Gulf Journal of Mathematics
Vol 8, Issue 1 (2020) 1-5

EXISTENCE OF EMBEDDINGS OF VARIETIES IN PROJECTIVE SPACES WHOSE POINTS ARE SPANNED BY LOW DEGREE SMOOTHABLE ZERO-DIMENSIONAL SUBSCHEMES

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ABSTRACT. Let X be an integral projective variety. Set $n := \dim X$. Let $e(X) \geq 2n+1$ be the embedding dimension of X (we may take $e(X) = 2n+1$ if X is smooth). Fix integers δ and $r \geq e(X)$. We prove the existence of many embeddings $j : X \hookrightarrow \mathbb{P}^r$ such that $\deg(X) \geq \delta$ and every point of $\mathbb{P}^r$ is spanned by a low degree smoothable zero-dimensional subscheme of X.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For each $q \in \mathbb{P}^r$ the X-rank $r_X(q)$ of q is the minimal cardinality of a subset $S \subset X$ such that $q \in \langle S\rangle$, where $\langle \quad \rangle$ denote the linear span. For any integer $k > 0$ the k-secant variety $\sigma_k(X)$ of X is the closure in $\mathbb{P}^r$ of the union of all points q with $r_X(q) \leq k$. A dimensional count shows that $\dim \sigma_k(X) \leq \min\{r, (n+1)k-1\}$. In many cases equality holds. The variety $X \subset \mathbb{P}^r$ is said to be *not defective* or *nondefective* if $\dim \sigma_k(X) = \min\{r, (n+1)k-1\}$ for all $k > 0$. When X is not-defective there is a non-empty open subset $U \subset X$ such that $r_X(q) = \lceil (r+1)/(n+1)\rceil$ for all $q \in U$. A zero-dimensional scheme $Z \subset X$ is said to be *smoothable inside* X is it is a flat limit of a family of subsets of X with cardinality $\deg(Z)$ ([5, 6, 7, **?**]). We recall that if X is smooth the notion of smoothability inside X does not depend on X, but just Z as an abstract scheme ([5, Proposition 2.1]). Let $r_X(\text{gen})$ be the first positive integer a such that $\sigma_a(X) = \mathbb{P}^r$. If X is not defective we have $r_X(\text{gen}) = \lceil (r+1)/(n+1)\rceil$.

Definition 1.1. We say that the n-dimensional embedded variety $X \subset \mathbb{P}^r$ is *boring* if it is not defective and for each $q \in \mathbb{P}^r$ there is a zero-dimensional scheme $Z \subset X$ smoothable in X such that $\deg(Z) \leq \lceil (r+1)/(n+1)\rceil$ and $q \in \langle Z\rangle$. Fix a positive integer k. We say that X is k-boring if for every $q \in \sigma_k(X)$ there is a zero-dimensional scheme $Z \subset X$ smoothable in X such that $\deg(Z) \leq k$. We say that X is *strongly boring* if it is boring and k-boring for all $k \leq r_X(\text{gen})$.

Date: Received: Dec 30, 2019; Accepted: Mar 13, 2020.
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2010 *Mathematics Subject Classification.* Primary 14N05;Secondary 14M99.
Key words and phrases. Embeddings of projective varieties; zero-dimensional scheme.

We think that “most variety ” or “a random variety ” is boring (if nondefective), but we do not dare to guess strong boredom. In a few cases the stronger condition would be satisfied (Remark 2.5), but it is too strong in general. To state our main result we need to introduce the following integer $e(X)$ for any integral projective variety X. Let X be an integral projective variety. Set $n := \dim X$. For each $i > n$ let $X(\mathrm{Sing}, t)$ denote the set of all $p \in \mathrm{Sing}(X)$ such that the Zariski tangent space of X at p has dimension t, i.e. the set of all $p \in X$ with embedding dimension t. The set $X(\mathrm{Sing}, t)$ is a locally closed subset of X contained in $\mathrm{Sing}(X)$. Let $e(X)$ denote the maximum between $2n + 1$ and all integers $t + \dim X(\mathrm{Sing}, t)$, with the convention $t + dimX(\mathrm{Sing}, t) = 0$ if $X(\mathrm{Sing}, t) = \emptyset$. It is well-known that X has many embeddings in $\mathbb{P}^r$ if $r \geq e(X)$ (Lemma 2.3). There are many non k-boring embeddings, even non-defective ones ([6, Theorem 1.3]).

Theorem 1.2. *Let X be an integral variety. Fix integers δ and $r \geq e(X)$. Then there is an embedding $j : X \to \mathbb{P}^r$ such that $\deg(j(X)) \geq \delta$, $j(X)$ is non-degenerate and strongly boring.*

Remark 1.3. Fix X and $r \geq e(X)$ as in Theorem 1.2. The integer $\deg(j(X))$ is the degree of a very ample line bundle on X. If $n = 1$ there is an integer $\delta(X, r)$ (we may take $\delta(X, r)) = 2r + 4 + 2p_a(X))$ such that for all integers $d \geq \delta(X, r)$ there is a boring embedding of X. Note that for any boring embedding $\deg(j(X))$ is the degree of a very ample line bundle on X. Thus the example of $\mathbb{P}^n$, $n \geq 2$, (and many other examples) shows that when $n \geq 2$ there is no integer $\delta(X, r)$ such that for all $d \geq \delta(X, r)$ there is an embedding $X \subset \mathbb{P}^r$ with $\deg(j(X)) = d$ and $j(X)$ strongly boring (or just boring).

For $n = 1, 2$ there is no integer c_n such that for n-dimensional all smooth projective varieties X and every linearly normal non-defective embedding $j : X \to \mathbb{P}^r$ we have $r_{j(X)}(q) \leq \lceil (r + 1)/(n + 1) \rceil$ for all $q \in \mathbb{P}^r$ (Remarks 2.1 and 2.2). In our opinion (after the fundamental [4]) it would be interesting to have examples for $n \geq 3$.

2. The proofs

Remark 2.1. Let Y be a smooth curve of genus g. Fix an integer $d \geq 2g + 1$ and any degree d line bundle L on X. By Riemann-Roch L is very ample and $|L|$ induces a non-degenerate embedding $j : Y \to \mathbb{P}^{d-g}$. Set $a := \lceil (d+1-g)/2 \rceil$. Since no non-degenerate curve is defective ([1, Remark 1.6]), we have $\sigma_a(X) = \mathbb{P}^{d-g}$ and $\sigma_{a-1}(j(X)) \neq \mathbb{P}^{d-g}$. Fix $q \in \mathbb{P}^r \setminus X$ which is in the tangential variety of $j(X)$, i.e. take $q \in \mathbb{P}^r \setminus X$ such that there is a connected degree 2 scheme $v \subset X$ such that $q \in \langle \nu(v) \rangle$. By Riemann-Roch every zero-dimensional subscheme $Z \subset j(X)$ with $\deg(Z) \leq d - 2g + 1$ is linearly independent. Since $\deg(v) = 2$ and v is connected, we get $r_{j(Y)}(q) \geq d - 2g - 1$. Note that $d - 2g - 1 - a$ increases linearly with d.

Remark 2.2. Fix an integer $d \geq 5$. Let $X_d \subset \mathbb{P}^r$, $r = \binom{d+2}{2} - 1$, denote the image of $\mathbb{P}^2$ by the order d Veronese embedding. Set $a := \lceil (d + 2)(d + 1)/6 \rceil$. By a very particular case of the Alexander-Hirschowitz theorem ([11, Ch. 15]) X_d is nondefective and in particular $\sigma_a(X_d) = \mathbb{P}^r$ and $\sigma_{a-1}(X_d) \neq \mathbb{P}^r$. By [10] and [8, Theorem 3.1] there is $q \in \mathbb{P}^r$ such that $r_{X_d}(q) = \lfloor (d^2+2d+5)/4 \rfloor$ (this is the highest X_d-rank by [9] if d is even). Note that $r_X(q) - a$ is a quadratic function of d.

Lemma 2.3. *Let X be an integral projective variety. Fix integers $N > r \ge e(X)$ and assume the existence of a non-degenerate embedding $j : X \hookrightarrow \mathbb{P}^N$. Then for a general $(N-r-1)$-dimensional linear space $V \subset \mathbb{P}^N$ we have $V \cap j(X) = \emptyset$ and $\ell_{V|j(X)}$ is an embedding, where $\ell_V : \mathbb{P}^N \setminus V \to \mathbb{P}^r$ is the linear projection from V.*

Proof. Since $r \ge e(X) \ge 2r+1$, a classical dimensional count shows that $\ell_{V|X}$ is injective and it is an embedding at each point of X_{reg}. By Nakayama's lemma to prove that $\ell_{V|j(X)}$ is an embedding it is sufficient to prove that a general V meets no Zariski tangent space $T_p(j(X))$, $p \in j(\mathrm{Sing}(X))$. This is a dimensional count (considering separately all points with Zariski tangent space with a fixed dimension t). It works by the definition of the integer $e(X)$. □

Definition 2.4. Let $X \subset \mathbb{P}^r$ be an integral variety. Let $\rho(X)$ (resp. $\rho(X)_1$) be the maximal positive integer such that each 0-dimensional scheme $Z \subset X$ (resp. which is smoothable in X) with $\deg(Z) \le \rho(X)$ (resp. $\deg(Z) \le \rho(X)_1$) is linearly independent, i.e. $\dim\langle Z\rangle = \deg(Z) - 1$.

We have $\rho(X) \ge \rho(X)_1$.

Remark 2.5. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. Fix a positive integer $k \le \rho(X)_1$. Since each point of $\sigma_k(X)$ is contained in linear space, which is limit of linear spaces spanned by k points of x and the Hilbert scheme of all degree k zero-dimensional subschemes of X is proper, there is a zero-dimensional scheme $Z \subset X$ smoothable in X, with $\deg(Z) = k$ such that $q \in \langle Z\rangle$ ([5],[6, 3.4]).

Remark 2.6. We list a few examples of boring varieties $X \subset \mathbb{P}^r$. Set $n := \dim X$.

(a) By Remark 2.5 X is strongly boring if X is non-defective and $\rho(X)_1 \ge \lceil (r+1)/(n+1)\rceil$.

(b) Every curve X is non-defective ([1, Remark 1.6]) and hence we may apply part (a) to all curves with $\rho(X)_1 \ge \lceil (r+1)/2\rceil$.

(c) Since $\mathcal{O}_X(1)$ is very ample, we have $\rho(X)_1 \ge 2$. Thus each X is 2-boring. Hence each X with $\sigma_2(X) = \mathbb{P}^r$ (and hence $r \le 2n+1$) is strongly boring.

(d) Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate linearly normal curve with $h^1(\mathcal{O}_X(1)) = 0$. Set $d := \deg(X)$ and $g := p_a(X)$. By Riemann-Roch we have $r = d - g$. By Riemann-Roch for rank 1 torsion free sheaves we have $\rho(X) \ge d - 2g + 1$. Recall that X is not defective.

The following example shows that there are boring varieties $X \subset \mathbb{P}^r$ in which to be boring one need to use zero-dimensional schemes smoothable inside X, but (sometimes) intersecting $\mathrm{Sing}(X)$.

Example 2.7. Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve with $p_a(X) = 1$, $\deg(X) = 4$ and a singular point $o \in X$ which is an ordinary cusp. By case (c) of Remark 2.6 X is boring. The curve X is a linearly normal embedding $j : T \to \mathbb{P}^3$ of a cuspidal plane cubic $T \subset \mathbb{P}^2$. Set $\{a\} := \mathrm{Sing}(T)$ and let $L \subset \mathbb{P}^2$ be the cuspidal tangent line of T, i.e. the only line of $\mathbb{P}^2$ such that the subscheme $L \cap T$ of L is a counted with multiplicity 3. Let $v \subset L$ be the degree 2 connected scheme with $\{a\}$ as its reduction. Set $R := \langle j(v) \rangle \subset \mathbb{P}^3$. R is a line meeting X only at o and with $j(v) = X \cap R$ (scheme-theoretic intersection). Fix $q \in R \setminus \{o\}$. We claim that there is no degree 2 zero-dimensional scheme $Z \subset X \setminus \{o\}$ such that $q \in \langle Z \rangle$. Assume that Z exists and take any $b \in Z_{\mathrm{red}}$. Since X is smooth at b, the linear projection $\ell_b : \mathbb{P}^3 \setminus \{b\} \to X$, extends to an embedding $m : X \to \mathbb{P}^2$ with $m(X)$ a cuspidal plane cubic. Moreover, the point $\ell(q)$ is contained in a line $D \subset \mathbb{P}^2$ containing $\ell_b(j(v))$, which is the cuspidal point of $m(X)$. Since $\ell_q(\langle j(v) \rangle$, the line $\ell_q(\langle j(v) \rangle$ meets $\mu(X)$ only at $\mu(o)$. Thus there is no degree 2 zero-dimensional scheme $Z \subset X \setminus \{o\}$ such that $q \in \langle Z \rangle$.

Proof of Theorem 1.2: Fix a very ample line bundle L on X. Set $y := \lceil (r+1)/(n+1) \rceil$. Since the set $\mathcal{B} := \cup_{1 \le i \le y} \mathrm{Hilb}^i(X)$ of all degree zero-dimensional subscheme of X with degree at most $2y$ is a projective scheme, it is easy to check the existence of an integer $b \ge 2y - 1$ such $h^0(L^{\otimes x}(-Z)) = h^0(L^{\otimes x}) - \deg(Z)$ for all $x \ge b$ and all $Z \in \mathcal{B}$. Fix an integer $x \ge b$ such that the intersection number $L \cdots L$ is at least δ/x^n. Let $u : X \to \mathbb{P}^N$, $N = h^0(L^{\otimes b}) - 1$, denote the embedding of X given by the complete linear system $|L^{\otimes x}|$. Set $Y := u(X)$. By construction $\rho(Y) \ge 2y$ and $\deg(Y) \ge \delta$. Since $\rho(X) \ge 2y - 1$, every finite set $S \subset Y$ with cardinality at most $2y - 1$ is linearly independent, for each $q \in \mathbb{P}^N$ with $r_Y(q) \le y$ there is a unique $S \subset X$ such that $\#(S) = r_Y(q)$ and $q \in \langle S \rangle$. Thus $\dim \sigma_k(Y) = (n+1)k - 1$ for all positive integers $k \le y$. Since $\rho(Y)_1 \ge a$ for each positive integer $k \le a$ and each $q \in \sigma_k(X)$ there is a degree k zero-dimensional scheme $Z \subset Y$ smoothable in Y such that $q \in \langle Z \rangle$. Let $V \subset \mathbb{P}^N$ be a general $(N-r-1)$-dimensional linear subspace. Since $\dim \sigma_a(Y) = (n+1)a - 1$ and V is general, we have $V \cap \sigma_k(Y) = \emptyset$ for all $k < a$ and $\dim V \cap \sigma_a(Y) = a(n+1) - r - 2$, with the convention $\dim \emptyset = -1$. By Lemma 2.3, the inequality $r \ge e(X)$ and the generality of V the linear projection $\ell_{|V} : \mathbb{P}^N \setminus V \to \mathbb{P}^r$ induces a non-degenerate embedding $\ell = \ell_{V|Y} : Y \to \mathbb{P}^r$ and hence a non-degenerate embedding $j = \ell \circ v : X \to \mathbb{P}^r$. Using ℓ we get that $\ell(Y)$ is k-boring for all $k < a$. If $r+1 \equiv 0 \pmod{n+1}$, then $V \cap \sigma_a(Y) = \emptyset$ and we immediately get $\ell_V(\sigma_a(Y)) = \mathbb{P}^r$. Thus if $r+1 \equiv 0 \mod n+1$, then $\ell(Y)$ is a-boring, concluding the proof in this case. Now assume $w := a(n+1) - r - 1 > 0$. To conclude the proof it is sufficient to prove that $\ell_V(\sigma_a(Y) \setminus \sigma_a(Y)) = \mathbb{P}^r$ for a general V. Take a general linear space $W \subset V$ with codimension w and let $\ell_W : \mathbb{P}^N \to \mathbb{P}^{r+w}$ denote the linear projection from W. As above we see that $\dim \sigma_k(\ell_W(Y)) = k(n+1) - 1$ (and so $\sigma_a(\ell_W(Y)) = \mathbb{P}^{r+w}$) and that $\ell_W(X)$ is k-boring for all $k \le a$. Set $M := \ell_W(V \setminus W)$. Call $\mu : \mathbb{P}^{r+w} \setminus M \to \mathbb{P}^r$ the linear projection from M. Fix $q \in \mathbb{P}^r$. Since μ is surjective, there is $q' \in \mathbb{P}^{r+w}$ such that $\mu(q') = q$. Take $Z \subset Y$ such that $\deg(Z) = a$, Z is smoothable in Y and $q' \in \langle \ell_W(Z) \rangle$. Since $\ell_V = \mu \circ \ell_W$, we have $q \in \langle \ell(Z) \rangle$, concluding the proof. □

Acknowledgement. The author was partially supported by MIUR and GN-SAGA of INdAM (Italy).

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