(\sqrt{\frac{\ln n}{n h_n}} \right). \tag{3.17}$$

• Starting with the term $\widehat{r}_l(x) - \widetilde{r}_l(x)$.
We have

$$\begin{aligned} \widehat{r}_l(x) - \widetilde{r}_l(x) &= \frac{1}{n h_n} \sum_{i=1}^{n} Y_i^{-l} K\left(\frac{x - X_i}{h_n} \right) - \frac{1}{n h_n} \sum_{i=1}^{n} Y_i^{-l} 1_{\{|Y_i|^{-l} \leq b_n\}} K\left(\frac{x - X_i}{h_n} \right) \\ &= \frac{1}{n h_n} \sum_{i=1}^{n} Y_i^{-l} 1_{\{|Y_i|^{-l} > b_n\}} K\left(\frac{x - X_i}{h_n} \right). \end{aligned} \tag{3.18}$$

Observe that for $\varepsilon > 0$

$$
\begin{aligned}
P\left(\left|\widehat{r}_l(x)-\widetilde{r}_l(x)\right|>\varepsilon\sqrt{\frac{\ln n}{nh_n}}\right) &= P\left(\left|\frac{1}{nh_n}\sum_{i=1}^{n}Y_i^{-l}1_{\{|Y_i|^{-l}>b_n\}}K\left(\frac{x-X_i}{h_n}\right)\right|>\varepsilon\sqrt{\frac{\log n}{nh_n}}\right)\\
&\leq P\left(\max_{1\leq i\leq n}|Y_i|^{-l}>b_n\right)\\
&\leq nP\left(|Y|^{-l}>b_n\right). \qquad (3.19)
\end{aligned}
$$

If $E\left(|Y|^{-l}\right)<\infty$, by employing the inequality of Markov, we get

$$
P\left(\left|\widehat{r}_l(x)-\widetilde{r}_l(x)\right|>\varepsilon\sqrt{\frac{\ln n}{nh_n}}\right)\leq\frac{n}{b_n}E\left(|Y|^{-l}\right), \qquad (3.20)
$$

for $\sum_{n\geq 1}\frac{n}{b_n}<\infty$, we choose $b_n=M_0n^p$, $M_0>0$, $p>2$.
If $E\left(\exp\left[|Y|^{-l}\right]\right)<\infty$, we obtain by using Markov's inequality,

$$
\begin{aligned}
P\left(\left|\widehat{r}_l(x)-\widetilde{r}_l(x)\right|>\varepsilon\sqrt{\frac{\ln n}{nh_n}}\right) &\leq nP\left(e^{|Y|^{-l}}>e^{b_n}\right)\\
&\leq ne^{-b_n}E\left(e^{|Y|^{-l}}\right). \qquad (3.21)
\end{aligned}
$$

for $\sum_{n\geq 1}ne^{-b_n}<\infty$, we take $b_n=M_0\ln n$, $M_0>2$.
Therefore, under assumption (H_6), we get

$$
\sum_{i\geq 1}P\left(\left|\widehat{r}_l(x)-\widetilde{r}_l(x)\right|>\varepsilon\sqrt{\frac{\ln n}{nh_n}}\right)<\infty. \qquad (3.22)
$$

• It is easy to see that

$$
\left|E\left(\widehat{r}_l(x)\right)-E\left(\widetilde{r}_l(x)\right)\right| \leq E\left|\widehat{r}_l(x)-\widetilde{r}_l(x)\right|, \qquad (3.23)
$$

we deduce that

$$
\left|E\left(\widehat{r}_l(x)\right)-E\left(\widetilde{r}_l(x)\right)\right|=O\left(\sqrt{\frac{\ln n}{nh_n}}\right). \qquad (3.24)
$$

• Finally, studying the term $\widetilde{r}_l(x)-E\left[\widetilde{r}_l(x)\right]$. Let us set

$$
\widetilde{r}_l(x)-E\left[\widetilde{r}_l(x)\right]:=\sum_{i=1}^{n}Z_i, \qquad (3.25)
$$

where, for $l=1,2$

$$
Z_i=\frac{1}{nh_n}\left[Y_i^{-l}1_{\{|Y_i|^{-l}\leq b_n\}}K\left(\frac{x-X_i}{h_n}\right)-E\left(Y_i^{-l}1_{\{|Y_i|^{-l}\leq b_n\}}K\left(\frac{x-X_i}{h_n}\right)\right)\right], \qquad (3.26)
$$

Notice that, the sequence Z_i is weakly dependent. Let $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function such that $F(X_i, Y_i) = Z_i$. Under the condition (H_4), we have

$$\|F\|_\infty \leq \frac{2b_n}{nh_n} \|K\|_\infty = O\left(\frac{b_n}{nh_n}\right) \tag{3.27}$$

and

$$LipF \leq \frac{2b_n}{nh_n^2} LipK = O\left(\frac{b_n}{nh_n^2}\right). \tag{3.28}$$

Furthermore, following Proposition 8 of [8], we can derive results analogous to relations (1)–(3) presented in this article, by taking $\Psi(u, v) = u + v$, $M = \|F\|_\infty$, $E^2 = MLipF$, $\mu = 1$, $\nu = 0$ and $L_1 = L_2 = \frac{1}{1-e^{-ar}}$.
Thus, we can apply the following exponential inequality, which is derived from [8]

$$P\left(\sum_{i=1}^{n} |Z_i| \geq \epsilon\right) \leq \exp\left(-\frac{\epsilon^2/2}{A_n + B_n^{1+\mu+\nu+2} \epsilon^{(2+2\mu+2\nu+3)(\mu+\nu+2)}}\right), \tag{3.29}$$

with A_n and B_n are specified as follows

$$A_n := \sigma^2 = Var\left(\sum_{i=1}^{n} Z_i\right) \tag{3.30}$$

and

$$B_n = 2(E \vee M)L_2\left(\frac{2^{4+\mu+\nu} nE^2 L_1}{A_n} \vee 1\right),$$

in view of Lemma 5.1, we get

$$A_n = O\left(\frac{1}{nh_n}\right) \text{ and } B_n = O\left(\frac{b_n^3}{nh_n^{7/2}}\right).$$

Therefore, taking $\varepsilon = \varepsilon_0 \sqrt{\frac{\ln n}{nh_n}}$ in the relation (3.29) and $b_n = M_0 \ln n$, $M_0 > 0$, we obtain

$$\begin{aligned}
\mathbb{P}\left(\left|\sum_{i=1}^{n} Z_i\right| > \varepsilon\right) &\leq \exp\left(-\frac{\varepsilon^2/2}{A_n + B_n^{1/3}\varepsilon^{5/3}}\right) \\
&\leq 2\exp\left\{\frac{-\frac{\varepsilon_0^2}{2}\ln n}{C\left(1 + \varepsilon_0^{5/3}\left(\frac{M_0^6(\ln n)^{11}}{nh_n^6}\right)^{1/6}\right)}\right\} \\
&\leq Cn^{-C'\varepsilon_0^2},
\end{aligned} \tag{3.31}$$

thanks to (H_3). □

Lemma 3.4. *Given the assumptions of Lemma 3.3, we derive*

$$\exists \zeta > 0, \quad \sum_{n \geq 1} P\left(\widehat{r}_2(x) \leq \frac{\zeta}{2}\right) < \infty.$$

Proof. We have

$$P\left(\widehat{r}_2(x) \leq \frac{g_2(x)}{2}\right) \leq P\left(|g_2(x) - \widehat{r}_2(x)| \geq \frac{g_2(x)}{2}\right).$$

The proof is completed by setting $\zeta = \frac{g_2(x)}{2}$ and using the Lemma 3.3. □

3.2. **Uniform almost complete convergence.** Consider $\mathcal{S}$ as a compact set of $\mathbb{R}$. To establish the uniform consistency of $\widehat{r}$, we introduce the following additional conditions.

- (H_9) For $l = 1, 2$, the functions $r_l(x)$ and f are continuous on $\mathcal{S}$.
- (H_9') For $l = 1, 2$, the functions $r_l(x)$ and f are m times continuously differentiable on $\mathcal{S}$.
- (H_{10}) $\exists C > 0$, $E\left(Y^{-4} \mid X = x\right) < C$ on $\mathcal{S}$
- (H_{11}) The function $f_{(X_1,X_2)}$ is continuous on $\mathcal{S} \times \mathcal{S}$.

Notice that these conditions are uniform version of the corresponding conditions (H_2), (H_2'), (H_7) and (H_8) in the pointwise case.

Theorem 3.5. *Suppose that assumptions (H_1), (H_3), (H_4), (H_6), (H_{10}) and (H_{11}) are fulfilled.*
(i) Under the additional hypothesis (H_9), we have

$$\sup_{x\in\mathcal{S}} |\widehat{r}(x) - r(x)| = o_{a.co}(1). \tag{3.32}$$

(ii) If, in addition, (H_9') and (H_5) are satisfied, we get

$$\sup_{x\in\mathcal{S}} |\widehat{r}(x) - r(x)| = O\left(h_n^m\right) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n}}\right). \tag{3.33}$$

It is clear that the demonstration of Theorem 3.5 follows directly from the decomposition (3.5) and the below lemmas, which represent the uniform forms of Lemmas 3.2, 3.3 and 3.4. A known method can be applied to derive the following results from Lemmas 3.2, 3.3 and 3.4, as shown in the proof of Lemmas 1-(ii), 2-(ii) and 4-(ii) in [11].

Lemma 3.6. *Assume that assumptions (H_3) and (H_4) hold.*
(i) Under the additional hypothesis (H_9), we get

$$\sup_{x\in\mathcal{S}} |E[\widehat{r}_l(x)] - g_l(x)| = o(1). \tag{3.34}$$

(ii) If, in addition, (H_9') and (H_5) are satisfied, we have

$$\sup_{x\in\mathcal{S}} |E[\widehat{r}_l(x)] - g_l(x)| = O\left(h_n^m\right). \tag{3.35}$$

Lemma 3.7. *Given hypotheses (H_1), (H_3), (H_4), (H_6), (H_9) (or (H_9')), (H_{10}) and (H_{11}), we obtain*

$$\sup_{x\in\mathcal{S}} |\widehat{r}_l(x) - E[\widehat{r}_l(x)]| = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n}}\right). \tag{3.36}$$

Lemma 3.8. *Given the assumptions of Lemma 3.7, we have*

$$\exists \zeta > 0, \quad \sum_{n\geq 1} P\left(\inf_{x\in\mathcal{S}} |\widehat{r}_2(x)| \leq \frac{\zeta}{2}\right) < \infty. \tag{3.37}$$

4. Simulation study

This section provides to evaluate the effectiveness of the relative error regression estimate (RER) defined by (2.4) for finite samples. Specifically, we compare the RER estimator $\widehat{r}(x)$ to the classical kernel estimator (CR) given by (2.3).
Similar to the simulation example presented by [14], we will consider Bernoulli shift process given by

$$Z_i = \frac{1}{2}\left(Z_{i-1} + \epsilon_i\right), \; 1 \leq i \leq n \tag{4.1}$$

with Bernoulli innovations $P(\epsilon_i = 0) = P(\epsilon_i = 1) = \frac{1}{2}$, which is an example of processes that are not mixing, moreover, it is proved that it satisfies the dependence condition introduced by [6]. To deal with the forecasting problem, the auto-regression function of order 1 is obtained by setting $X_i = Z_i$ and $Y_i = Z_{i+1}$.
Now, for computing the relative regression estimate, we employ the Epanechnikov kernel $K(t) = \frac{3}{4}(1-t)^2 1_{[-1,1]}$ and the smoothing bandwidth $h_n = Cn^{(-1/5)}$, $C > 0$.
To show the benefits of the relative estimator when outliers are present, we initially create artificial outliers by multiplying certain variables Z_i, $i = \overline{1,n}$ by the coefficients $M = 20$, 50, 200, the number of outliers is denoted by the constant k taking different increasing values and depending on the sample size n. We split then the sample into two parts, $n-p$-learning variables to construct the predicted values $\widehat{Z}_i$, and p-test ones Z_i (true values).
The performance of the regression estimators (RER) and (CR) is substantiated over $s = 50$ realizations of the generated process (Z_i) by the empirical relative mean square error given by

$$ERMSE = \frac{1}{s}\sum_{j=1}^{s}\frac{1}{p}\sum_{i=1}^{p}\left(\frac{Z_{i,j} - \widehat{Z}_{i,j}}{Z_{i,j}}\right)^2, \tag{4.2}$$

where $\widehat{Z}$ is calculated using the two regression estimators, RER and CR. The obtained values are given in tables 1, 2 and 3.
In figures 1, 2 and 3, the black lines (resp. the red ones) correspond to the true observations (resp. the forecasted ones obtained by the RER and CR methods).

Table 1. Effect of number of outliers k on prediction error, $n = 100$

$n = 100$	$k = 10$	$k = 6$	k=4	k=0
RER	0.42	0.34	0.25	0.22
CR	21.15	4.13	1.05	0.79

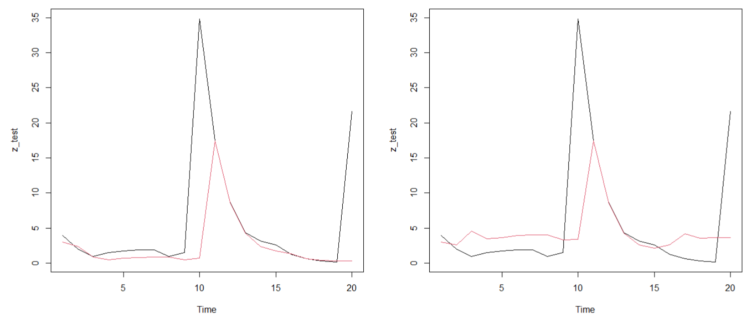

FIGURE 1. From left to right: the RER and CR estimates with outliers for $n = 100$

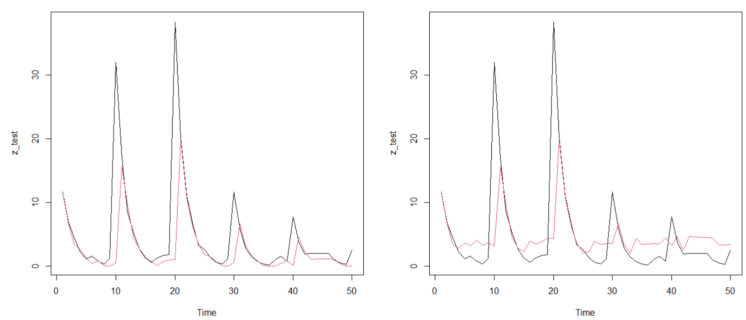

FIGURE 2. From left to right: the RER and CR estimate with outliers for $n = 300$,

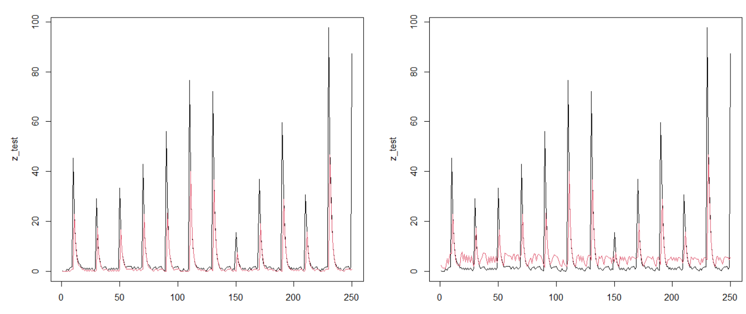

FIGURE 3. From left to right: the RER and CR estimates with outliers for $n = 500$

Comments.

- The predicted values are nearer to the true values of the dependent process using the RER estimator comparing to the CR estimator in the presence of outliers (see figures 1,2, 3).

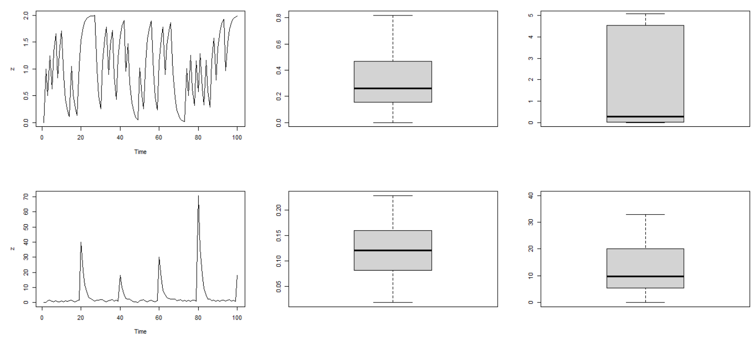

FIGURE 4. From left to right: Distribution of errors, The boxplot of the RER and the CR estimator with and without outliers for $n = 100$

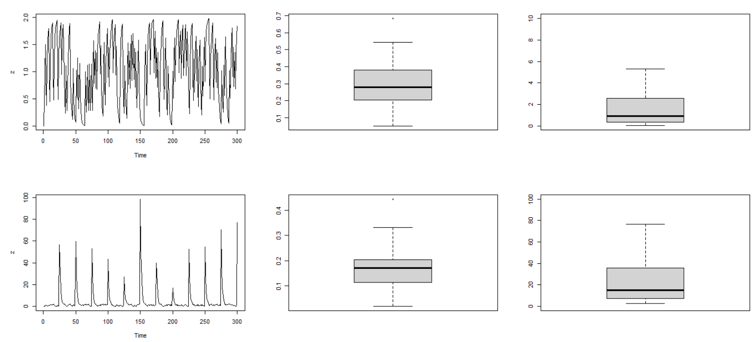

FIGURE 5. From left to right: Distribution of errors, The boxplot of the RER and the CR estimator with and without outliers for $n = 300$

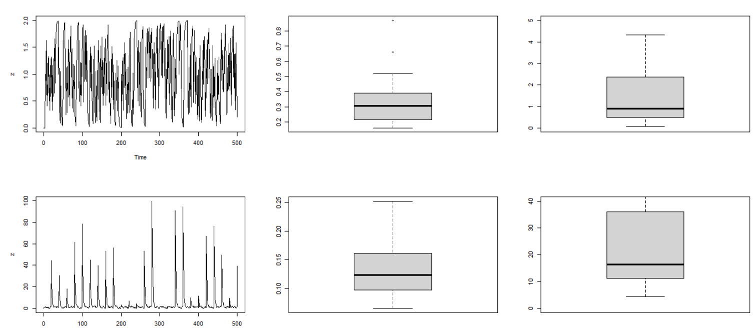

FIGURE 6. rom left to right: Distribution of errors, The boxplot of the RER and the CR estimator with and without outliers for $n = 500$

- The distribution error confirms that RER is advantageous in the two cases, with and without outliers, but the difference is quite apparent in the presence of outliers. (See figures 4, 5 and 6).

TABLE 2. Effect of number of outliers k on prediction error, $n = 300$

n=300	k=15	k=10	k=7	k=0
RER	0.37	0.32	0.26	0.15
CR	69.09	31.41	2.52	0.23

TABLE 3. Effect of number of outliers k on prediction error, $n = 500$

n=500	k=50	k=8	k=5	k=0
RER	0.73	0.54	0.51	0.093
CR	48.67	12.95	10.67	0.14

- When sample sizes increases, we get better results.
- The quantity of outliers affects also the quality of prediction, more the number of outliers is higher more the superiority of RER estimator is obvious (see tables 1, 2 and 3).

5. Appendix

Lemma 5.1. *If assumptions (H_1), (H_2) (or (H_2')) and (H_3), (H_4), (H_7) and (H_8) are satisfied, we get*

$$A_n = O\left(\frac{1}{nh_n}\right), \tag{5.1}$$

where A_n is defined in (3.30).

Proof. We have

$$\begin{aligned} A_n &= Var\left(\sum_{i=1}^{n} Z_i\right) \\ &= nVar\left(Z_1\right) + I_{1,n}(x) + I_{2,n}(x), \end{aligned} \tag{5.2}$$

where

$$I_{1,n}(x) := \sum_{S_1} Cov\left(Z_i, Z_j\right), \quad S_1 = \left\{(i,j)\,/1 \le |i-j| \le m_n\right\} \tag{5.3}$$

and

$$I_{2,n}(x) := \sum_{S_2} Cov\left(Z_i, Z_j\right), \quad S_2 = \left\{(i,j)\,/m_n + 1 \le |i-j| \le n-1\right\}, \tag{5.4}$$

with the sequence m_n will be specified below.
• Study of $Var(Z_1)$.

We have

$$\begin{aligned} n^2 h_n Var(Z_1) &\leq \frac{1}{h_n} E\left[K^2\left(\frac{x-X_1}{h_n}\right) E\left(Y_1^{-2l} \mid X_1\right)\right] \\ &\leq \frac{C}{h_n} E\left[K^2\left(\frac{x-X_1}{h_n}\right)\right] \\ &\leq \frac{C}{h_n}\int_{\mathbb{R}} K^2\left(\frac{z}{h_n}\right) f_X(x-z)dz \\ &\to C f_X(x)\int_{\mathbb{R}} K^2(z)dz, \text{ as } n\to\infty, \end{aligned} \tag{5.5}$$

thanks to Bockner's theorem and the conditions (H_2)-(H_4) and (H_7), thus

$$Var(Z_1) = O\left(\frac{1}{n^2 h_n}\right). \tag{5.6}$$

• Study of $I_{1,n}(x)$.
As $E(Z_i)=0$, we get

$$\begin{aligned} I_{1,n}(x) &= \frac{1}{n^2 h_n^2}\sum_{S_1} Cov\left(K\left(\frac{x-X_i}{h_n}\right) Y_i^{-l} 1_{\{|Y_i|^{-l}\leq b_n\}},\ K\left(\frac{x-X_j}{h_n}\right) Y_j^{-l} 1_{\{|Y_j|^{-l}\leq b_n\}}\right) \\ &\leq \frac{b_n^2}{n^2 h_n^2}\sum_{S_1} E\left[K\left(\frac{x-X_i}{h_n}\right) K\left(\frac{x-X_j}{h_n}\right)\right], \end{aligned} \tag{5.7}$$

Bockner's theorem and condition (H_8) lead to

$$\begin{aligned} \frac{1}{h_n^2} E\left[K\left(\frac{x-X_i}{h_n}\right) K\left(\frac{x-X_j}{h_n}\right)\right] &= \frac{1}{h_n^2}\int\int K\left(\frac{u}{h_n}\right) K\left(\frac{v}{h_n}\right) f_{(X_1,X_2)}(x-u,x-v)dudv \\ &\to f_{(X_1,X_2)}(x,x)\int\int K(u)\,K(v)\,dudv \text{ as } n\to\infty \end{aligned} \tag{5.8}$$

thus,

$$E\left[K\left(\frac{x-X_i}{h_n}\right) K\left(\frac{x-X_j}{h_n}\right)\right] \leq C h_n^2. \tag{5.9}$$

Then,

$$I_{1,n}(x) = O\left(\frac{m_n b_n^2}{n}\right). \tag{5.10}$$

• Study of $I_{2,n}(x)$.
In view of the η-dependence condition (H_1), the relation (2.2) states that

$$Cov\left(K\left(\frac{x-X_i}{h_n}\right), K\left(\frac{x-X_j}{h_n}\right)\right) \leq \frac{2}{h_n}\|K\|_\infty\, LipK\ \eta(|i-j|) \leq \frac{C}{h_n} e^{-a|i-j|} \tag{5.11}$$

so,

$$\begin{aligned}
I_{2,n}(x) &\leq \frac{b_n^2}{n^2h_n^2}\sum_{S_2} Cov\left(K\left(\frac{x-X_i}{h_n}\right), K\left(\frac{x-X_j}{h_n}\right)\right)\\
&\leq \frac{Cb_n^2}{n^2h_n^3}\sum_{S_2} e^{-a|i-j|}\\
&\leq \frac{Cb_n^2}{nh_n^3}\int_{m_n}^{n} e^{-au}du\\
&\leq \frac{Cb_n^2e^{-am_n}}{nh_n^3}. \qquad (5.12)
\end{aligned}$$

Thus,

$$I_{2,n}(x) = O\left(\frac{b_n^2e^{-am_n}}{nh_n^3}\right). \qquad (5.13)$$

Choosing $b_n = M_0 \ln n$, with $M_0 > 0$ and $m_n = \lfloor \frac{1}{b_n^2 h_n} \rfloor$, we obtain

$$I_{1,n}(x) = O\left(\frac{1}{nh_n}\right) \text{ and } I_{2,n}(x) = O\left(\frac{1}{nh_n}\right). \qquad (5.14)$$

We readily derive the claimed result from (5.2), (5.6) and (5.14).

□

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