Gulf Journal of Mathematics
Vol 19, Issue 2 (2025) 328-339

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v19i2.2779

KRONECKER FUNCTION RING OF A RING EXTENSION IN PULLBACK

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Abstract. In this paper, we study the Kronecker function ring of a ring extension arising from a pullback diagram of type $\square$ corresponding to e.a.b. star operation.

1. Introduction

In [6], Knebusch and Kaiser introduced the notion of the Kronecker function ring of a ring extension with respect to a star operation. This approach has generalized the classical notion of the Kronecker function ring.
Recall that a star operation on a ring extension $R \subseteq S$ is a map $* : \Im(R,S) \to \Im(R,S)$, where $\Im(R,S)$ is the set of all R-submodules of S, satisfying the following conditions: for all $A, B \in \Im(R,S)$, we have:

(1) $A \subseteq A^*$ and $(A^*)^* = A^*$.
(2) $AB^* \subseteq (AB)^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

A star operation $*$ on a ring extension $R \subseteq S$ is said to be e.a.b. if the following holds: if A, B, and C are finitely generated R-submodules of S such that $AS = S$ and $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$ [6, Definition 1].

The Kronecker function ring of a ring extension arising from a pullback diagram of type $\square$ corresponding to $R \subseteq S$ Prüfer extension was introduced by Paudel and Tchamna in [9]. According to [6], if the extension $R \subseteq S$ is Prüfer, the star operation $*$ on the extension $R \subseteq S$ is e.a.b. star operation. The purpose of this paper is to study the Kronecker function ring corresponding to the e.a.b. star operation in pullback. We provide a more general result showing the relation between commutative diagrams and Kronecker function rings, which, unlike in [9, Theorem 2.12], does not assume that $R \subseteq S$ is an extension of Prüfer.

2. preliminaries

In this paper, all rings are assumed to be commutative with identity. Throughout the paper, for any ring R, we denote by $\mathrm{Jac}(R)$ the Jacobson radical of R.

Date: Received: Mar 18, 2025; Accepted: Apr 7, 2025
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2020 *Mathematics Subject Classification.* Primary 13A15; Secondary 13A18, 13B02.
Key words and phrases. Star operation, Kronecker function ring, ring extension, pullback.

Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. For any $g \in S[X]$, and any ring A between R and S, we denote by $c_A(g)$ the A-submodule of S generated by the coefficients of g.

Definition 2.1. ([7, Definition 1]) An R-submodule I of S is called regular in S (or S-regular) if $IS = S$.

Lemma 2.2. *([9, Remark 1.3]) Let S be a ring, and let X be an indeterminate over S. Let $\alpha = \frac{f}{g} \in S(X)$ with $f, g \in S[X]$ and $c_S(g) = S$. Then $\alpha = 0$ if and only if $f = 0$.*

By the definition of localization and the fact that $c_S(g) = S$ ensures g is not a zero divisor.

Lemma 2.3. *([5, Proposition 33.1]) Let S be a ring, and X be an indeterminate over S. Let $\{M_\beta\}_{\beta \in B}$ be the set of maximal ideals of S, and $T = \{g \in S[X] : c_S(g) = S\}$. Then the following statements hold:*

(1) *If for each $\beta \in B$, $M_\beta[X]$ denotes the ideal of $S[X]$ generated by M_β, then*

$$T = S[X] - \left(\bigcup_{\beta \in B} M_\beta \right).$$

(2) *T is a multiplicative system in $S[X]$ containing only regular elements; $S(X)$ denotes the quotient ring $(S[X])_T$.*
(3) *$\{M_\beta[X]S(X)\} = \{M_\beta S(X)\}$ is the set of maximal ideals of $S(X)$.*

Definition 2.4. ([6, Definition 3]) Let $R \subseteq S$ be a ring extension, and let $*$ be a star operation on the extension $R \subseteq S$. We denote by $Kr(*)$ the set of all quotients $\frac{f}{g} \in S(X)$ such that $f, g \in S[X]$, $c_S(g) = S$, and $(c_R(f)H)^* \subseteq (c_R(g)H)^*$ for some finitely generated S-regular R-submodule H of S. The ring $Kr(*)$ is called the Kronecker function ring of $R \subseteq S$ with respect to $*$.

Lemma 2.5. *([9, Remark 2.1]) Let $R \subseteq S$ be a ring extension. Then for each e.a.b. star operation $*$ on the extension $R \subseteq S$, we have*

$$Kr(*) = \left\{ \frac{f}{g} \in S(X) : f, g \in S[X],\, c_S(g) = S,\, c_R(f)^* \subseteq c_R(g)^* \right\}.$$

We now define what a pullback diagram is, which is the main tool in this paper.

Definition 2.6. Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions, and consider the following commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \Psi} \\ L & \longrightarrow & T \end{array}$$

such that

a) $\ker \Psi$ is an ideal of R.
b) $\Psi : S \to T$ is surjective.

c) The restriction $\alpha : R \to L$ of Ψ is also surjective, and the horizontal mappings are inclusions.

If $\ker \Psi$ is a maximal ideal of S, then the diagram above is called a pullback diagram of type $\square$.

Pullback diagrams of type $\square$ are studied by Gabelli and Houston in [4].

Next, we position ourselves in the following situation. Let $R \subseteq S$ be a ring extension, and let I be a common ideal of R and S. Consider a pullback diagram of type $\square$:

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Psi} \\ R/I & \longrightarrow & S/I \end{array}$$

Now, we state some important remarks:

Remark 2.7. ([9, Remark 2.4]) Consider a pullback diagram of type $\square$. If H is an R-submodule of S such that $HS = S$, then $I \subseteq H$.

Remark 2.8. ([9, Remark 2.5]) Let $R \subseteq S$ be a ring extension, and $\Psi : S \to T$ a ring homomorphism, and A, B two R-submodules of S such that $\Psi(A) \subseteq \Psi(B)$. If Ψ is surjective and $\ker \Psi \subseteq B$, then $A \subseteq B$.

Finally, we define the following:

Definition 2.9. If $f : A \to C$, $g : B \to C$ are ring homomorphisms, the subring $T := A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$ of the direct product $A \times B$ is called the pullback (or fiber product) of A and B over C.

3. RESULTS

In this section, we study the behavior of Kronecker function rings in pullback diagrams. Let $R \subseteq S$ be a ring extension, and let I be a common ideal of R and S. Consider a pullback diagram of type $\square$:

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Psi} \\ R/I & \longrightarrow & S/I \end{array}$$

We prove that $Kr(*)$ and $S(X)$ share the same extension of I, that is $IKr(*) = IS(X)$.

Proposition 3.1. *Let $R \subseteq S$ be a ring extension, and let I be a common ideal of R and S. Then for each e.a.b. star operation $*$ on the extension $R \subseteq S$, we have:*

$$IKr(*) = IS(X).$$

Proof. Since the $*$ star operation is e.a.b., by Lemma 2.5, we have

$$Kr(*) = \left\{ \frac{f}{g} \in S(X) : f, g \in S[X],\, c_S(g) = S,\, c_R(f)^* \subseteq c_R(g)^* \right\},$$

so $Kr(*) \subseteq S(X)$. This shows that $IKr(*) \subseteq IS(X)$.

For the containment $IS(X) \subseteq IKr(*)$, it suffices to show that each element of the form

$$a = \sum_{i=1}^{r} u_i \frac{f_i}{g_i},$$

with $u_i \in I$ and $\frac{f_i}{g_i} \in S(X)$, is also an element of $IKr(*)$.

Write $f_i = \sum_{k=0}^{n} a_k^i X^k$ and $g_i = \sum_{l=0}^{m} b_l^i X^l$, with $a_k^i \in S$, $b_l^i \in S$ for $0 \leq k \leq n$ and $0 \leq l \leq m$. Since $c_S(g_i) = S$, then

$$1 = s_0^i b_0^i + \cdots + s_m^i b_m^i,$$

with $s_l^i \in S$.

Then,

$$u_i \frac{f_i}{g_i} = \left(u_i s_0^i b_0^i + \cdots + u_i s_m^i b_m^i\right) \frac{f_i}{g_i} = u_i s_0^i b_0^i \frac{f_i}{g_i} + \cdots + u_i s_m^i b_m^i \frac{f_i}{g_i}.$$

This expression can be rewritten as:

$$u_i s_0^i f_i \frac{b_0^i}{g_i} + \cdots + u_i s_m^i f_i \frac{b_m^i}{g_i}.$$

Since I is an ideal of S and $u_i \in I$, $s_l^i \in S$, and $a_k^i \in S$, it follows that $u_i s_l^i f_i \in I[X]$. For $0 \leq l \leq m$, we have $b_l^i \in S[X]$ and $c_R(b_l^i) = b_l^i R \subseteq c_R(g_i) = b_0^i R + \cdots + b_m^i R$, so

$$c_R(b_l^i)^* \subseteq c_R(g_i)^*.$$

Since $c_S(g_i) = S$, we conclude that $\frac{b_l^i}{g_i} \in Kr(*)$. This implies that

$$u_i \frac{f_i}{g_i} \in I[X]Kr(*).$$

Notice that if $h = \sum_{k=0}^{d} c_k X^k \in I[X]$ and $\frac{A}{B} \in Kr(*)$, then

$$h\left(\frac{A}{B}\right) = \sum_{k=0}^{d} c_k \left(\frac{X^k A}{B}\right) \in IKr(*).$$

Since $\frac{A}{B} \in Kr(*)$ and $(c_R(X^k A))^* = c_R(A)^*$, we have

$$(c_R(X^k A))^* \subseteq c_R(B)^*.$$

Therefore, $\frac{X^k A}{B} \in Kr(*)$, and thus

$$h\left(\frac{A}{B}\right) \in IKr(*).$$

Thus, $I[X]Kr(*) \subseteq IKr(*)$, so $u_i \frac{f_i}{g_i} \in IKr(*)$. Therefore, $a \in IKr(*)$.

Thus, we conclude that

$$IS(X) \subseteq IKr(*).$$

It follows that $IKr(*) = IS(X)$.

□

If $f = \sum_{k=0}^{n} a_k X^k \in S[X]$, write

$$\overline{f} = \sum_{k=0}^{n} \overline{a_k} X^k = \sum_{k=0}^{n} \Psi(a_k) X^k \in (S/I)[X].$$

Let $\overline{\Psi} : S(X) \to (S/I)(X)$ be defined by $\frac{f}{g} \mapsto \frac{\overline{f}}{\overline{g}}$, the extension of the canonical map $\Psi : S \to S/I$. Let $*$ be a star operation e.a.b. on $R \subseteq S$, and let $*'$ be a star operation e.a.b. on $R/I \subseteq S/I$. We denote by $\overline{\alpha} : Kr(*) \to Kr(*')$, given by $\frac{f}{g} \mapsto \frac{\overline{f}}{\overline{g}}$, the restriction of $\overline{\Psi}$ to $Kr(*)$. The maps $Kr(*) \hookrightarrow S(X)$ and $Kr(*') \hookrightarrow (S/I)(X)$ are inclusion maps.

Lemma 3.2. *Consider a diagram of type* □. *Let* $*$ *be a star operation e.a.b. on* $R \subseteq S$, *and let* $*'$ *be a star operation e.a.b. on* $R/I \subseteq S/I$ *such that for any* A *in* $\Im(R, S)$, $\Psi(A)^{*'} = \Psi(A^*)$. *Then,* $\overline{\Psi}$ *and* $\overline{\alpha}$ *are well-defined.*

Proof. Let $f = \sum_{k=0}^{n} a_k X^k$ and $g = \sum_{k=0}^{m} b_k X^k$ with $a_k, b_k \in S$. Write $\overline{f} = \sum_{k=0}^{n} \overline{a_k} X^k$ and $\overline{g} = \sum_{k=0}^{m} \overline{b_k} X^k$.

$$\begin{aligned}
\frac{f}{g} \in S(X) &\Rightarrow c_S(g) = S \\
&\Rightarrow \Psi(c_S(g)) = \Psi(S) \\
&\Rightarrow \Psi(b_0 S + \cdots + b_m S) = \Psi(S) \\
&\Rightarrow \Psi(b_0)\Psi(S) + \cdots + \Psi(b_m)\Psi(S) = S/I \\
&\Rightarrow \Psi(b_0)(S/I) + \cdots + \Psi(b_m)(S/I) = S/I \\
&\Rightarrow c_{S/I}(\overline{g}) = S/I \\
&\Rightarrow \frac{\overline{f}}{\overline{g}} \in (S/I)(X)
\end{aligned}$$

It is clear that if $\frac{f}{g} = \frac{f'}{g'}$ with $f, g, f', g' \in S[X]$, then $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f'}}{\overline{g'}}$. This shows that $\overline{\Psi}$ is well defined.

Now, let $\frac{f}{g} \in Kr(*)$. Since $*$ is a star operation e.a.b. on $R \subseteq S$, by Lemma 2.5, we have $\frac{f}{g} \in S(X)$, $c_S(g) = S$, and $c_R(f)^* \subseteq c_R(g)^*$. Therefore, $\overline{f}, \overline{g} \in (S/I)[X]$, $\Psi(c_S(g)) = \Psi(S)$, and $\Psi(c_R(f)^*) \subseteq \Psi(c_R(g)^*)$. Hence, $\overline{f}, \overline{g} \in (S/I)[X]$, $c_{S/I}(\overline{g}) = S/I$, and $(\Psi(c_R(f)))^{*'} \subseteq (\Psi(c_R(g)))^{*'}$. This implies that $\frac{\overline{f}}{\overline{g}} \in (S/I)(X)$ and $(c_{R/I}(\overline{f}))^{*'} \subseteq (c_{R/I}(\overline{g}))^{*'}$, so $\frac{\overline{f}}{\overline{g}} \in Kr(*')$.

It is clear that if $\frac{f}{g} = \frac{f'}{g'}$ with $f, g, f', g' \in S[X]$, then $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f'}}{\overline{g'}}$. This shows that $\overline{\alpha}$ is well defined.

□

Proposition 3.3. *Consider a pullback diagram of type* □, *and suppose that* $I \subseteq Jac(R)$ *or* $I \subseteq Jac(S)$. *Let* $*'$ *be a star operation e.a.b. on* $R/I \subseteq S/I$, *and*

let $*$ be the map from $\Im(R,S)$ to $\Im(R,S)$ defined in the following way: for any $A \in \Im(R,S)$, we have $A^* = \Psi^{-1}(\Psi(A)^{*'})$, such that the diagram

$$\begin{array}{ccc} Kr(*) & \xrightarrow{i_1} & S(X) \\ \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \overline{\Psi}} \\ Kr(*') & \xrightarrow{i_2} & (S/I)(X) \end{array}$$

commutes. Then:

(1) *$*$ is a star operation e.a.b. on $R \subseteq S$.*
(2) *For each $A \in \Im(R,S)$, $\Psi(A)^{*'} = \Psi(A^*)$.*
(3) *$Kr(*') \cong Kr(*)/IKr(*)$.*

Proof. 1) and 2) come from [9], Proposition 2.8.

3) We show that $\overline{\alpha}$ is surjective and $\mathrm{Ker}(\overline{\alpha}) = IKr(*)$.

Let $\frac{u}{v} \in Kr(*')$. By Lemma 2.5, $\frac{u}{v} \in (S/I)(X)$ with $u, v \in (S/I)[X]$, $c_{S/I}(v) = S/I$, and $(c_{R/I}(u))^{*'} \subseteq (c_{R/I}(v))^{*'}$.

Write $u = \sum_{i=0}^{n} \Psi(a_i)X^i = \sum_{i=0}^{n} \overline{a_i}X^i$, and $v = \sum_{j=0}^{m} \Psi(b_j)X^j = \sum_{j=0}^{m} \overline{b_j}X^j$, where $a_i, b_j \in S$.

Let $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$. We have $c_{S/I}(v) = S/I$, so $c_{R/I}(v)S/I = S/I$. Hence, we can write:

$$\overline{1} = \sum_{i=1}^{l} y_i \overline{s_i}, \quad y_i \in c_{R/I}(v), \quad s_i \in S.$$

Since $y_i \in c_{R/I}(v)$, then $y_i = \sum_{k=0}^{m} \overline{b_k}\overline{r_k^i}$, where $b_k \in S$ and $r_k^i \in R$. Thus:

$$\overline{1} = \overline{\sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i s_i}.$$

Now, since $1 - \sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i s_i \in I \subseteq \mathrm{Jac}(R)$ (or $I \subseteq \mathrm{Jac}(S)$), we conclude that $\sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i s_i$ is a unit in R (or S). Therefore, there exists $x \in S$ such that:

$$\left(\sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i s_i\right) x = 1.$$

Thus, $S \subseteq c_R(g)S$. Since $c_R(g)S \subseteq S$, we have $c_R(g)S = S$, and hence $c_S(g) = S$. This implies that $\frac{f}{g} \in S(X)$.

Next, we show that $\frac{f}{g} \in Kr(*)$. Notice that $\Psi(c_R(f)) = c_{R/I}(u)$ and $\Psi(c_R(g)) = c_{R/I}(v)$. Since $(c_{R/I}(u))^{*'} \subseteq (c_{R/I}(v))^{*'}$, we get:

$$(\Psi(c_R(f)))^{*'} \subseteq (\Psi(c_R(g)))^{*'},$$

which implies:

$$\Psi(c_R(f)^*) \subseteq \Psi(c_R(g)^*).$$

Since $c_R(g)$ is an R-submodule of S satisfying $c_R(g)S = S$, by Remark 2.7, we have $I \subseteq c_R(g)$. Therefore, $I \subseteq c_R(g)^*$, and $\mathrm{Ker}(\Psi) = I \subseteq c_R(g)^*$. This shows that:

$$\Psi(c_R(f)^*) \subseteq \Psi(c_R(g)^*).$$

Since Ψ is surjective, by Remark 2.8, we conclude that $c_R(f)^* \subseteq c_R(g)^*$, and hence $\frac{f}{g} \in Kr(*)$. Since $\overline{\alpha}\left(\frac{f}{g}\right) = \frac{\overline{f}}{\overline{g}} = \frac{u}{v}$, we conclude that $\overline{\alpha}$ is surjective.

Now, let $\frac{f}{g} \in Kr(*)$ such that $\overline{\alpha}\left(\frac{f}{g}\right) = \overline{0}$. Then $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$, and we have:

$$\frac{\overline{f}}{\overline{g}} = \frac{\sum_{i=0}^{n} \overline{a_i} X^i}{\sum_{j=0}^{n} \overline{b_j} X^j} = \overline{0}.$$

Therefore, $\frac{\overline{f}}{\overline{g}} \in Kr(*') \subseteq (S/I)(X)$. Since $c_{S/I}(\overline{g}) = S/I$ (if $c_S(g) = S$, then $c_{S/I}(\overline{g}) = S/I$), by Lemma 2.2, we have $\overline{f} = \overline{0}$. Since S/I is a domain, it follows that $\overline{a_i} = \overline{0}$, so $a_i \in I$. Since $c_S(g) = S$, we conclude that:

$$\frac{f}{g} = a_0 \frac{1}{g} + a_1 \frac{X}{g} + \cdots + a_n \frac{X^n}{g} \in IS(X).$$

Thus, $\mathrm{Ker}(\overline{\alpha}) \subseteq IS(X)$. From Proposition 3.1, we have $\mathrm{Ker}(\overline{\alpha}) \subseteq IKr(*)$.

For the reverse inclusion $IKr(*) \subseteq \mathrm{Ker}(\overline{\alpha})$, let $\frac{f}{g} = \sum_{i=1}^{r} u_i \frac{f_i}{g_i} \in IKr(*)$, where $u_i \in I$ and $\frac{f_i}{g_i} \in Kr(*) \subseteq S(X)$. We have $u_i f_i \in I[X]$.

If $x \in I$, we have $x = \frac{x}{1}$, and since $x, 1 \in S[X]$ with $c_S(1) = S$, we have $c_R(x) = xR \subseteq IR = I \subseteq R = c_R(1)$, so:

$$c_R(x)^* \subseteq c_R(1)^*.$$

Thus, $I \subseteq Kr(*)$, and therefore $u_i \frac{f_i}{g_i} \in Kr(*)$. Moreover, $\overline{\alpha}\left(u_i \frac{f_i}{g_i}\right) = \frac{\overline{u_i f_i}}{\overline{g_i}}$. Since $u_i f_i \in I[X]$ and S/I is a domain, we have $\overline{u_i f_i} = \overline{0}$, so:

$$\frac{\overline{u_i f_i}}{\overline{g_i}} = \overline{0}.$$

Thus, $\overline{\alpha}\left(u_i \frac{f_i}{g_i}\right) = \overline{0}$, and:

$$\overline{\alpha}\left(\sum_{i=1}^{r} u_i \frac{f_i}{g_i}\right) = \overline{0}.$$

Therefore, $\overline{\alpha}\left(\frac{f}{g}\right) = \overline{0}$, and $\frac{f}{g} \in \mathrm{Ker}(\overline{\alpha})$. Hence, $IKr(*) \subseteq \mathrm{Ker}(\overline{\alpha})$. This proves that:

$$\mathrm{Ker}(\overline{\alpha}) = IKr(*).$$

Since $\overline{\alpha}$ is surjective, we conclude that:

$$Kr(*)/\mathrm{Ker}(\overline{\alpha}) \cong Kr(*').$$

It follows that:

$$Kr(*)/IKr(*) \cong Kr(*').$$

□

Theorem 3.4. *Consider a diagram of type □. Let $*'$ be an e.a.b. star operation of $R/I \subseteq S/I$, and let $*$ be the e.a.b. star operation of $R \subseteq S$ such that $A^* = \Psi^{-1}(\Psi(A)^{*'})$ for each $A \in \Im(R,S)$. If $I \subseteq Jac(R)$ or $I \subseteq Jac(S)$, then the diagram*

$$\begin{array}{ccc} Kr(*) & \xrightarrow{\quad i_1 \quad} & S(X) \\ \Big\downarrow{\scriptstyle \overline{\alpha}} & & \Big\downarrow{\scriptstyle \overline{\Psi}} \\ Kr(*') & \xrightarrow{\quad i_2 \quad} & (S/I)(X) \end{array}$$

is a pullback diagram of type □, and in particular,

$$Kr(*) \cong Kr(*') \times_{(S/I)(X)} S(X).$$

Proof. Let $\frac{f}{g} \in Kr(*) \subseteq S(X)$. We have $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in (S/I)(X)$ and $\overline{\alpha}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in Kr(*') \subseteq (S/I)(X)$. Thus, $\overline{\Psi}(\frac{f}{g}) = \overline{\alpha}(\frac{f}{g})$.

Since $\frac{f}{g} \in Kr(*)$, we have $i_1(\frac{f}{g}) = \frac{f}{g}$, so $\overline{\Psi}(i_1(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$. Since $\frac{f}{g} \in Kr(*)$, we also have $\overline{\alpha}(\frac{f}{g}) \in Kr(*')$. Therefore, $i_2(\overline{\alpha}(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$, and we conclude that:

$$\overline{\Psi} \circ i_1(\frac{f}{g}) = i_2 \circ \overline{\alpha}(\frac{f}{g}).$$

This shows that the diagram is commutative.

Now, we show that $S(X)/IS(X) \cong (S/I)(X)$. Since $\overline{\Psi}$ is surjective, we have $S(X)/\mathrm{Ker}(\overline{\Psi}) \cong (S/I)(X)$.

Let $\frac{f}{g} = \sum_{i=1}^{l} u_i \frac{f_i}{g_i} \in IS(X)$, where $u_i \in I$ and $\frac{f_i}{g_i} \in S(X)$. We have:

$$\overline{\Psi}\left(\frac{f}{g}\right) = \sum_{i=1}^{l} \overline{\Psi}(u_i \frac{f_i}{g_i}),$$

where $f_i = \sum_{k=0}^{l} a_k^i X^k \in S[X]$, and $u_i f_i = \sum_{k=0}^{n} u_i a_k^i X^k \in I[X]$. Since $u_i a_k^i \in I$, we get $\overline{u_i a_k^i} = \overline{0}$, so:

$$\sum_{k=0}^{l} \overline{u_i a_k^i} X^k = \overline{0}.$$

By Lemma 2.2, we then have:

$$\overline{\Psi}\left(\frac{u_i f_i}{g_i}\right) = \frac{\sum_{k=0}^{l} \overline{u_i a_k^i} X^k}{\overline{g_i}} = \overline{0}.$$

Thus, $\overline{\Psi}\left(\sum_{i=1}^{l} \frac{u_i f_i}{g_i}\right) = \overline{0}$, and consequently:

$$\overline{\Psi}\left(\frac{f}{g}\right) = \overline{0}.$$

This shows that $\frac{f}{g} \in \mathrm{Ker}(\overline{\Psi})$, and therefore:

$$IS(X) \subseteq \mathrm{Ker}(\overline{\Psi}).$$

Next, we prove the reverse inclusion $\mathrm{Ker}(\overline{\Psi}) \subseteq IS(X)$. Let $\frac{f}{g} \in \mathrm{Ker}(\overline{\Psi}) \subseteq S(X)$, with $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$. Since $\overline{\Psi}\left(\frac{f}{g}\right) = \overline{0}$, we have:

$$\frac{\overline{f}}{\overline{g}} = \frac{\sum_{i=0}^{n} \overline{a_i} X^i}{\sum_{j=0}^{m} \overline{b_j} X^j} = \overline{0}.$$

By Lemma 2.2, we obtain:

$$\sum_{i=0}^{n} \overline{a_i} X^i = \overline{0},$$

which implies that $\sum_{i=0}^{n} \overline{a_i} X^i = \overline{0} \in (S/I)[X]$. Since S/I is a domain, we conclude that $\overline{a_i} = \overline{0}$, so $a_i \in I$. Since $c_S(g) = S$, we then have:

$$\frac{f}{g} = a_0 \frac{1}{g} + a_1 \frac{X}{g} + \cdots + a_n \frac{X^n}{g} \in IS(X).$$

Thus, $\mathrm{Ker}(\overline{\Psi}) \subseteq IS(X)$, and therefore:

$$\mathrm{Ker}(\overline{\Psi}) = IS(X).$$

Consequently, we obtain:

$$S(X)/IS(X) \cong (S/I)(X).$$

Since $*$ is a star operation e.a.b. on $R \subseteq S$, it follows from Proposition 3.1 that $IS(X)$ is a common ideal of $Kr(*)$ and $S(X)$. Furthermore, by Lemma 2.3, $IS(X)$ is a maximal ideal of $S(X)$. By Proposition 3.3, we have $Kr(*') \cong Kr(*)/IKr(*)$, and thus:

$$Kr(*') \cong Kr(*)/IS(X).$$

Therefore, we have a pullback diagram of type $\square$, and in particular:

$$Kr(*) \cong Kr(*') \times_{(S/I)(X)} S(X).$$

□

Finally, we present an example to illustrate our results.

Example 3.5. ($D + M$ constructions)

Let T be a valuation domain of the form $K+M$, where M is the maximal ideal of T and K is a subring of T canonically isomorphic to the field T/M, and D be a subring of K with field of quotients $F_r(D)$. Let $R := D+M$, and suppose that D is a Prüfer domain.

(1) Let $\{D_\alpha\}_\alpha$ be a family of all valuation overrings of D such that $D = \cap_\alpha D_\alpha$, and define the map $*' : \Im(D,K) \to \Im(D,K)$, $A \to \cap_\alpha (D_\alpha A)$. By applying [8, Proposition 2.2], we know that the map $*'$ is a star operation on $D \subseteq K$. Since D is a Prüfer domain, by [7, Example 5.17], we have $D \subseteq K$ is a Prüfer extension. So by [6, Proposition 5.7], the star operation $*'$ on $D \subseteq K$ is e.a.b. and

$$Kr(*') = \left\{ \frac{f}{g} \in K(X) : f,g \in K[X],\, c_K(g) = K,\, c_D(f)^{*'} \subseteq c_D(g)^{*'} \right\}$$

$$= \left\{ \frac{f}{g} \in K(X) : f,g \in K[X],\, c_D(f)^{*'} \subseteq c_D(g)^{*'} \right\}.$$

(2) We want to define a map $* : \Im(R,T) \to \Im(R,T)$ by

$$A^* = \Psi^{-1}\left((\Psi(A))^{*'} \right).$$

Since $M = \mathrm{Jac}(T)$, it follows from Proposition 3.3 that $*$ is a star operation e.a.b. on $R \subseteq T$. Therefore,

$$Kr(*) = \left\{ \frac{f}{g} \in T(X) : f,g \in T[X],\, c_T(g) = T,\, c_R(f)^* \subseteq c_R(g)^* \right\}.$$

(3) Since $M = \mathrm{Jac}(T)$, by Theorem 3.4, we have the diagram

$$\begin{array}{ccc} Kr(*) & \longrightarrow & T(X) \\ \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \overline{\Psi}} \\ Kr(*') & \longrightarrow & (T/M)(X) = K(X) \end{array}$$

is a pullback diagram of type □ and in particular

$$Kr(*) \cong Kr(*') \times_{K(X)} T(X).$$

(4) Let $\mathbb{Q}$ be the set of rational numbers and X an indeterminate over $\mathbb{Q}$, and set $T = \mathbb{Q}[X]_{(X)} = \mathbb{Q} + X\mathbb{Q}[X]_{(X)}$ as a valuation domain of $\mathbb{Q}(X)$. Set $M = X\mathbb{Q}[X]_{(X)}$ as the maximal ideal of T, and $D = \mathbb{Z}_{(2\mathbb{Z})} \cap \mathbb{Z}_{(3\mathbb{Z})}$ as

the subring of $\mathbb{Q}$. We then have $F_r(D) = \mathbb{Q}$. Set $R = D + M$. With the hypotheses of Theorem 3.4 and D being a Prüfer domain, we have

$$Kr(*) \cong Kr(*') \times_{\mathbb{Q}(Y)} \mathbb{Q}[X]_{(X)}(Y).$$

4. Conclusion

This work studies Kronecker function rings for ring extensions arising from pullback diagrams of type $\square$, extending previous work on Prüfer extensions to the context of e.a.b. star operations. It provides a clear definition of key concepts, such as pullback diagrams and Kronecker function rings, and proves an important isomorphism result: if $I \subseteq \mathrm{Jac}(R)$ or $I \subseteq \mathrm{Jac}(S)$, then an isomorphism

$$Kr(*) \cong Kr(*') \times_{(S/I)(X)} S(X).$$

holds, describing the Kronecker function ring in the pullback setting.

The paper also places these results in the context of prior literature, demonstrating how these findings generalize previous work. Finally, it concludes with a concrete example illustrating the practical application of the theoretical results in commutative algebra.

Acknowledgement. The authors would like to express their sincere gratitude to the editor and the anonymous referee for their careful reading and thoughtful feedback on an earlier version of this manuscript. Their valuable suggestions and comments have been instrumental in improving the clarity and overall quality of the paper.

References

1. Bastida, E., & Gilmer, R. (1973). Overrings and divisorial ideals of rings of the form $D+M$. Michigan Math. J. 20(1): 79-95. https://doi.org/10.1307/mmj/1029001014
2. Chibloun, R., & Eddamane, E. Pairs of rings with the same ideal and Kronecker function rings. Gulf Journal of Mathematics, 18(2), 16-28. https://doi.org/10.56947/gjom.v12i2.677
3. Dobbs, D. E., & Fontana, M. (1986). Kronecker function rings and abstract Riemann surfaces. J. Algebra 99, no. 1, 263–274 https://doi.org/10.1016/0021-8693(86)90067-0
4. Gabelli, S., & Houston, E. (2000). Ideal theory in pullbacks, Non-Noetherian commutative ring theory, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, pp. 199–227.
5. Gilmer, R. (1992). Multiplicative ideal theory, Corrected reprint of the 1972 edition. Queen's Papers in Pure and Applied Mathematics, 90. Queen's University, Kingston, Ontario.
6. Knebusch, M., & Kaiser, T. (2014). Manis Valuations and Prüfer Extensions II, Lectures Notes in Mathematics, 2103, Springer, Cham. https://doi.org/10.1007/978-3-319-03212-2
7. Knebusch, M., & Zhang, D. (2002). Manis valuations and Prüfer extensions. I. A new chapter in commutative algebra. Lecture Notes in Mathematics, 1791. Springer-Verlag, Berlin.
8. Paudel, L., & Tchamna, S. (2018). Kronecker function rings of ring extensions. J. Algebra Appl. 17 (2018), no. 2, 1850021, 15 pp. https://doi.org/10.1142/S0219498818500214
9. Paudel, L., & Tchamna, S. (2019). Pullback diagrams and Kronecker function rings. Rocky Mountain J. Math. 49, no. 7, 2267-2279. https://doi.org/10.1216/RMJ-2019-49-7-2267

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