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NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING THE WRIGHT FUNCTION ASSOCIATED WITH THE JUNG-KIM-SRIVASTAV OPERATOR

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Abstract. In this paper, we introduce a novel operator formed by the convolution of the Jung-Kim-Srivastav operator and the Wright function. This operator is used to construct classes of analytic functions within the open unit disk. The primary objective is to establish precise estimates for the basic coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series for functions belonging to this class. Furthermore, we present new results related to this study.

1. Introduction

The Wright function, introduced by E. M. Wright in the early 20th century [34], plays a crucial role in asymptotic expansions and differential equations. The Mittag-Leffler function is often used to describe solutions to fractional differential equations due to its ability to interpolate between exponential functions and power-law behaviors. The Wright function extends this by introducing additional parameters that allow for greater flexibility in capturing complex phenomena, such as anomalous diffusion and memory effects in physical systems. For instance, in certain parameter regimes, the Wright function reduces to the Mittag-Leffler function, thereby encompassing it as a special case. This makes the Wright function an indispensable tool for modeling and solving a wider variety of problems in fractional calculus and applied mathematics, particularly in areas involving non-local and time-fractional dynamics.

There are two main forms of the Wright function: the entire Wright function and the Wright generalized Bessel function, both defined for complex parameters. These functions are valuable for solving fractional differential equations, modeling memory-dependent systems, and describing processes that diverge from classical Brownian motion. Additionally, the Wright function is important in the analysis of Levy stable distributions and scaling laws in statistical mechanics [20].

Let $\mathcal{A}$ be the class of all functions $\mathcal{F}$ which are analytic in the open unit disk

$$\nabla = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

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and normalized by the conditions:

$$\mathcal{F}(0)=0 \quad \text{and} \quad \mathcal{F}'(0)=1.$$

Thus, the function $\mathcal{F}\in\mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$\mathcal{F}(z)=z+\sum_{n=2}^{\infty}a_nz^n, \quad (z\in\nabla). \tag{1.1}$$

For two functions $\mathcal{F}, g\in\mathcal{A}$, the function $\mathcal{F}$ is said to be subordinate to the function g in ∇, denoted by

$$\mathcal{F}(z)\prec g(z) \quad (z\in\nabla), \tag{1.2}$$

if there exists a function $w\in\mathcal{B}_0:=\{w: w\in\mathcal{A}, w(0)=0 \text{ and } |w(z)|<1\,(z\in\nabla)\}$ such that

$$\mathcal{F}(z)=g(w(z)) \quad (z\in\nabla). \tag{1.3}$$

In the case when the function g is univalent in ∇, The following equivalence is established:

$$\mathcal{F}(z)\prec g(z) \quad (z\in\nabla) \quad\Leftrightarrow\quad \mathcal{F}(0)=g(0) \text{ and } \mathcal{F}(\nabla)\subset g(\nabla). \tag{1.4}$$

Next, consider a function $\mathcal{F}\in\mathcal{A}$ defined by equation (1.1), along with another function $g\in\mathcal{A}$ defined by

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n \quad (z\in\nabla), \tag{1.5}$$

the convolution of the functions $\mathcal{F}$ and g is defined by

$$(\mathcal{F}*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*\mathcal{F})(z). \tag{1.6}$$

It is well known that every univalent function $\mathcal{F}$ has an inverse $\mathcal{F}^{-1}$, defined by

$$\mathcal{F}^{-1}(\mathcal{F}(z))=z=\mathcal{F}(\mathcal{F}^{-1}(z)) \quad (z\in\nabla),$$

and

$$\mathcal{F}(\mathcal{F}^{-1}(w))=w \quad \left(|w|<r_0(\mathcal{F}); \quad r_0(\mathcal{F})\geq\frac{1}{4}\right),$$

where

$$\mathcal{F}^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_3^2-5a_2a_3+a_4)w^4+\cdots \tag{1.7}$$

A function $\mathcal{F}\in\mathcal{A}$ is said to be bi-univalent in ∇ if both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are univalent in ∇. It is denoted the class of all such functions by Σ. In recent years, the groundbreaking research by Srivastava *et al.* [31] has effectively revitalized the exploration of different subclasses within the analytic and bi-univalent function class Σ. Many authors have introduced and studied various subclasses of the analytic and bi-univalent function class Σ in numerous sequels to the pioneering work of Srivastava *et al.* [32]. For examples of this research, see references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 28, 30, 31, 32, 33, 35].

However, only non-sharp estimates regarding the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were reported in these recent studies see references [13, 18, 27].

In 2022, [19] introduced a special function which is named as Wright function and defined in the following way:

$$\Upsilon_{\lambda,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \vartheta)}, \tag{1.8}$$

where $\lambda > -1$, $\vartheta \in \mathbb{C}$, and $\Gamma(\cdot)$ stands for the usual Gamma function. The series given by (1.1) is absolutely convergent for all $z \in \mathbb{C}$, while for $\lambda = -1$ it is absolutely convergent in ∇. That was also proven that it is an entire function for $\lambda > -1$. For more basic properties on Wright functions, one may refer to [14, 19, 21, 23, 25, 29].

To see [22] it is easy to see that the series (1.1) is not in normalized form, thus it is normalized

$$\Upsilon_{\lambda,\vartheta}(z) = z\Gamma(\vartheta)W_{\lambda,\vartheta}(z).$$

Thus, It has been found that

$$\Upsilon_{\lambda,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\vartheta) z^{n+1}}{n!\Gamma(\lambda n + \vartheta)}, \tag{1.9}$$

where $\lambda > -1$, $\vartheta > 0$, and $z \in \nabla$.

Now, An introduction is made to the Wright distribution in the following way. First, The series has been defined as follows

$$\Upsilon_{\lambda,\vartheta}(m) = \sum_{n=0}^{\infty} \frac{\Gamma(\vartheta) m^{n+1}}{n!\Gamma(\lambda n + \vartheta)}, \tag{1.10}$$

which is convergent for all $\lambda, \vartheta, m > 0$.

The probability mass function of the Wright distribution is given by

$$p(n) = \frac{\Gamma(\vartheta) m^{n+1}}{n!\Gamma(\lambda n + \vartheta)\Upsilon_{\lambda,\vartheta}(m)}, \quad m, \vartheta, \lambda > 0,\ n = 0, 1, 2, 3, \ldots, \tag{1.11}$$

It is worthy to note that for $\lambda = 0$, it reduces to the Poisson distribution.

In 1993, Jung, Kim, and Srivastav introduced the Jung-Kim-Srivastav operator in the field of geometric function theory. This operator creates new classes of analytic functions and examines properties such as univalence, starlikeness, and convexity (see [1, 12, 22, 24, 26]).

The operator is defined as:

$$\mathcal{K}_{n,\gamma}\mathcal{F}(z) = z + \sum_{n=1}^{\infty} \left(\frac{2}{n+2}\right)^{\gamma} a_{n+1} z^{n+1}, \tag{1.12}$$

where $\gamma > 1$ are constant, and $\mathcal{F}(z)$ is analytic in the unit disk ∇.

Now introduce a new operator that is defined as the convolution (or Hadamard product) between the Jung-Kim-Srivastav operator and the Wright function. This operator, denoted by $\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)$, is constructed to explore further properties of analytic functions. The new operator is defined as follows:

$$\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z) = \mathcal{K}_{n,\gamma}\mathcal{F}(z) * \Upsilon_{\lambda,\mu}(z) = z + \sum_{n=1}^{\infty} \mathfrak{L}_{\lambda,\vartheta,\gamma}(n+1) a_{n+1} z^{n+1}, \tag{1.13}$$

where the coefficients $\mathfrak{L}_{\lambda,\vartheta,\gamma}(n+1)$ are given by:

$$\mathfrak{L}_{\lambda,\vartheta,\gamma}(n+1) = \left(\frac{2}{n+2}\right)^{\gamma} \frac{\Gamma(\vartheta)}{n!\Gamma(\lambda n + \vartheta)}. \tag{1.14}$$

where $\mathfrak{L}_{\lambda,\vartheta,\gamma}(1) = 1$, $\mathfrak{L}_{\lambda,\vartheta,\gamma}(2) = \left(\frac{2}{3}\right)^{\gamma} \frac{\Gamma(\vartheta)}{\Gamma(\lambda+\vartheta)}$ and $\mathfrak{L}_{\lambda,\vartheta,\gamma}(3) = \left(\frac{1}{2}\right)^{\gamma} \frac{\Gamma(\vartheta)}{2\Gamma(\lambda 2+\vartheta)}$

Definition 1.1. A function $\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)$ given by (1.1) is said to be in the class $S^{\gamma}_{\lambda,\vartheta}(\alpha,\delta,\kappa,\rho)$ if the following conditions are satisfied:

$$\mathcal{F} \in \Sigma \quad \text{and} \quad \left|\arg\left((1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)}{z}\right)^{\alpha} + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}\mathcal{F}(z)\right)^{\delta}\right)\right| < \frac{\rho\pi}{2}, \tag{1.15}$$

and

$$\left|\arg\left((1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}g(w)}{w}\right)^{\alpha} + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}g(w)\right)^{\delta}\right)\right| < \frac{\rho\pi}{2}, \tag{1.16}$$

where $0 < \rho \leq 1, \alpha, \delta \geq 0, \kappa \geq 1, (\gamma > 1, \lambda > -1, \vartheta > 0), z, w \in \nabla$, and $g = \mathcal{F}^{-1}$ is given by (1.7) .

Definition 1.2. A function $\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)$ given by (1.1) is said to be in the class $M^{\vartheta,\gamma}_{\lambda}(\beta,\alpha,\delta,\kappa)$ if the following conditions are satisfied:

$$\mathcal{F} \in \Sigma \quad \text{and} \quad \Re\left((1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)}{z}\right)^{\alpha} + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}\mathcal{F}(z)\right)^{\delta}\right) > \beta, \tag{1.17}$$

and

$$\Re\left((1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}g(w)}{w}\right)^{\alpha} + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}g(w)\right)^{\delta}\right) > \beta, \tag{1.18}$$

where $0 \leq \beta < 1, \alpha, \delta \geq 0, \kappa \geq 1, (\gamma > 1, \lambda > -1, \vartheta > 0), w, z \in \nabla$, and $g = \mathcal{F}^{-1}$ is given by (1.7).

To prove our theorem, the following lemma will be used:

Lemma 1.3. *If $h \in H$, where H represents all analytic functions in ∇ and satisfy $\Re(h(z)) > 0$, where*

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots, \tag{1.19}$$

then $|h_i| \leq 2$ for each index i.

2. Coefficients bounds for classes $S^{\gamma}_{\kappa,\vartheta}(\alpha,\delta,\kappa,\rho)$ and $M^{\vartheta,\gamma}_{\lambda}(\beta,\alpha,\delta,\kappa)$

Theorem 2.1. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $S^{\gamma}_{\kappa,\vartheta}(\alpha,\delta,\kappa,\rho)$, where $0 < \rho \leq 1, \alpha, \delta \geq 0, \kappa \geq 1$. Then*

$$|a_2| \leq \frac{2\rho}{\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\sqrt{\rho\left[(1-\kappa)\left(\alpha(\alpha-1)+2\alpha\right)+2\kappa\delta\left(2\delta+1\right)\right]-(\rho-1)\left[\alpha(1-\kappa)+2\kappa\delta\right]^2}}. \tag{2.1}$$

and

$$|a_3| \leq \frac{2\rho}{[\alpha(1-\kappa)+3\kappa\delta]\left(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)\right)} + \frac{4\rho^2}{[\alpha(1-\kappa)+2\kappa\delta]^2\left(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)\right)}. \tag{2.2}$$

Proof. To prove the theorem, by referring to (1.15) and (1.16), It is derived that:

$$(1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)}{z}\right)^{\alpha} + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}\mathcal{F}(z)\right)^{\delta} = [\mathcal{P}(z)]^{\rho}, \tag{2.3}$$

and

$$(1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}g(w)}{w}\right)^{\alpha} + \kappa(\mathfrak{S}'_{\lambda,\vartheta,\gamma}g(w))^{\delta}) = [\mathcal{Q}(w)]^{\rho}, \tag{2.4}$$

where $\mathcal{P}(z)$ and $\mathcal{Q}(w)$ are in the class H and satisfy the conditions defined in (1.19). These functions can be written as:

$$\mathcal{P}(z) = 1 + \mathcal{P}_1 z + \mathcal{P}_2 z^2 + \mathcal{P}_3 z^3 + \cdots,$$

and

$$\mathcal{Q}(w) = 1 + \mathcal{Q}_1 w + \mathcal{Q}_2 w^2 + \mathcal{Q}_3 w^3 + \cdots.$$

By equating coefficients in equations (2.3) and (2.4), he following relations are obtained:

$$[\alpha(1-\kappa)+2\kappa\delta]\,\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)a_2 = \rho\mathcal{P}_1, \tag{2.5}$$

$$[\alpha(1-\kappa)+3\kappa\delta]\,(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))a_3 + \left[(1-\kappa)\frac{\alpha(\alpha-1)}{2} + 2\kappa\delta(\delta-1)\right]\left(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\right)^2 a_2^2 = \rho\mathcal{P}_2 + \frac{\rho(\rho-1)}{2}\mathcal{P}_1^2, \tag{2.6}$$

$$-[\alpha(1-\kappa)+2\kappa\delta]\,(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))a_2 = \rho\mathcal{Q}_1, \tag{2.7}$$

and

$$[\alpha(1-\kappa)+3\kappa\delta]\,(2\left(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\right)^2 a_2^2 - (\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))a_3) + \left[(1-\kappa)\frac{\alpha(\alpha-1)}{2} + 2\kappa\delta(\delta-1)\right]\left(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\right)^2 a_2^2$$

$$= \rho\mathcal{Q}_2 + \frac{\rho(\rho-1)}{2}\mathcal{Q}_1^2. \tag{2.8}$$

Utilizing equations (2.5) and (2.7), It is derived that:

$$\mathcal{P}_1 = -\mathcal{Q}_1, \tag{2.9}$$

and

$$2\left[\alpha(1-\kappa)+2\kappa\delta\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2 a_2^2 = \rho^2\left(\mathcal{P}_1^2+\mathcal{Q}_1^2\right). \tag{2.10}$$

Subsequently, combining equations (2.6), (2.8), and (2.10), it is acquired that:

$$[(1-\kappa)(\alpha(\alpha-1)+2\alpha)+2\kappa\delta(2(\delta-1)+3)]\,(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2 a_2^2 = \rho(\mathcal{P}_2+\mathcal{Q}_2) + \frac{\rho(\rho-1)}{2}\left(\mathcal{P}_1^2+\mathcal{Q}_1^2\right)$$

$$[(1-\kappa)(\alpha(\alpha-1)+2\alpha)+2\kappa\delta(2(\delta-1)+3)]\,({}_{\lambda,\vartheta,\gamma}(2))^2 a_2^2$$
$$= \rho(P_2+Q_2) + \tfrac{(\rho-1)[\alpha(1-\kappa)+2\kappa\delta]^2({}_{\lambda,\vartheta,\gamma}(2))^2 a_2^2}{\rho}$$

Thus, it is derived that:

$$a_2^2 = \frac{\rho^2 (\mathcal{P}_2 + \mathcal{Q}_2)}{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2(\rho\left[(1-\kappa)(\alpha(\alpha-1)+2\alpha)+2\kappa\delta(2\delta+1)\right]-(\rho-1)\left[\alpha(1-\kappa)+2\kappa\delta\right]^2)} \tag{2.11}$$

Consequently, it is ascertained that:

$$|a_2| \le \frac{2\rho}{\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\sqrt{\rho\left[(1-\kappa)(\alpha(\alpha-1)+2\alpha)+2\kappa\delta(2\delta+1)\right]-(\rho-1)\left[\alpha(1-\kappa)+2\kappa\delta\right]^2}}. \tag{2.12}$$

Next, for the purpose of establishing the constraint on $|a_3|$, it is subtracted that (2.6) from (2.8), it is obtained that:

$$2\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))a_3 - 2\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2 a_2^2 = \rho(\mathcal{P}_2-\mathcal{Q}_2) + \frac{\rho(\rho-1)}{2}(\mathcal{P}_1^2-\mathcal{Q}_1^2). \tag{2.13}$$

By applying equation (2.9) and replacing it in equation (2.10) within equation (2.13), it is derived that:

$$2\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))a_3 = \rho(\mathcal{P}_2-\mathcal{Q}_2) + \left[\alpha(1-\kappa)+3\kappa\delta\right]\frac{\rho^2(\mathcal{P}_1^2+\mathcal{Q}_1^2)}{\left[\alpha(1-\kappa)+2\kappa\delta\right]^2}. \tag{2.14}$$

Conversely, it can be stated as:

$$a_3 = \frac{\rho(\mathcal{P}_2-\mathcal{Q}_2)}{2\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{\rho^2(\mathcal{P}_1^2+\mathcal{Q}_1^2)}{2\left[\alpha(1-\kappa)+2\kappa\delta\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{2.15}$$

The following upper bound for $|a_3|$ can be established:

$$|a_3| \le \frac{2\rho}{\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{4\rho^2}{\left[\alpha(1-\kappa)+2\kappa\delta\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{2.16}$$

which completes the proof.

□

Theorem 2.2. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $M_\lambda^{\vartheta,\gamma}(\beta,\alpha,\delta,\kappa)$, where $0 \le \beta < 1, \alpha, \delta \ge 0$ and $\kappa \ge 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{((\alpha(1-\kappa)+3\kappa\delta)+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right])(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{2.17}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{\left[\alpha(1-\kappa)+2\kappa\delta\right]^2\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{\left[\alpha(1-\kappa)+3\kappa\delta\right]\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} \tag{2.18}$$

Proof. It can be inferred from inequalities (1.17) and (1.18) that there exist $\mathcal{P}(z)$ and $\mathcal{Q}(w)$ are in the class H such that

$$(1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}\mathcal{F}(z)}{z}\right)^\alpha + \kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}\mathcal{F}(z)\right)^\delta = \beta + (1-\beta)p(z), \tag{2.19}$$

and

$$((1-\kappa)\left(\frac{\mathfrak{S}_{\lambda,\vartheta,\gamma}g(w)}{w}\right)^{\alpha}+\kappa\left(\mathfrak{S}'_{\lambda,\vartheta,\gamma}g(w)\right)^{\delta}=\beta+(1-\beta)q(w). \tag{2.20}$$

Equating coefficients in (2.19) and (2.20), this leads to:

$$[\alpha(1-\kappa)+2\kappa\delta]\,\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)a_2=(1-\beta)\mathcal{P}_1 \tag{2.21}$$

$$[\alpha(1-\kappa)+3\kappa\delta]\,\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)a_3+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2=(1-\beta)\mathcal{P}_2 \tag{2.22}$$

$$-[\alpha(1-\kappa)+2\kappa\delta]\,\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)a_2=(1-\beta)\mathcal{Q}_1 \tag{2.23}$$

and

$$[\alpha(1-\kappa)+3\kappa\delta]\,(2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2-\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)a_3)+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2 = (1-\beta)\mathcal{Q}_2. \tag{2.24}$$

Utilizing equations (2.21) and (2.23), it is derived that:

$$\mathcal{P}_1=-\mathcal{Q}_1 \tag{2.25}$$

and

$$2\left[\alpha(1-\kappa)+2\kappa\delta\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2=(1-\beta)^2(\mathcal{P}_1^2+\mathcal{Q}_1^2) \tag{2.26}$$

From equations (2.22) and (2.24), it is derived that:

$$2\left([\alpha(1-\kappa)+3\kappa\delta]+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right]\right)(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2=(1-\beta)(\mathcal{P}_2+\mathcal{Q}_2) \tag{2.27}$$

Consequently, we obtain the following:

$$a_2=\sqrt{\frac{(1-\beta)(\mathcal{P}_2+\mathcal{Q}_2)}{2((\alpha(1-\kappa)+3\kappa\delta)+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right])(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{2.28}$$

This determines the upper bound for $|a_2|$.

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{((\alpha(1-\kappa)+3\kappa\delta)+\left[(1-\kappa)\frac{\alpha(\alpha-1)}{2}+2\kappa\delta(\delta-1)\right])(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{2.29}$$

Next, for the purpose of establishing the constraint on $|a_3|$, we subtract (2.21) from (2.23), it is obtained that:

$$2\left[\alpha(1-\kappa)+3\kappa\delta\right]\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)a_3-2\left[\alpha(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2a_2^2=(1-\beta)(\mathcal{P}_2-\mathcal{Q}_2), \tag{2.30}$$

Alternatively, it can be expressed as:

$$a_3=\frac{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}{\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)}a_2^2+\frac{(1-\beta)(\mathcal{P}_2-\mathcal{Q}_2)}{2\left[\alpha(1-\kappa)+3\kappa\delta\right]\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)}. \tag{2.31}$$

By using equation (2.25) in equation (2.31), It has been found that:

$$a_3 = \frac{(1-\beta)^2(\mathcal{P}_1^2+\mathcal{Q}_1^2)}{2\left[\alpha(1-\kappa)+2\kappa\delta\right]^2 \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{(1-\beta)(\mathcal{P}_2-\mathcal{Q}_2)}{2\left[\alpha(1-\kappa)+3\kappa\delta\right] \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)}. \tag{2.32}$$

The following upper bound for $|a_3|$ can be established:

$$|a_3| \leq \frac{4(1-\beta)^2}{\left[\alpha(1-\kappa)+2\kappa\delta\right]^2 \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{\left[\alpha(1-\kappa)+3\kappa\delta\right] \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} \tag{2.33}$$

Which completes the proof. ∎

3. Corollaries and Consequences

By substituting $\alpha = 1$ in Theorem (2.1) and Theorem (2.2), the following results can be derived.

Corollary 3.1. *Consider a function $\mathcal{F} \in \mathcal{A}$ of the form described in (1.1) that belongs to the function class $S^{\gamma}_{\kappa,\vartheta}(1,\delta,\kappa,\rho)$, where $0 < \rho \leq 1, \delta \geq 0, \kappa \geq 1$. Then*

$$|a_2| \leq \frac{2\rho}{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))\sqrt{\rho\left[2(1-\kappa)+2\kappa\delta(2\delta+1)\right]-(\rho-1)\left[(1-\kappa)+2\kappa\delta\right]^2}}. \tag{3.1}$$

and

$$|a_3| \leq \frac{2\rho}{\left[(1-\kappa)+3\kappa\delta\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{4\rho^2}{\left[(1-\kappa)+2\kappa\delta\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{3.2}$$

Corollary 3.2. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $M^{\vartheta,\gamma}_{\lambda}(\beta,1,\delta,\kappa)$, where $0 \leq \beta < 1, \delta \geq 0, \kappa \geq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\left[(1-\kappa)+3\kappa\delta+2\kappa\delta(\delta-1)\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}}. \tag{3.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{\left[(1-\kappa)+2\kappa\delta\right]^2 \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{\left[(1-\kappa)+3\kappa\delta\right] \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)}. \tag{3.4}$$

By substituting $\delta = 1$ in Theorem (2.1) and Theorem (2.2), the following corollaries are arrived at:

Corollary 3.3. *Consider a function $\mathcal{F} \in \mathcal{A}$ of the form described in (1.1) that belongs to the function class $S^{\gamma}_{\kappa,\vartheta}(\alpha,1,\kappa,\rho)$, where $0 < \rho \leq 1, \alpha \geq 0, \kappa \geq 1$. Then*

$$|a_2| \leq \frac{2\rho}{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))\sqrt{\rho\left[(1-\kappa)\left(\alpha\left(\alpha-1\right)+2\alpha\right)+6\kappa\right]-(\rho-1)\left[\alpha\left(1-\kappa\right)+2\kappa\right]^2}}. \tag{3.5}$$

$$|a_3| \leq \frac{2\rho}{\left[\alpha(1-\kappa)+3\kappa\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{4\rho^2}{\left[\alpha(1-\kappa)+2\kappa\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{3.6}$$

Corollary 3.4. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $M_{\lambda}^{\vartheta,\gamma}(\beta,\alpha,1,\kappa)$, where $0 \leq \beta < 1, \alpha \geq 0, \kappa \geq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{((\alpha(1-\kappa)+3\kappa)+(1-\kappa)\left[\frac{\alpha(\alpha-1)}{2}\right])(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{3.7}$$

$$|a_3| \leq \frac{4(1-\beta)^2}{\left[\alpha(1-\kappa)+2\kappa\right]^2 \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{\left[\alpha(1-\kappa)+3\kappa\right] \mathfrak{L}_{\lambda,\vartheta,\gamma}(3)}. \tag{3.8}$$

By considering the case when $\delta = 1$ and $\alpha = 1$, the following results can be derived:

Corollary 3.5. *Consider a function $\mathcal{F} \in \mathcal{A}$ of the form described in (1.1) that belongs to the function class $S_{\kappa,\vartheta}^{\gamma}(1,1,\kappa,\rho)$, where $0 < \rho \leq 1$ and $\kappa \geq 1$. Then:*

$$|a_2| \leq \frac{2\rho}{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))\sqrt{\rho\left[2(1-\kappa)+6\kappa\right]-(\rho-1)\left[(1-\kappa)+2\kappa\right]^2}}. \tag{3.9}$$

and

$$|a_3| \leq \frac{2\rho}{\left[1+2\kappa\right](\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{4\rho^2}{\left[1+\kappa\right]^2(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{3.10}$$

Corollary 3.6. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $M_{\lambda}^{\vartheta,\gamma}(\beta,1,1,\kappa)$, where $0 \leq \beta < 1, \kappa \geq 1$. Then:*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1+2\kappa)(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{3.11}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\kappa)^2\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{(1+2\kappa)\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} \tag{3.12}$$

By substituting $\kappa = 1$ in the previous corollaries, the following corollaries are arrived at:

Corollary 3.7. *Consider a function $\mathcal{F} \in \mathcal{A}$ of the form described in (1.1) that belongs to the function class $S_{\kappa,\vartheta}^{\gamma}(1,1,1,\rho)$, where $0 < \rho \leq 1$. Then:*

$$|a_2| \leq \frac{2\rho}{\mathfrak{L}_{\lambda,\vartheta,\gamma}(2)\sqrt{6\rho-4(\rho-1)}} \tag{3.13}$$

and

$$|a_3| \leq \frac{2\rho}{3(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))} + \frac{\rho^2}{(\mathfrak{L}_{\lambda,\vartheta,\gamma}(3))}. \tag{3.14}$$

Corollary 3.8. *Let $\mathcal{F}(z)$ given by (1.1) be in the class $M_{\lambda}^{\vartheta,\gamma}(\beta,1,1,1)$, where $0 \leq \beta < 1$. Then:*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3(\mathfrak{L}_{\lambda,\vartheta,\gamma}(2))^2}} \tag{3.15}$$

$$|a_3| \leq \frac{(1-\beta)^2}{\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} + \frac{2(1-\beta)}{3\mathfrak{L}_{\lambda,\vartheta,\gamma}(3)} \tag{3.16}$$

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