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SOME PROPERTIES OF BESOV-FOFANA TYPES SPACES

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Abstract. In this paper, we define a Banach Besov-Fofana type space and we give some of its properties which allows us give some relationship with other well-known spaces.

1. Introduction and preliminaries

Let d be a natural number or zero and assume that $\mathbb{R}^d$ is equipped with its usual Euclidean structure and its Lebesgue measure. For any subset A of $\mathbb{R}^d$, χ_A and $|A|$ denote the characteristic function and the Lebesgue measure of A, respectively. Let L^0 denote the set of equivalence classes (modulo equality almost everywhere) of measurable complex functions on $\mathbb{R}^d$, endued with its usual algebra structure on $\mathbb{C}$. Let L^1_{loc} denote the set of all elements f of L^0 for which $\|f\chi_K\|_1 < \infty$ for any compact subset K of $\mathbb{R}^d$.

For $1 \leq \alpha \leq \infty, L^\alpha$ will be the classical Lebesgue space on $\mathbb{R}^d$, endued with its usual norm $\|.\|_\alpha$ and α' denotes the conjugate exponent of α : $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ with the convention $\frac{1}{\infty} = 0$.

Suppose $1 \leq q \leq \alpha \leq p \leq \infty$. We let, for every element f of L^0

$$\|f\|_{\mathcal{M}_q^\alpha} = \sup_{y\in\mathbb{R}^d,\rho>0} \rho^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\|f\chi_{Q(y,\rho)}\|_q, \quad \text{where} \quad Q(y,\rho) = \prod_{j=1}^{d}\left[y_j - \frac{\rho}{2}, y_j + \frac{\rho}{2}\right], \quad 0 < \rho < \infty.$$

The Morrey space $\mathcal{M}_q^\alpha$ is characterized as the vector subspace $\left\{f \in L^0,\ \|f\|_{\mathcal{M}_q^\alpha} < \infty\right\}$.

$(\mathcal{M}_q^\alpha, \|f\|_{\mathcal{M}_q^\alpha})$ is a complex Banach space introduce in 1938 by C. B. Morrey (see [14]), it is used in the study of regularity problems in the theory of partial differential equations and the calculation of variation.

We let, for any $f \in L^0$

$$\|f\|_{q,p,\alpha} = \sup_{\rho>0} \rho^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_\rho\|f\|_{q,p}$$

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where

$$
{}_{\rho}\|f\|_{q,p} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^d} \|f\chi_{I_k^\rho}\|_q^p \right)^{\frac{1}{p}} & \text{if} \quad p < \infty \\ \sup_{y\in\mathbb{R}^d} \|f\chi_{J_y^\rho}\|_q & \text{if} \quad p = \infty, \end{cases}
$$

with $I_k^\rho = \prod_{j=1}^{d} [k_j\rho, (k_j+1)\rho)$ and $J_y^\rho = \prod_{j=1}^{d} \left(y_j - \frac{\rho}{2}, y_j + \frac{\rho}{2}\right)$,
$k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d$,$y = (y_1, y_2, ..., y_d) \in \mathbb{R}^d$ and $\rho \in \mathbb{R}_+^*$. The space $(L^q, l^p)^\alpha$ is defined as being the vector subspace
$\{f \in L^0 : \ \|f\|_{q,p,\alpha} < \infty\}$ of L^0, endowed with its norm $f \longmapsto \|f\|_{q,p,\alpha}$.

Functional spaces have been broadly used in various fields of analysis such as harmonic analysis, partial differential equations, Navier-Stokes equations, and others.

Over the past few years, there has been an increase of interest in a new family of functional spaces that generalize Morrey's spaces [1, 16, 18]. The spaces $(L^q, l^p)^\alpha$ $(1 \leq q \leq \alpha \leq p \leq \infty)$ has been introduced in 1988 by I. Fofana in connection with the problem of Fourier multipliers in Lebesgue spaces. It is well known that $(L^q, l^p)^\alpha$ is a complex Banach space [5, 6, 8] and these spaces turned out to be a good framework for the study of the properties of the Fourier transformation [3, 4, 7, 12, 17].These spaces generalizes Lebesgue spaces since (for $\alpha = p$ or $\alpha = q$, we have $(L^q, l^p)^\alpha = L^\alpha$ and $L^\alpha \subsetneq (L^q, l^p)^\alpha$ if $q < \alpha < p$) and Morrey spaces when $p = \infty$, $(L^q, l^p)^\alpha = \mathcal{M}_q^\alpha$.

For a while now, there has been some interest in smoothness spaces associated with Morrey spaces, specifically Morrey-type Besov spaces and Morrey-type triebel Lizorkin spaces [2, 13, 21, 23], which use a potent harmonic analysis technique like the Fourier transform.

We define the Fourier transform and its inverse by

$$
\mathcal{F}f(t) = \widehat{f}(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-ix.t} dx, \quad t \in \mathbb{R}^d
$$

$$
\mathcal{F}^{-1}f(t) = \check{f}(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{ix.t} dx, \quad t \in \mathbb{R}^d
$$

for $f \in L^1$.

We equip the Schwartz space $\mathcal{S}$ (rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$) with the topology defined by the family of norms $(P_N)_{N\geq 0}$ with

$$
P_N(\phi) = \sup_{x\in\mathbb{R}^d} (1+|x|)^N \sum_{|\beta|\leq N} |\partial^\beta \phi(x)|, \quad \phi \in \mathcal{S} \tag{1.1}
$$

where $N \in \mathbb{N}$ and $\partial^\beta = (\frac{\partial}{\partial x_1})^{\beta_1} ... (\frac{\partial}{\partial x_d})^{\beta_d}$, for all $\beta = (\beta_1, ..., \beta_d) \in \mathbb{N}^d$.

It is important to note that L^1 contains the Schwartz space $\mathcal{S}$ of test functions. Moreover, the restriction of $\mathcal{F}$ to $\mathcal{S}$ has an extension, in the sense of duality, to the space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}^d$. This extension is a linear operator also called Fourier transform and denoted again by $\mathcal{F}$ such that $\mathcal{F} : T \mapsto \mathcal{F}(T) = \widehat{T}$. Some common properties of this operator are reported below.

Proposition 1.1. *([9],[11],[20]).*

(1) *The space* L^α *is contained in* $\mathcal{S}'$ $(1 \leq \alpha \leq \infty)$.
(2) *For* $1 \leq \alpha \leq 2$, $\mathcal{F}$ *maps* L^α *into* $L^{\alpha'}$ *and there exists a real constant* M_α *not dependent on* f *such that*

$$\|\widehat{f}\|_{\alpha'} \leq M_\alpha \|f\|_\alpha, \quad f \in L^\alpha \ \ (Hausdorff - Young\ inequality).$$

(3) $\mathcal{F}$ *is an isometry on* L^2, *namely*

$$\|\widehat{f}\|_2 = \|f\|_2, \quad f \in L^2 \ \ (Plancherel\ identity).$$

(4) *The Fourier transform* $\mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$ *is a linear, continuous, bijective, and inverse continuous transformation*

The aim of this paper is to define Besov-Fofana types spaces and give some of these properties.

We organize the remainder of this paper as follows: Section 2 is devoted to the construction and properties of $U_k^r, \quad k \in \mathbb{Z}^d$ and $r > 0$, which is a partition of unity on $\mathbb{R}^d$. Section 3 contains the definition and some properties of our new spaces $X(q, p, \alpha)$ and we devoted section 4 to the construction of the besov space and to the completeness of the space $X(q, p, \alpha)$.

2. Partition of unity on $\mathbb{R}^d$

It is well-known that $t \mapsto \theta_0(t) = \begin{cases} \exp\left(-\frac{1}{\left(\frac{1}{2}-t\right)\left(t+\frac{1}{2}\right)}\right) & \text{if } \ t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ 0 & \text{if } \ t \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right), \end{cases}$

is a non negative and continuous function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \theta_0(t) dt = c_0 \in (0, \infty).$$

Moreover, it is well-known that $\theta_0 \in \mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$ (see Section 1.2 in [10]).

Let $u = c_0^{-1} \chi_{[0,2]} * \theta_0$ where $*$ is a convolution. The function u defined above satisfies the following properties

i)

$$\begin{cases} u \in \mathcal{C}^\infty \\ u(t) > 0, \ \ t \in \left(-\dfrac{1}{2}, \dfrac{5}{2}\right) \text{ and } \ \operatorname{supp}(u) = \left[-\dfrac{1}{2}, \dfrac{5}{2}\right]. \\ u(t) = 1, \quad t \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right]. \end{cases} \tag{2.1}$$

ii)

$$u(t) + u(t-2) = 1, \quad t \in \left[\frac{3}{2}, \frac{5}{2}\right] \tag{2.2}$$

iii)

$$u(t) \geq \frac{1}{2}, \qquad t \in [0, 2]. \tag{2.3}$$

It is easy to verify that, for any element (k, r) of $(0,\ \infty) \times \mathbb{Z}$, we have

$$\left[2kr - \frac{r}{2},\ 2(k+1)r + \frac{r}{2}\right] = \bigcup_{l \in \{2k, 2k+1, 2k+2\}} \left[lr - \frac{r}{2},\ (l+1)r - \frac{r}{2}\right]. \tag{2.4}$$

Definition 2.1. For any element (r, k) of $(0,\ \infty) \times \mathbb{Z}^d$. We let

$$U_k^r(x) = \prod_{j=1}^{d} u(r^{-1}x_j - 2k_j), \quad x \in \mathbb{R}^d.$$

It is easy to see that U_k^r inherits properties of u. Thus, we have

$$\begin{cases} U_k^r \in \mathcal{C}_c^\infty = \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ (the set of all elements of } \mathcal{C}^\infty \\ \text{with compact support in } \mathbb{R}^d\text{), positive on } \mathring{P}_k^r \text{ and } \operatorname{supp}(U_k^r) = P_k^r \\ U_k^r(x) = 1, \quad x \in Q_k^r \\ U_k^r(x) \geq 2^{-d}, \quad x \in \overline{I_k^{2r}} \\ \displaystyle\sum_{k \in \mathbb{Z}^d} U_k^r(x) = 1, \quad x \in \mathbb{R}^d \\ \#\left\{k \in \mathbb{Z}^d : U_k^r(x) \neq 0\right\} \leq 2^d, \quad x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} Q_k^r &= \prod_{j=1}^{d} \left[\left(2k_j + \frac{1}{2}\right) r, \left(2(k_j+1) - \frac{1}{2}\right) r\right] \\ P_k^r &= \prod_{j=1}^{d} \left[\left(2k_j - \frac{1}{2}\right) r, \left(2(k_j+1) + \frac{1}{2}\right) r\right]. \end{aligned}$$

A consequence of (2.4) is,

$$P_k^r = \bigcup_{\substack{l \in \mathbb{Z}^d \\ 0 \leq l_j \leq 2}} \prod_{j=1}^{d} \left[(2k_j + l_j)r - \frac{r}{2}, (2k_j + l_j + 1)r - \frac{r}{2}\right]$$

$$P_k^r = \bigcup_{\substack{l \in \mathbb{Z}^d \\ 0 \leq l_j \leq 2}} \overline{I_{2k+l}^r} - \frac{rw}{2} \quad \text{where } w = (1,\ 1, ..., 1). \tag{2.6}$$

Therefore, P_k^r is the union of $3^d \overline{I_k^r}$.

For any real $r > 0$, $\{U_k^r :\ k \in \mathbb{Z}^d\}$ is a partition of unity on $\mathbb{R}^d$ composed of positive elements of $\mathcal{C}_c^\infty$ and locally finite. The following remark will be used in the proof of the Proposition 3.1.

Remark 2.2.

(i) Let $U(x) = \prod_{j=1}^{d} u(x_j),\ \ x \in \mathbb{R}^d$.

It's clear that:

$$\begin{cases} U \in \mathcal{C}_c^\infty, \text{ positive on } \mathring{Q}(w,3) \text{ and } \operatorname{supp}(U) = Q(w,3) \text{ therefore inherits properties of } u. \\ U(x) = 1 \quad \text{ on } Q(w,1). \end{cases}$$

(ii) Let

$$\psi(x) = U(x+w), \quad x \in \mathbb{R}^d. \tag{2.7}$$

It follows immediately from (i) that:

$$\begin{cases} \psi \in \mathcal{C}_c^\infty \text{ and positive on } \mathring{Q}(0,3) \\ \psi(x) = 1 \text{ on } Q(0,1). \end{cases}$$

We are now going to define and give some properties of our new functional space.

3. Definition and basic properties of $X(q,p,\alpha)$

Consider an element f of $\mathcal{S}'$.

For any real $r > 0$, $\{U_k^r \mathcal{F} f : \ k \in \mathbb{Z}^d\}$ is a countable family of distributions with compact support, such that

$$\sum_{k\in\mathbb{Z}^d} U_k^r \mathcal{F} f = \mathcal{F} f$$

and therefore

$$\sum_{k\in\mathbb{Z}^d} \mathcal{F}^{-1}(U_k^r \mathcal{F} f) = f.$$

We define for, $1 \leq q, p, \alpha \leq \infty$

$$\begin{aligned} {}_r\|f\|_{X(q,p)} &= \left\|\left\{\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}\right\}\right\|_p = \left\|\left\{\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}\right\}_{k\in\mathbb{Z}^d}\right\|_p, \quad r \in \mathbb{R}_+^* \\ \|f\|_{X(q,p,\alpha)} &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_r\|f\|_{X(q,p)} \\ X(q,p) &= \left\{f \in \mathcal{S}' : \|f\|_{X(q,p)} = {}_1\|f\|_{X(q,p)} < \infty\right\} \\ X(q,p,\alpha) &= \left\{f \in \mathcal{S}' : \|f\|_{X(q,p,\alpha)} < \infty\right\}. \end{aligned}$$

The following proposition proves that the space $\mathcal{M}_{\mathcal{F}_q^\alpha}$ given by Nakamura-Sawano (see [15]) is a particular case of $X(q,p,\alpha)$.

Proposition 3.1. *Let $1 \leq q, \alpha \leq \infty$. We have*

$$\|f\|_{X(q,\infty,\alpha)} \leq \|f\|_{\mathcal{M}_{\mathcal{F}_q^\alpha}} \leq 3^d \|\mathcal{F}^{-1}\psi\|_1 \|f\|_{X(q,\infty,\alpha)}, \quad f \in \mathcal{S}',$$

where ψ satisfies (2.7).

Proof.

Consider an element f of $\mathcal{S}'$.

(a) For any element (k,r) of $\mathbb{Z}^d \times \mathbb{R}_+^*$, we have:

$$U_k^r(x) = U(r^{-1}x - 2k) = U\left(\frac{x-(2k+w)r}{r} + w\right) = \psi\left(\frac{x-(2k+w)r}{r}\right)$$

and therefore

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\|\mathcal{F}^{-1}[U_k^r\mathcal{F}f]\|_{q'} = r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left\|\mathcal{F}^{-1}\left[\psi\left(\frac{.-(2k+w)r}{r}\right)\right]*f\right\|_{q'} \leq \|f\|_{\mathcal{M}_{\mathcal{F}_q^\alpha}}.$$

Thus,

$$\|f\|_{X(q,\infty,\alpha)} = \sup_{\substack{r>0\\k\in\mathbb{Z}^d}} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\|\mathcal{F}^{-1}[U_k^r\mathcal{F}f]\|_{q'} \leq \|f\|_{\mathcal{M}_{\mathcal{F}_q^\alpha}}.$$

(b) Let (c,r) be an element of $\mathbb{R}^d\times\mathbb{R}_+^*$.

We have

$$supp\ \psi\left(\frac{.-c}{r}\right)\subset Q(c,3r) \text{ and } supp(U_k^r) = Q\left((2k+w)r,3r\right), \quad k\in\mathbb{Z}^d.$$

So,

$$\begin{aligned}\#\left\{k\in\mathbb{Z}^d : \left|supp\ \psi\left(\frac{.-c}{r}\right)\cap supp(U_k^r)\right| > 0\right\} &\leq \#\{k\in\mathbb{Z}^d : |Q(c,3r)\cap Q\left((2k+w)r,3r\right)| > 0\}\\ &\leq 3^d\end{aligned}$$

and therefore

$$\psi\left(\frac{.-c}{r}\right)\mathcal{F}f = \sum_{k\in K(c,r)}\psi\left(\frac{.-c}{r}\right)U_k^r\mathcal{F}f,\ K(c,r) = \{k\in\mathbb{Z}^d : |Q(c,3r)\cap Q\left((2k+w)r,3r\right)| > 0\}.$$

Therefore,

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[\psi\left(\frac{.-c}{r}\right)\mathcal{F}f\right]\right\|_{q'} &\leq \sum_{k\in K(c,r)}\left\|\left[\mathcal{F}^{-1}\psi\left(\frac{.-c}{r}\right)\right]*\left[\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right]\right\|_{q'}\\ &\leq \sum_{k\in K(c,r)}\left\|\mathcal{F}^{-1}\psi\left(\frac{.-c}{r}\right)\right\|_1\left\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right\|_{q'}\\ &= \sum_{k\in K(c,r)}\|\mathcal{F}^{-1}\psi\|_1\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}\\ &\leq 3^d\|\mathcal{F}^{-1}\psi\|_{1r}\|f\|_{X(q,\infty)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|_{\mathcal{M}_{\mathcal{F}_q^\alpha}} &= \sup_{\substack{r>0\\x\in\mathbb{R}^d}} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left\|\mathcal{F}^{-1}\left[\psi\left(\frac{.-c}{r}\right)\mathcal{F}f\right]\right\|_{q'}\\ &\leq 3^d\|\mathcal{F}^{-1}\psi\|_1\sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{X(q,\infty)}\\ &\leq 3^d\|\mathcal{F}^{-1}\psi\|_1\|f\|_{X(q,\infty,\alpha)}.\end{aligned}$$

Thus,

$$\|f\|_{X(q,\infty,\alpha)} \leq \|f\|_{\mathcal{M}_{\mathcal{F}_q^\alpha}} \leq 3^d\|\mathcal{F}^{-1}\psi\|_1\|f\|_{X(q,\infty,\alpha)}, \quad f\in\mathcal{S}'.$$

$\square$

The following remark shows that the space $X(q,p,\alpha)$ is increasing with respect to the second index.

Remark 3.2. It is well-known that if $1 < p_1 < p_2 < \infty$ then:

$$\|\{(a_k)\}\|_{l^\infty} \leq \|\{(a_k)\}\|_{l^{p_2}} \leq \|\{(a_k)\}\|_{l^{p_1}}, \quad (a_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}.$$

Therefore, if $1 \leq q, \alpha \leq \infty$ and $1 < p_1 < p_2 < \infty$ then:

$$\|f\|_{X(q,\infty,\alpha)} \leq \|f\|_{X(q,p_2,\alpha)} \leq \|f\|_{X(q,p_1,\alpha)}, \quad f \in \mathcal{S}'.$$

Thus,

$$X(q,p_1,\alpha) \subset X(q,p_2,\alpha) \subset X(q,\infty,\alpha). \tag{3.1}$$

We will use the following lemma to prove the Proposition 3.4.

Lemma 3.3. ([22]) *Suppose that $0 < q_1 \leq q_2 \leq \infty$. Then there exists a real number C such that*

$$\|\phi\|_{q_2} \leq Cr^{d\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\|\phi\|_{q_1}, \quad (x,r) \in \mathbb{R}^d \times \mathbb{R}_+^*, \quad \phi \in \mathcal{S}, \quad supp(\mathcal{F}\phi) \subset Q(x,r).$$

Proposition 3.4.

a) *Suppose that: $1 \leq q_1, q_2, \alpha, p \leq \infty$ and $q_1 \leq q_2$. Then, there exists a real constant C such that:*

$$\|f\|_{X(q_1,p,\alpha)} \leq C\|f\|_{X(q_2,p,\alpha)}, \quad f \in \mathcal{S}'. \tag{3.2}$$

b) *Suppose that: $1 \leq q \leq \alpha \leq \infty$. Then exists a real constant C such that*

$$\|f\|_{X(\alpha,\infty,\alpha)} \leq \|\mathcal{F}^{-1}U\|_1\|f\|_{\alpha'}, \quad f \in L^1_{loc} \cap \mathcal{S}'.$$

Proof. a) Suppose that: $1 \leq q_1, q_2, \alpha, p \leq \infty$ with $q_1 \leq q_2$ and consider an element f of $\mathcal{S}'$.

It is easy to verify that

$$\begin{array}{rcl} q_2' & \leq & q_1' \\ \mathcal{F}\left[\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right] & = & U_k^r\mathcal{F}f, \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^* \\ supp\left\{\mathcal{F}\left[\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right]\right\} & \subset & \overline{Q}\left((2k+w),3r\right), \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^*. \end{array}$$

Therefore, by the Lemma 3.3, there exists a real constant C such that

$$\begin{array}{rcl} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'} & \leq & Cr^{d\left(\frac{1}{q_2'}-\frac{1}{q_1'}\right)}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_2'}, \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^*. \\ \left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}^p\right)^{\frac{1}{p}} & \leq & Cr^{d\left(\frac{1}{q_2'}-\frac{1}{q_1'}\right)}\left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_2'}^p\right)^{\frac{1}{p}}, \quad r > 0 \\ r^{d\left(\frac{1}{\alpha}-\frac{1}{q_1}\right)}\left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}^p\right)^{\frac{1}{p}} & \leq & Cr^{d\left(\frac{1}{\alpha}-\frac{1}{q_2}\right)}\left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_2'}^p\right)^{\frac{1}{p}} \end{array}$$

and therefore

$$\|f\|_{X(q_1,p,\alpha)} \leq C\|f\|_{X(q_2,p,\alpha))}.$$

b) Suppose that $1 \leq q \leq \alpha \leq \infty$.

By a simple calculation, we have

$$\mathcal{F}^{-1}U_k^r(x) = \mathcal{F}^{-1}\left[U(r^{-1}. - 2k)\right](x) = e^{2ikrx}(\mathcal{F}^{-1}U)_r(x), \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^*, \ \ x \in \mathbb{R}^d$$

with

$$(\mathcal{F}^{-1}U)_r(x) = r^d \mathcal{F}^{-1}U(rx).$$

Consequently,

$$\|\mathcal{F}^{-1}U_k^r\|_1 = \|(\mathcal{F}^{-1}U)_r\|_1 = \|\mathcal{F}^{-1}U\|_1 < \infty, \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^*.$$

Hence, by Young's inequality, we have

$$\begin{aligned}\|\mathcal{F}^{-1}U_k^r * f\|_{\alpha'} &\leq \|\mathcal{F}^{-1}U_k^r\|_1\|f\|_{\alpha'}, \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^* \\ &\leq \|\mathcal{F}^{-1}U\|_1\|f\|_{\alpha'}.\end{aligned}$$

Thus,

$$\|f\|_{X(\alpha,\infty,\alpha)} = \sup_{\substack{k\in\mathbb{Z}^d \\ r>0}} r^{\left(\frac{1}{\alpha}-\frac{1}{\alpha}\right)}\|(\mathcal{F}^{-1}U_k^r) * f\|_{\alpha'} \leq \|\mathcal{F}^{-1}U\|_1\|f\|_{\alpha'}.$$

Moreover, according to the result obtained in $a)$

$$\|f\|_{X(q,\infty,\alpha)} \leq C\|f\|_{X(\alpha,\infty,\alpha)}$$

where C is a real number not dependent of f.

□

The following proposition gives the link between $X(q,p,\alpha)$ spaces and that of Lebesgue.

Proposition 3.5. *Let $1 \leq \alpha < \infty$. Then*

$$X(\alpha,\infty,\alpha) = L^{\alpha'}.$$

Proof. i) By Proposition 3.4, b),

$L^{\alpha'}$ is included in $X(\alpha,\infty,\alpha)$.

ii) Consider an element f of $X(\alpha,\infty,\alpha)$.

We have

$$\begin{aligned}\left\|\left\{\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{\alpha'}\right\}\right\|_\infty &\leq \|f\|_{X(\alpha,\infty,\alpha)} < \infty, \quad r > 0 \\ \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{\alpha'} &\leq \|f\|_{X(\alpha,\infty,\alpha)} < \infty, \ \ k \in \mathbb{Z}^d \ \text{ and } \ r > 0.\end{aligned}$$

Consider an element k_0 of $\mathbb{Z}^d$.

The sequence $\left(\mathcal{F}^{-1}(U_{k_0}^{r_j}\mathcal{F}f)\right)_{j\geq 0}$ is bounded in $L^{\alpha'} = (L^\alpha)^*$. According to Banach Alaoglu's theorem, there exists a subsequence $(r_{j_i})_{i\geq 0}$ of $(r_j)_{j\geq 0}$ and an element g of $L^{\alpha'}$ such that $\left(\mathcal{F}^{-1}(U_{k_0}^{r_{j_i}}\mathcal{F}f)\right)_{i\geq 0}$ converges weakly to g in $L^{\alpha'}$ when i tends to ∞.

Moreover, for any element h of $\mathcal{S}$, we have

$$\begin{aligned}\langle g, h\rangle &= \lim_{i\longrightarrow\infty}\left\langle \mathcal{F}^{-1}(U_{k_0}^{r_{j_i}}\mathcal{F}f), h\right\rangle \\ &= \lim_{i\longrightarrow\infty}\left\langle f, \mathcal{F}(U_{k_0}^{r_{j_i}}\mathcal{F}^{-1}h)\right\rangle \\ &= \langle f, h\rangle.\end{aligned}$$

Therefore, f is the regular distribution represented by g.
In other words $f = g \in L^{\alpha'}$.
Moreover, we have

$$\|f\|_{\alpha'} = \|g\|_{\alpha'} \leq \lim_{i\to\infty} \|\mathcal{F}^{-1}\left(U_{k_0}^{r_{j_i}}\mathcal{F}f\right)\|_{\alpha'} \leq \|f\|_{X(\alpha,\infty,\alpha)}.$$

In conclusion
$f \in X(\alpha,\infty,\alpha) \Longrightarrow f \in L^{\alpha'}(\mathbb{R}^d)$ and $\|\mathcal{F}^{-1}\left(U_{k_0}^{r_{j_i}}\mathcal{F}f\right)\|_{\alpha'} \leq \|f\|_{X(\alpha,\infty,\alpha)}$,
therefore $X(\alpha,\infty,\alpha)$ is included in $L^{\alpha'}$.
Thus, $X(\alpha,\infty,\alpha) = L^{\alpha'}$. □

The following proposition shows that $X(q,p,\alpha)$ is a normed vector space.

Proposition 3.6. *Let q,p and α be elements of $[1,\infty]$. $X(q,p,\alpha)$ is a vector subspace of $\mathcal{S}'$ on which $f \mapsto \|f\|_{X(q,p,\alpha)}$ is a norm.*

Proof.

a) It is obvious that the zero of $\mathcal{S}'$ verifies $\|0\|_{X(q,p,\alpha)} = 0$ and therefore belongs to $X(q,p,\alpha)$.

Consider an element f of $\mathcal{S}'$ satisfying $\|f\|_{X(q,p,\alpha)} = 0$. We have

$$\|(\mathcal{F}^{-1}U_k^r) * f\|_{q'} = \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'} = 0, \quad k \in \mathbb{Z}^d.$$

Moreover, for any element k of $\mathbb{Z}^d$, $\mathcal{F}^{-1}U_k^r$ belongs to $\mathcal{S}$ and therefore $(\mathcal{F}^{-1}U_k^r) * f$ is a function of class $\mathcal{C}^\infty$ on $\mathbb{R}^d$. Thus

$$\mathcal{F}^{-1}(U_k^r\mathcal{F}f) = (\mathcal{F}^{-1}U_k^r) * f = 0 \text{ (punctually and therefore in } \mathcal{S}',\ k \in \mathbb{Z}^d)$$

$$U_k^r\mathcal{F}f = 0 \text{ in } \mathcal{S}',\ \ k \in \mathbb{Z}^d \Longrightarrow \sum_{k\in\mathbb{Z}^d}(U_k^r\mathcal{F}f) = 0 \Longrightarrow \mathcal{F}f = 0 \text{ in } \mathcal{S}'.$$

This implies, $f = 0$ in $\mathcal{S}'$.

b) It is easy to verify that

$$\begin{aligned} \|f+g\|_{X(q,p,\alpha)} &\leq \|f\|_{X(q,p,\alpha)} + \|g\|_{X(q,p,\alpha)}, \qquad f,g \in \mathcal{S}' \\ \|\lambda f\|_{X(q,p,\alpha)} &= |\lambda|\|f\|_{X(q,p,\alpha)}, \qquad (\lambda,f) \in \mathbb{C}\times\mathcal{S}'. \end{aligned}$$

It follows from the above that $X(q,p,\alpha)$ is a vector subspace of $\mathcal{S}'$ on which $f \mapsto \|f\|_{X(q,p,\alpha)}$ is a norm.

□

It is well-known that the Fofana space $(L^q, l^p)^\alpha = \{0\}$ when $p < \alpha$. What can be said for $X(q,p,\alpha)$ under the same condition? The following proposition partially answers this question.

Proposition 3.7. *Suppose that $2 \leq \alpha$ and $p < \alpha$. Then $X(q,p,\alpha) = \{0\}$ for $q \geq 2$.*

Proof. Let $f \in X(q,p,\alpha)$.
We know that

$$\overline{I}_k^{2r} \subset \mathring{P}_k^r, \quad k \in \mathbb{Z}^d, \quad r > 0$$

and for any $x \in I_k^{2r}$, $\quad U_k^r(x) \geq 2^{-d}$, $\quad k \in \mathbb{Z}^d$, $\quad r > 0$

$$\begin{aligned}
2^{-d}\|\chi_{I_k^{2r}}\mathcal{F}f\|_2 &\leq \|U_k^r\mathcal{F}f\|_2, \quad k \in \mathbb{Z}^d, \quad r > 0 \\
2^{-d}\left[\sum_{k\in\mathbb{Z}^d}\|\chi_{I_k^{2r}}\mathcal{F}f\|_2^p\right]^{\frac{1}{p}} &\leq \left[\sum_{k\in\mathbb{Z}^d}\|U_k^r\mathcal{F}f\|_2^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_2^p\right]^{\frac{1}{p}}, \quad r > 0 \\
2^{-d}{}_{2r}\|\mathcal{F}f\|_{2,p} &\leq {}_r\|f\|_{X(2,p)} \\
(2r)^{d\left(\frac{1}{\alpha}-\frac{1}{2}\right)}{}_{2r}\|\mathcal{F}f\|_{2,p} &\leq 2^d 2^{d\left(\frac{1}{\alpha}-\frac{1}{2}\right)} r^{d\left(\frac{1}{\alpha}-\frac{1}{2}\right)}{}_r\|f\|_{X(2,p)}, \quad r > 0 \\
\|\mathcal{F}f\|_{2,p,\alpha} &\leq 2^{d\left(\frac{1}{\alpha}+\frac{1}{2}\right)}\|f\|_{X(2,p,\alpha)} < \infty.
\end{aligned}$$

Therefore,

$\mathcal{F}f = 0$ and thus $f = 0$ because $\mathcal{F}$ is an injective application see Proposition 1.1.

Hence, $X(2,p,\alpha) = \{0\}$.

On the other hand, according to the Proposition 3.4,

$$2 < q \text{ and } p < \alpha \implies X(q,p,\alpha) \subset X(2,p,\alpha) = \{0\}.$$

Therefore, $X(q,p,\alpha) = \{0\}$. □

Like the Fofana's spaces $(L^q, l^p)^\alpha$, the following proposition shows that the space $X(q,p,\alpha)$ becomes trivial when $\alpha < q$.

Proposition 3.8. *Let $q,p,\alpha \in [1,\infty]$ such that $\alpha < q$. Then*

$$X(q,p,\alpha) = \{0\}.$$

Proof. Suppose that $\alpha < q$ and consider an element f of $X(q,p,\alpha)$.

According to the definition of $X(q,p,\alpha)$, we have

$$\left\|\left\{\|\mathcal{F}^{-1}[U_k^r\mathcal{F}f]\|_{q'}\right\}\right\|_p \leq r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)}\|f\|_{X(q,p,\alpha)}, \quad r > 0.$$

Since $\alpha < q$, we have

$$\lim_{r\to\infty}\left\|\left\{\|\mathcal{F}^{-1}[U_k^r\mathcal{F}f]\|_{q'}\right\}\right\|_p = 0.$$

It means that:

$$\lim_{r\to\infty}\left\{\|\mathcal{F}^{-1}[U_k^r\mathcal{F}f]\|_{q'}\right\} = 0 \text{ in } l^p.$$

Therefore, for any element k of $\mathbb{Z}^d$

$$\mathcal{F}^{-1}[U_k^r\mathcal{F}f] \longrightarrow 0 \text{ in } L^{q'} \text{ when } r \text{ tends to } \infty.$$

Consequently, we have for any element k of $\mathbb{Z}^d$

$$U_k^r\mathcal{F}f \longrightarrow 0 \text{ in } \mathcal{S}', \text{when } r \text{ tends to } \infty.$$

Let ϕ be an element of $\mathcal{D}$, $(\mathcal{D} = \mathcal{C}^\infty)$ there exists a positive real R such that: $supp(\phi) \subset \left[-\frac{3R}{4}, \frac{3R}{4}\right]^d$.

Consider an element ψ of $\mathcal{D}$ such that:

$$\begin{cases} \psi = 1 \quad \text{on} \quad \left[-\frac{3R}{4}, \frac{3R}{4}\right]^d \\ supp(\psi) \subset \left[-\frac{3R}{2}, \frac{3R}{2}\right]^d \end{cases}$$

We have $\phi = \phi\psi$ and for any element T of $\mathcal{D}'$

$$< T, \phi > = < T, \phi\psi > = < \psi T, \phi > .$$

In addition, for $r \geq R$

$$\sum_{k\in\mathbb{Z}^d} \psi U_k^r \mathcal{F} f = \sum_{k\in\{-1,0\}^d} \psi U_k^r \mathcal{F} f.$$

Therefore, for any element $r \geq R$, we have

$$\begin{aligned} \left\langle \sum_{k\in\mathbb{Z}^d} U_k^r \mathcal{F} f,\ \phi \right\rangle &= \left\langle \sum_{k\in\mathbb{Z}^d} \psi U_k^r \mathcal{F} f,\ \phi \right\rangle \\ &= \left\langle \sum_{k\in\{-1,0\}^d} U_k^r \mathcal{F} f,\ \phi \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{r\to\infty} \left\langle \sum_{k\in\mathbb{Z}^d} U_k^r \mathcal{F} f,\ \phi \right\rangle &= \left\langle \sum_{k\in\{-1,0\}^d} \lim_{r\to\infty} U_k^r \mathcal{F} f,\ \phi \right\rangle \\ &= 0. \end{aligned}$$

Thus in $\mathcal{D}'$, we have

$$0 = \lim_{r\to\infty} \sum_{k\in\mathbb{Z}^d} U_k^r \mathcal{F} f = \mathcal{F} f$$

it means that

$$\mathcal{F} f = 0 \quad \text{in } \mathcal{D}' \text{ and therefore in } \mathcal{S}'.$$

Hence,

$$f = 0 \ \text{ in } \ \mathcal{S}'.$$

□

We can describe the boundedness of the Fourier transform on spaces $X(q, p, \alpha)$.

Proposition 3.9.

a) *Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$ *with* $q \leq 2$ *then there exists a real constant* C *such that*

$$\|\mathcal{F} f\|_{X(q,p,\alpha)} \leq C \|f\|_{q,p,\alpha}, \quad f \in (L^q, l^p)^\alpha.$$

b) *If* $2 \leq \alpha \leq p \leq \infty$. *there exists two real constants* C_1 *and* C_2 *such that:*

$$C_1 \|f\|_{2,p,\alpha} \leq \|\mathcal{F} f\|_{X(2,p,\alpha)} \leq C_2 \|f\|_{2,p,\alpha}, \quad f \in (L^2, l^p)^\alpha.$$

Proof. a) Consider an element f of $(L^q, l^p)^\alpha$.

It is easy to see that $\widetilde{f} = f(-.)$ is an element of $(L^q, l^p)^\alpha$ such that

$$\|\widetilde{f}\|_{q,p,\alpha} = \|f\|_{q,p,\alpha}.$$

For any element (k, r) of $\mathbb{Z}^d \times \mathbb{R}^*_+$, we have

$$\|\mathcal{F}^{-1}[U_k^r \mathcal{F}(\mathcal{F}f)]\|_{q'} = \|\mathcal{F}^{-1}(U_k^r \widetilde{f})\|_{q'} \leq M_q \|U_k^r \widetilde{f}\|_q \ \text{ (Hausdorff-Young inequality)}$$

where M_q is a real constant not dependent of f, k and r.

Moreover,

$$0 \leq U_k^r \leq \chi_{P_k^r} \leq \chi_{\underset{l\in L(k)}{\bigcup}\left(I_l^r - \frac{rw}{2}\right)} \ \text{ where } \ (k, r) \in \mathbb{Z}^d \times \mathbb{R}^*_+, \ \ L(k) = \left\{l \in \mathbb{Z}^d, \ \left(I_l^r - \frac{rw}{2}\right) \subset P_k^r\right\}.$$

Therefore

$$\|U_k^r \widetilde{f}\|_q \leq \sum_{l\in L(k)} \left\|\chi_{\left(I_l^r - \frac{rw}{2}\right)} \widetilde{f}\right\|_q.$$

Thus, if $p < \infty$ then for any $r > 0$,

$$\begin{aligned}
\left[\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}[U_k^r \mathcal{F}(\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} &\leq M_q \left[\sum_{k\in\mathbb{Z}^d} \left(\sum_{l\in L(k)} \left\|\chi_{\left(I_l^r - \frac{rw}{2}\right)} \widetilde{f}\right\|_q\right)^p\right]^{\frac{1}{p}} \\
&\leq M_q 3^{\frac{d}{p'}} \left[\sum_{k\in\mathbb{Z}^d} \sum_{l\in L(k)} \left\|\chi_{\left(I_l^r - \frac{rw}{2}\right)} \widetilde{f}\right\|_q^p\right]^{\frac{1}{p}} \quad \text{(Hölder inequality).}
\end{aligned}$$

Moreover, for any element (l, r) of $\mathbb{Z}^d \times \mathbb{R}^*_+$,

$$\#\left(\{k \in \mathbb{Z}^d : \ l \in L(k)\}\right) \leq 3^d.$$

Therefore, if $p < \infty$ then for any real $r > 0$, we have

$$\begin{aligned}
{}_r\|\mathcal{F}f\|_{X(q,p)} &\leq M_q 3^{\frac{d}{p'}} 3^{\frac{d}{p}} \left[\sum_{l\in\mathbb{Z}^d} \left\|\chi_{\left(I_l^r - \frac{rw}{2}\right)} \widetilde{f}\right\|_q^p\right]^{\frac{1}{p}} \\
&= M_q 3^d \left[\sum_{l\in\mathbb{Z}^d} \left\{\sum_{k\in K(l)} \int_{I_k^r \cap \left(I_l^r - \frac{rw}{2}\right)} |\widetilde{f}(x)|^q dx\right\}^{\frac{p}{q}}\right]^{\frac{1}{p}}
\end{aligned}$$

$$\text{where} \quad K(l) = \left\{k \in \mathbb{Z}^d : \ I_k^r \cap \left(I_l^r - \frac{rw}{2}\right) \neq \emptyset\right\}$$

It is well-known that:

$$\text{if } 0 < \beta < \infty, \ \text{ then } \ \left(\sum_{i=1}^n a_i\right)^\beta \leq n^\beta \sum_{i=1}^n a_i^\beta.$$

From the above, we have

$$\left(\sum_{k\in K(l)}\int_{I_k^r\cap(I_l^r-\frac{rw}{2})}|\widetilde{f}(x)|^q dx\right)^{\frac{p}{q}} \leq 2^{\frac{dp}{q}}\sum_{k\in K(l)}\left(\int_{I_k^r\cap(I_l^r-\frac{rw}{2})}|\widetilde{f}(x)|^q dx\right)^{\frac{p}{q}}$$

Therefore,

$$\begin{aligned}
{}_r\|\mathcal{F}f\|_{X(q,p)} &\leq M_q 3^d 2^{\frac{d}{q}}\left[\sum_{l\in\mathbb{Z}^d}\sum_{k\in K(l)}\left(\int_{I_k^r\cap(I_l^r-\frac{rw}{2})}|\widetilde{f}(x)|^q dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}\\
&= M_q 3^d 2^{\frac{d}{q}}\left[\sum_{k\in\mathbb{Z}^d}\left\{\sum_{\substack{k\in K(l)\\ l\in\mathbb{Z}^d}}\left(\int_{I_k^r\cap(I_l^r-\frac{rw}{2})}|\widetilde{f}(x)|^q dx\right)^{\frac{p}{q}}\right\}^{\frac{q}{p}\times\frac{p}{q}}\right]^{\frac{1}{p}}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\leq M_q 3^d 2^{\frac{d}{q}}\left[\sum_{k\in\mathbb{Z}^d}\left(\int_{I_k^r}|\widetilde{f}(x)|^q dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}\\
&= M_q 3^d 2^{\frac{d}{q}}\,{}_r\|\widetilde{f}\|_{q,p}.
\end{aligned}$$

So, if $p<\infty$ then

$$\|\mathcal{F}f\|_{X(q,p,\alpha)} \leq M_q 3^d 2^{\frac{d}{q}}\|\widetilde{f}\|_{q,p,\alpha} = M_q 3^d 2^{\frac{d}{q}}\|f\|_{q,p,\alpha}.$$

We show in an analogous way the case $p=\infty$.

b) Consider an element f of $(L^2,l^p)^\alpha$.
Fix $r>0$ and $k\in\mathbb{Z}^d$.
We have

$$\|\mathcal{F}^{-1}[U_k^r\mathcal{F}(\mathcal{F}f)]\|_2 = \|\mathcal{F}^{-1}(U_k^r\widetilde{f})\|_2 = \|U_k^r\widetilde{f}\|_2.$$

We know that U_k^r is continuous with support P_k^r, equal to 1 on Q_k^r, positive on $\mathring{P}_k^r$ and

$$2^{-d} \leq U_k^r(x), \quad x\in\ I_k^{2r}.$$

Therefore, we have

$$\|U_k^r\widetilde{f}\|_2 \geq 2^{-d}\|\chi_{I_k^{2r}}\widetilde{f}\|_2.$$

Consequently,

$${}_r\|\mathcal{F}f\|_{X(2,p)} = \left\|\left\{\left\|U_k^r\widetilde{f}\right\|_2\right\}\right\|_p \geq 2^{-d}\left\|\left\{\left\|\chi_{I_k^{2r}}\widetilde{f}\right\|_2\right\}\right\|_p = 2^{-d}\,{}_{2r}\|\widetilde{f}\|_{2,p}$$

and thus

$$\|\mathcal{F}f\|_{X(2,p,\alpha)} \geq 2^{-d}2^{d\left(\frac{1}{2}-\frac{1}{\alpha}\right)}\|\widetilde{f}\|_{2,p,\alpha} = 2^{-d\left(\frac{1}{2}+\frac{1}{\alpha}\right)}\|f\|_{2,p,\alpha}.$$

$\square$

The following proposition gives the property of the dilation in $X(q,p,\alpha)$.

Proposition 3.10. *Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$, $R \in \mathbb{R}_+^*$, $f \in \mathcal{S}'$ *and* $f_R = f(R.)$ *is a tempered distribution on* $\mathbb{R}^d$ *defined by:*

$$< f_R, \phi > = < f, \phi_R >, \quad \phi \in \mathcal{S}.$$

Then

$$\|f_R\|_{X(q,p,\alpha)} = R^{-\frac{d}{\alpha'}} \|f\|_{X(q,p,\alpha)}.$$

Proof. We have for any element (k,r) of $\mathbb{Z}^d \times \mathbb{R}_+^*$.

$$\begin{aligned}
(\mathcal{F}^{-1}U_k^r) * f_R(y) &= \left\langle f_R, \widetilde{(\mathcal{F}^{-1}U_k^r)}(. - y) \right\rangle = \langle f_R, (\mathcal{F}U_k^r)(. - y) \rangle \\
&= \left\langle f, R^{-d}(\mathcal{F}U_k^r)(R^{-1}. - y) \right\rangle = \langle f, (\mathcal{F}U_k^r)_R(. - Ry) \rangle \\
&= \left\langle f, \widetilde{(\mathcal{F}^{-1}U_k^r)}_R(. - Ry) \right\rangle = [(\mathcal{F}^{-1}U_k^r)_R * f](Ry) \\
&= \mathcal{F}^{-1}(U_k^r(R.)) * f(Ry), \quad y \in \mathbb{R}^d
\end{aligned}$$

as,

$$U_k^r(Rz) = U(r^{-1}Rz - 2k) = U_k^{R^{-1}r}(z).$$

we have:

$$(\mathcal{F}^{-1}U_k^r) * f_R = (\mathcal{F}^{-1}U_k^{rR^{-1}}) * f(R.).$$

And therefore

$$\begin{aligned}
\|(\mathcal{F}^{-1}U_k^r) * f_R\|_{q'} &= R^{-\frac{d}{q'}} \|(\mathcal{F}^{-1}U_k^{rR^{-1}}) * f\|_{q'}, \quad (k,r) \in \mathbb{Z}^d \times \mathbb{R}_+^* \\
r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left(\sum_{k \in \mathbb{Z}^d} \|(\mathcal{F}^{-1}U_k^r) * f_R\|_{q'}^p \right)^{\frac{1}{p}} &= r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} R^{-\frac{d}{q'}} \left(\sum_{k \in \mathbb{Z}^d} \|(\mathcal{F}^{-1}U_k^{rR^{-1}}) * f\|_{q'}^p \right)^{\frac{1}{p}}, \quad r > 0.
\end{aligned}$$

Thus,

$$\|f_R\|_{X(q,p,\alpha)} = R^{-\frac{d}{\alpha'}} \|f\|_{X(q,p,\alpha)}.$$

□

Fatou's property will be useful in the proof of the completeness of the space $X(q,p,\alpha)$.

Proposition 3.11. *(Fatou's property) Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$.
Let $(f_j)_{j \geq 0} \subset \mathcal{S}'$ *converge to* f *in* $\mathcal{S}'$. *Then*

$$\|f\|_{X(q,p,\alpha)} \leq \liminf_{j \to \infty} \|f_j\|_{X(q,p,\alpha)}.$$

Proof.
Note that if (f_j) converges to f in $\mathcal{S}'$, then $\mathcal{F}^{-1}(U_k^r \mathcal{F} f_j)(x) = \mathcal{F}^{-1}U_k^r * f_j(x) \to \mathcal{F}^{-1}U_k^r * f(x)$ for each x of $\mathbb{R}^d$, $k \in \mathbb{Z}^d$ and $r > 0$.

(f_j) converges to f in $\mathcal{S}'$ is equivalent to for all ϕ of $\mathcal{S}$,

$$\int_{\mathbb{R}^d} f_j(x)\phi(x)dx \quad \longrightarrow \quad \int_{\mathbb{R}^d} f(x)\phi(x)dx.$$

We know that

$$\mathcal{F}^{-1}U_k^r * f_j(x) = \int_{\mathbb{R}^d} \mathcal{F}^{-1}U_k^r(x-y)f_j(y)dy.$$

For any x of $\mathbb{R}^d$, the maps $y \longmapsto \mathcal{F}^{-1}U_k^r(x-y)$ belongs to $\mathcal{S}$.

As (f_j) converges to f in $\mathcal{S}'$, according to Fatou's Lemma, for all $r>0$, we have for $p<\infty$

$$\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}^p \leq \liminf_{j\to\infty} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f_j)\|_{q'}^p.$$

Therefore,

$$\left(\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}^p\right)^{\frac{1}{p}} \leq \liminf_{j\to\infty} \left(\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f_j)\|_{q'}^p\right)^{\frac{1}{p}}$$

$$\begin{aligned} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left(\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q'}^p\right)^{\frac{1}{p}} &\leq \liminf_{j\to\infty} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left(\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f_j)\|_{q'}^p\right)^{\frac{1}{p}} \\ &\leq \liminf_{j\to\infty} \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left(\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^r\mathcal{F}f_j)\|_{q'}^p\right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$\|f\|_{X(q,p,\alpha)} \leq \liminf_{j\to\infty} \|f_j\|_{X(q,p,\alpha)}.$$

For $p=\infty$ see ([15]). ☐

The following result shows that the space $X(q,p,\alpha)$ is translation invariant.

Proposition 3.12. *Suppose that $1\leq q\leq\alpha\leq p\leq\infty$. Then for any $y\in\mathbb{R}^d$ and $f\in X(q,p,\alpha)$, we have*

$$\|\tau_y f\|_{X(q,p,\alpha)} = \|f\|_{X(q,p,\alpha)}.$$

Proof. Let $y\in\mathbb{R}^d$ and $f\in X(q,p,\alpha)$, we have

$$\begin{aligned} \|\tau_y f\|_{X(q,p,\alpha)} &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|\left\|\mathcal{F}^{-1}(U_k^r\mathcal{F}(\tau_y f))\right\|_{q'}\right\|_p \\ &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|\left\|\mathcal{F}^{-1}(U_k^r e^{-i(.)y}\mathcal{F}f)\right\|_{q'}\right\|_p \\ &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|\left\|\mathcal{F}^{-1}(e^{-i(.)y}U_k^r\mathcal{F}f)\right\|_{q'}\right\|_p \\ &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|\left\|\tau_y\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right\|_{q'}\right\|_p \\ &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|\left\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\right\|_{q'}\right\|_p \text{ because } \tau_y \text{ is translation invariant in } L^{q'}. \end{aligned}$$

Therefore,

$$\|\tau_y f\|_{X(q,p,\alpha)} = \|f\|_{X(q,p,\alpha)}.$$

☐

We prove the following convolution inequality.

Proposition 3.13. *Suppose that $1 \leq q \leq \alpha \leq p \leq \infty$. Then there exists C such that for any $g \in L^1$ and $f \in X(q,p,\alpha)$, we have*

$$\|g * f\|_{X(q,p,\alpha)} \leq C\|g\|_1 \|f\|_{X(q,p,\alpha)}.$$

Proof. Let $g \in L^1$ and $f \in X(q,p,\alpha)$, we have

$$\begin{aligned}
\|g * f\|_{X(q,p,\alpha)} &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| \left\| \mathcal{F}^{-1}(U_k^r \mathcal{F}(g * f)) \right\|_{q'} \right\|_p \\
&= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| \left\| \mathcal{F}^{-1}(U_k^r \mathcal{F}g\mathcal{F}f) \right\|_{q'} \right\|_p \\
&= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| \left\| \mathcal{F}^{-1}(U_k^r \mathcal{F}f) * g \right\|_{q'} \right\|_p \quad \text{Young's inequality in } L^{q'} \\
&\leq C\|g\|_1 \|f\|_{X(q,p,\alpha)}.
\end{aligned}$$

Therefore,

$$\|g * f\|_{X(q,p,\alpha)} \leq C\|g\|_1 \|f\|_{X(q,p,\alpha)}.$$

□

Remark 3.14. The above proposition proves that the spaces $X(q,p,\alpha)$ is an L^1-module. Therefore constitute a good framework to study the continuity of potentials whose kernel belongs to L^1. As an example of an operator whose kernel belongs to L^1, we have Bessel's operator (see [20]). Indeed, if T is an operator such that $T(f) = k * f$ with $k \in L^1$ then T is bounded in $X(q,p,\alpha)$.

Proposition 3.15. *Let $1 \leq q \leq \alpha \leq p \leq \infty$, $1 \leq q_1 \leq \alpha_1 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq \alpha_2 \leq p_2 \leq \infty$ satisfy*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},\ \frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \ \text{ and } \ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then there exists a positive constant K such that, for any $f \in X(q_1,p_1,\alpha_1)$ and $g \in X(q_2,p_2,\alpha_2)$

$$\|f * g\|_{X(q,p,\alpha)} \leq K\|f\|_{X(q_1,p_1,\alpha_1)} \|g\|_{X(q_2,p_2,\alpha_2)}.$$

Proof. Let $f \in X(q_1,p_1,\alpha_1)$ and $g \in X(q_2,p_2,\alpha_2)$.

It is clear that for any $r > 0$, $U_k^r = U_k^r \sum\limits_{l\in L_k} U_l^r$ where $L_k = \left\{l \in \mathbb{Z}^d,\ \ \mathrm{supp}(U_k^r) \cap \mathrm{supp}(U_l^r) \neq \emptyset\right\}$.

$$\begin{aligned}
\|\mathcal{F}^{-1}(U_k^r\mathcal{F}(f*g))\|_{q'} &= \left\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f) * \mathcal{F}^{-1}\left(\sum_{l\in L_k} U_l^r\mathcal{F}g\right)\right\|_{q'} \\
&\leq C\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}\left\|\sum_{l\in L_k}\mathcal{F}^{-1}(U_l^r\mathcal{F}g)\right\|_{q_2'} \quad \text{(Young Inequality)} \\
&\leq C\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}\sum_{l\in L_k}\|\mathcal{F}^{-1}(U_l^r\mathcal{F}g)\|_{q_2'} \quad \text{(Minkowski Inequality)} \\
\|\mathcal{F}^{-1}(U_k^r\mathcal{F}(f*g))\|_{q'}^p &\leq C^p(\#L_k)^{\frac{p}{p'}}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}^p\sum_{l\in L_k}\|\mathcal{F}^{-1}(U_l^r\mathcal{F}g)\|_{q_2'}^p \quad \text{(Hölder Inequality)} \\
\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}(f*g))\|_{q'}^p &\leq C^p(\#L_k)^{\frac{p}{p'}}\left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}^{p_1}\right)^{\frac{p}{p_1}}\left[\sum_{k\in\mathbb{Z}^d}\sum_{l\in L_k}\|\mathcal{F}^{-1}(U_l^r\mathcal{F}g)\|_{q_2'}^{p_2}\right]^{\frac{p}{p_2}} \\
&\leq C^p(\#L_k)^{\frac{p}{p'}+\frac{p}{p_1}}\left(\sum_{k\in\mathbb{Z}^d}\|\mathcal{F}^{-1}(U_k^r\mathcal{F}f)\|_{q_1'}^{p_1}\right)^{\frac{p}{p_1}}\left[\sum_{l\in\mathbb{Z}^d}\sum_{k\in L_l}\|\mathcal{F}^{-1}(U_l^r\mathcal{F}g)\|_{q_2'}^{p_2}\right]^{\frac{p}{p_2}} \\
{}_r\|f*g\|_{X(q,p)} &\leq C^p(\#L_k)^{\frac{p}{p'}+\frac{p}{p_1}+\frac{p}{p_2}}\,{}_r\|f\|_{X(q_1,p_1)}\,{}_r\|g\|_{X(q_2,p_2)}
\end{aligned}$$

Therefore,

$$\|f*g\|_{X(q,p,\alpha)} \leq K\|f\|_{X(q_1,p_1,\alpha_1)}\|g\|_{X(q_2,p_2,\alpha_2)}$$

where $K = C3^{d(1+\frac{1}{p'})}$. □

4. Preliminary result on Besov spaces

In this section we define, the Besov space and high ligth some of its relations with other spaces.

a) The function $t \longmapsto \delta_0(t) = \begin{cases} e^{-\frac{1}{4\left(\frac{1}{8}-t\right)\left(\frac{1}{8}+t\right)}} & \text{if } -\dfrac{1}{8} < t < \dfrac{1}{8} \\ 0 & \text{otherwise} \end{cases}$

belongs to $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R})$ and verifies

$$\delta_0(t) \geq 0, \quad supp(\delta_0) = \left[-\frac{1}{8}, \frac{1}{8}\right], \quad \int_{\mathbb{R}}\delta_0(t)dt = a_0 > 0.$$

Consequently $\delta = a_0^{-1}\delta_0$ belongs to $\mathcal{C}^\infty$ and verifies

$$\delta(t) \geq 0, \quad supp(\delta) = \left[-\frac{1}{8}, \frac{1}{8}\right], \quad \int_{\mathbb{R}}\delta(t)dt = 1.$$

Let be $\Delta = \frac{1}{2}\delta\left(\frac{1}{2}.\right)$, $v_g = \chi_{[1,2]} * \delta$ and $v_d = \chi_{[1,2]} * \Delta$.

It is clear that v_g and v_d belongs to $\mathcal{C}^\infty$ and verifies:

$$\chi_{\left[1+\frac{1}{8},2-\frac{1}{8}\right]} \leq v_g \leq \chi_{\left[1-\frac{1}{8},2+\frac{1}{8}\right]}.$$

$$1-\frac{1}{8}\leq t\leq 1+\frac{1}{8} \implies v_g(t)=\int_{-\frac{1}{8}}^{t-1}\delta(s)ds$$

$$2-\frac{1}{8}\leq t\leq 2+\frac{1}{8} \implies v_g(t)=\int_{t-2}^{\frac{1}{8}}\delta(s)ds$$

$$\chi_{\left[1+\frac{1}{4},2-\frac{1}{4}\right]}\leq v_d\leq \chi_{\left[1-\frac{1}{4},2+\frac{1}{4}\right]}.$$

$$1-\frac{1}{4}\leq t\leq 1+\frac{1}{4} \implies v_d(t)=\int_{-\frac{1}{4}}^{t-1}\Delta(s)ds$$

$$2-\frac{1}{4}\leq t\leq 2+\frac{1}{4}\implies\begin{cases} v_d(t)=\displaystyle\int_{t-2}^{\frac{1}{4}}\Delta(s)ds\ (*) \\ v_g\left(\dfrac{t}{2}\right)=\displaystyle\int_{-\frac{1}{8}}^{\frac{t}{2}-1}\delta(s)ds=\int_{-\frac{1}{8}}^{\frac{t}{2}-1}2\Delta(2s)ds=\int_{-\frac{1}{4}}^{t-2}\Delta(s)ds\ (**)\end{cases}$$

Summing $(*)$ and $(**)$, we get:

$$v_d(t)+v_g\left(\frac{t}{2}\right)=\int_{-\frac{1}{4}}^{\frac{1}{4}}\Delta(s)ds=1.$$

Noting that v_d and v_g coincide on $\left[1+\frac{1}{4},2-\frac{1}{4}\right]$, we see that the function

$$t\longmapsto v(t)=\begin{cases} v_g(t)\ \text{ for }\ t\leq 2-\dfrac{1}{4} \\ v_d(t)\ \text{ for }\ t\geq 1+\dfrac{1}{4}\end{cases}$$

is of class $\mathcal{C}^\infty$ and verifies

$$\begin{cases}\chi_{\left[1+\frac{1}{8},2-\frac{1}{4}\right]}\leq v\leq \chi_{\left[1-\frac{1}{8},2+\frac{1}{4}\right]} \\ v(t)+v\left(\dfrac{t}{2}\right)=1,\quad t\in\left[2-\dfrac{1}{4},2+\dfrac{1}{4}\right]\end{cases}$$

b) For any element x of $\mathbb{R}^d$, we let

$$\begin{cases} V(x)=v(|x|) \\ V_n(x)=V(2^{-n}x)\ \text{ for }\ n\geq 1 \\ V_0=1-\displaystyle\sum_{n\geq 1}V_n\end{cases}$$

We notice that V is in $\mathcal{C}^\infty=\mathcal{C}^\infty(\mathbb{R}^d)$ and verifies

$$\begin{cases}\chi_{\left[\overline{B}\left(0,2\left(1-\frac{1}{8}\right)\right)\setminus B\left(0,1+\frac{1}{8}\right)\right]}\leq V\leq \chi_{\left[\overline{B}\left(0,2\left(1+\frac{1}{8}\right)\right)\setminus B\left(0,1-\frac{1}{8}\right)\right]} \\ V(x)+V\left(\dfrac{x}{2}\right)=1\ \text{ for }\ x\in\overline{B}\left(0,2\left(1+\dfrac{1}{8}\right)\right)\setminus B\left(0,2\left(1-\dfrac{1}{8}\right)\right)\end{cases}$$

Consequently, $(V_n)_{n\geq 0}$ is a sequence of functions belonging $\mathcal{C}^\infty$ such that

$$\chi_{\overline{B}\left(0,2\left(1-\frac{1}{8}\right)\right)}\leq V_0\leq \chi_{\overline{B}\left(0,2\left(1+\frac{1}{8}\right)\right)}$$

$$\begin{cases} \chi_{\left[\overline{B}\left(0,2^{n+1}\left(1-\frac{1}{8}\right)\right)\setminus B\left(0,2^{n}\left(1+\frac{1}{8}\right)\right)\right]} \leq V_n \leq \chi_{\left[\overline{B}\left(0,2^{n+1}\left(1+\frac{1}{8}\right)\right)\setminus B\left(0,2^{n}\left(1-\frac{1}{8}\right)\right)\right]} \\ V_n(x) + V_n\left(\dfrac{x}{2}\right) = 1 \text{ for } x \in \overline{B}\left(0, 2^{n+1}\left(1+\dfrac{1}{8}\right)\right) \setminus B\left(0, 2^{n+1}\left(1-\dfrac{1}{8}\right)\right), \quad n \geq 1 \end{cases}$$

$$\sum_{n\geq 0} V_n(x) = 1, \quad x \in \mathbb{R}^d.$$

Thus, $(V_n)_{n\geq 1}$ is a partition of unity, each element of which belongs to $\mathcal{C}^\infty$ and with compact support (crown).

Definition 4.1. Suppose that: $s \in \mathbb{R}$ and $0 < q, p \leq \infty$. We define

$$\|f\|_{B^s_{q,p}} = \begin{cases} \left[\sum_{n\geq 0} \left(2^{ns}\|\mathcal{F}^{-1}(V_n\mathcal{F}f)\|_q\right)^p\right]^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{n\geq 0} 2^{ns}\|\mathcal{F}^{-1}(V_n\mathcal{F}f)\|_q & \text{if } p = \infty \end{cases}$$

where $f \in \mathcal{S}'$ and

$$B^s_{q,p} = \left\{ f \in \mathcal{S}' \ / \ \|f\|_{B^s_{q,p}} < \infty \right\}.$$

Remark 4.2. We find that, for any integer $n \geq 1$,

$$\overline{B}(0, 2^{n+1}) \subset \bigcup_{k\in K^e(n)} P^1_k$$

$$\begin{aligned} \text{with } k \in K^e(n) &\iff -2^{n+1} \leq 2(k_j+1) + \frac{1}{2} \text{ and } 2k_j - \frac{1}{2} \leq 2^{n+1}, \quad j \in \{1,2,...,d\} \\ k \in K^e(n) &\implies -2^n - 1 \leq k_j \leq 2^n, \quad j \in \{1,2,...,d\} \end{aligned}$$

$$\overline{B}(0, 2^n\frac{7}{8}) \supset \bigcup_{k\in K^i(n)} P^1_k$$

$$\begin{aligned} \text{with } k \in K^i(n) &\iff -2^n\frac{7}{8} \leq 2k_j - \frac{1}{2} \text{ and } 2(k_j+1) + \frac{1}{2} \leq 2^n\frac{7}{8}, \quad j \in \{1,2,...,d\} \\ k \in K^i(n) &\implies -2^{n-1} \leq k_j \leq 2^{n-1} - 1, \quad j \in \{1,2,...,d\} \end{aligned}$$

and therefore

$$\overline{B}\left(0, 2^{n+1}\left(1+\frac{1}{8}\right)\right) \setminus B\left(0, 2^n\left(1-\frac{1}{8}\right)\right) \subset \bigcup_{k\in K(n)} P^1_k$$

with

$$\begin{cases} K(n) = K^e(n) \setminus K^i(n) \\ \#K(n) = [2(2^n+1)]^d - [2(2^{n-1})]^d \leq 2^{(n-1)d}(2^{nd}-1) \end{cases}$$

Moreover,

$$\overline{B}(0,2) \subset \bigcup_{k\in K(0)} P^1_k \text{ with } k \in K(0) \iff -2 \leq 2(k_j+1) + \frac{1}{2} \text{ and } 2k_j - \frac{1}{2} \leq 2$$

$$k \in K(0) \implies -2 \leq k_j \leq 1, \quad j \in \{1,2,...,d\} \implies \#K(0) = 4^d.$$

Thus, we have

$$V_n = \sum_{k \in K(n)} U_k^1 V_n, \qquad \# K(n) \lesssim 2^{nd}$$

with

$$K(n) = \left\{ k \in \mathbb{Z}^d : P_k^1 \cap \overline{B}\left(0, 2^{n+1}\left(1+\frac{1}{8}\right)\right) \setminus B\left(0, 2^n\left(1-\frac{1}{8}\right)\right) \neq \emptyset \right\}.$$

Since there is a strong similarity between the Besov space and the space $X(q,p,\alpha)$. The following proposition compares the two spaces.

Proposition 4.3. *For any element* (q,p) *of* $((0,\infty])^2$, *there is a real number* M *(depending only on* d *and* p*) such that:*

$$\|f\|_{B_{q',p}^{-\frac{d}{p'}}} \leq M \|f\|_{X(q,p)}, \quad f \in \mathcal{S}'.$$

and therefore $X(q,p)$ *continuously injects into* $B_{q',p}^{-\frac{d}{p'}}$.

Proof.

Suppose that $p < \infty$.

For any element f of $\mathcal{S}'$, we have

$$\begin{aligned}
\|f\|_{B_{q',p}^{-\frac{d}{p'}}} &= \left[\sum_{n\geq 0}\left(2^{-n\frac{d}{p'}}\left\|\mathcal{F}^{-1}(V_n\mathcal{F}f)\right\|_{q'}\right)^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{n\geq 0}\left(2^{-n\frac{d}{p'}}\left\|\sum_{k\in K(n)}\mathcal{F}^{-1}(V_nU_k^1\mathcal{F}f)\right\|_{q'}\right)^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{n\geq 0}\left\{2^{-n\frac{d}{p'}}\sum_{k\in K(n)}\left\|(\mathcal{F}^{-1}V_n) * \mathcal{F}^{-1}(U_k^1\mathcal{F}f)\right\|_{q'}\right\}^p\right]^{\frac{1}{p}}
\end{aligned}$$

By using Young inequality and Hölder inequality

$$\begin{aligned}
\|f\|_{B_{q',p}^{-\frac{d}{p'}}} &\lesssim \left[\sum_{n\geq 0}\left(2^{-n\frac{d}{p'}}\|\mathcal{F}^{-1}V_n\|_1\right)^p \#[K(n)]^{\frac{p}{p'}}\sum_{k\in K(n)}\|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} \\
&\lesssim \left[\sum_{n\geq 0}\left(2^{-n\frac{d}{p'}}\|\mathcal{F}^{-1}V_n\|_1 2^{\frac{nd}{p'}}\right)^p\sum_{k\in K(n)}\|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} \\
&\lesssim \|\mathcal{F}^{-1}V\|_1\left[\sum_{n\geq 0}2^{np\left(-\frac{d}{p'}+\frac{d}{p'}\right)}\sum_{k\in K(n)}\|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} \quad \left(\text{because } \|\mathcal{F}^{-1}V_n\|_1 = \|\mathcal{F}^{-1}V\|_1\right)
\end{aligned}$$

and therefore,

$$\begin{aligned}
\|f\|_{B^{-\frac{d}{p'}}_{q',p}} &\lesssim \left[\sum_{n\geq 0}\sum_{k\in K(n)} \|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}^p\right]^{\frac{1}{p}} \lesssim \|f\|_{X(q,p)} \quad \left(\text{because } \sup_{k\in\mathbb{Z}^d}[\#\{n\geq 0,\ \ k\in K(n)\}] < \infty\right).
\end{aligned}$$

For $p=\infty$

$$\begin{aligned}
\|\|_{B^{-d}_{q',\infty}} &= \sup_{n\geq 0} 2^{-nd}\|\mathcal{F}^{-1}(V_n\mathcal{F}f)\|_{q'} \\
&= \sup_{n\geq 0} 2^{-nd}\left\|\sum_{k\in K(n)} \mathcal{F}^{-1}(V_nU_k^1\mathcal{F}f)\right\|_{q'} \\
&\leq \sup_{n\geq 0} 2^{-nd}\sum_{k\in K(n)} \|\mathcal{F}^{-1}(V_n) * \mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'} \\
&\lesssim \sup_{n\geq 0} 2^{-nd}\|\mathcal{F}^{-1}V_n\|_1 \sum_{k\in K(n)} \|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'} \\
&\lesssim \|\mathcal{F}^{-1}V\|_1 \sup_{n\geq 0} 2^{-nd}2^{nd}\left\|\|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}\right\|_\infty \\
&\lesssim \|\mathcal{F}^{-1}V\|_1 \sup_{n\geq 0} 2^{n(-d+d)}\left\|\|\mathcal{F}^{-1}(U_k^1\mathcal{F}f)\|_{q'}\right\|_\infty .
\end{aligned}$$

And so

$$\|f\|_{B^{-d}_{q',\infty}} \lesssim \|f\|_{X(q,\infty)}.$$

□

From Proposition 3.9 and Proposition 4.3, we deduce the following result.

Corollary 4.4. *Suppose that $1\leq q\leq \alpha\leq p\leq\infty$ and $q\leq 2$, there exists a constant K depending only on d,p,q such that:*

$$\|\mathcal{F}f\|_{B^{-\frac{d}{p'}}_{q',p}} \leq K\|f\|_{q,p,\alpha}, \quad f\in (L^q,l^p)^\alpha.$$

Proof. It is immediate from the Proposition 3.9 and Proposition 4.3. □

Remark 4.5. The previous corollary allows us to find Theorem 4 and Corollary 5 of ([19]) in the framework of Fofana spaces by just taking $<z>=c$.

Now we will show the completeness of the space $X(q,p,\alpha)$.

Proposition 4.6. *For $1\leq q\leq\alpha\leq p\leq\infty$, $\left(X(q,p,\alpha),\|.\|_{X(q,p,\alpha)}\right)$ is a banach space.*

Proof. a) $(X(q,p,\alpha),\|.\|_{X(q,p,\alpha)})$ is a normed vector space by Proposition 3.6

b) We prove the completeness.

Let $(f_n)_{n\geq 1}$ be a Cauchy sequence. Since, we have $X(q,p,\alpha) \hookrightarrow B^{-\frac{d}{p'}}_{q',p}$. Then the sequence $(f_n)_{n\geq 1}$ converges to an element f of $B^{-\frac{d}{p'}}_{q',p}$ in the sense of $\mathcal{S}'$.

Thus, using Fatou's property for the space $X(q,p,\alpha)$, we get

$$\|f - f_n\|_{X(q,p,\alpha)} \leq \liminf_{m\to\infty} \|f_m - f_n\|_{X(q,p,\alpha)}, \quad n \geq 1.$$

Let $\varepsilon > 0$. There exists n_ε, $n \geq n_\varepsilon$ and $m \geq m_\varepsilon$ then $\|f_m - f_n\|_{X(q,p,\alpha)} < \varepsilon$.

Therefore for any $n \geq n_\varepsilon$

$$\liminf_{m\to\infty} \|f_m - f_n\|_{X(q,p,\alpha)} \leq \varepsilon.$$

Therefore, we have

$$\|f - f_n\|_{X(q,p,\alpha)} \leq \varepsilon, \quad n \geq n_\varepsilon.$$

Hence the space $X(q,p,\alpha)$ is complete.

□

Proposition 4.7. *Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$ *and* $f \in X(q,p,\alpha)$. *Then,* $\mathcal{S} \hookrightarrow X(q,p,\alpha)$.

Proof. Let $1 \leq q \leq \alpha \leq p \leq \infty$, N be a positive integer and $f \in \mathcal{S}$.

By the relation (1.1), we have:

$$\begin{aligned} (1+|x|)^N |f(x)| &\leq P_N(f), \quad x \in \mathbb{R}^d. \text{ Therefore} \\ |f(x)| &\leq (1+|x|)^{-N} P_N(f), \quad x \in \mathbb{R}^d. \text{ And thus} \\ \|f\|_{X(q,\infty,\alpha)} &\leq P_N(f) \|(1+|.|)^{-N}\|_{X(q,\infty,\alpha)}. \end{aligned}$$

Using Proposition 3.4, we have

$$\|f\|_{X(q,\infty,\alpha)} \leq P_N(f) \|\mathcal{F}^{-1}U\|_1 \|(1+|.|)^{-N}\|_{\alpha'}.$$

Since $\|(1+|.|)^{-N}\|_{\alpha'} = C(d,N,\alpha')$, whenever $N\alpha' > d$.

Therefore

$$\|f\|_{X(q,\infty,\alpha)} \leq C P_N(f), \quad N \geq \frac{d}{\alpha'}.$$

Thus, $\mathcal{S} \hookrightarrow X(q,p,\alpha)$. □

Proposition 4.8. *suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$ *and* $f \in X(q,p,\alpha)$. *Then,* $X(q,p,\alpha) \hookrightarrow \mathcal{S}'$.

Proof. Let $1 \leq q \leq \alpha \leq p \leq \infty$. We know that $X(q,p,\alpha) \hookrightarrow B^{-\frac{d}{p'}}_{q',p}$ (see Proposition 4.3) moreover, $B^{-\frac{d}{p'}}_{q',p} \hookrightarrow \mathcal{S}'$ (see [22]) . Hence the result. □

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