Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 1 (2025) 190-221 https://doi.org/10.56947/gjom.v20i.2856

ON KATUGAMPOLA-PRABHAKAR FRACTIONAL INTEGRAL-DIFFERENTIAL OPERATORS

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Abstract. This work systematically investigates the Katugampola-Prabhakar (K-P) fractional operators, introducing rigorous definitions for K-P integrals and integro-differential operators while establishing their fundamental properties. We develop analytical solutions to Cauchy problems for fractional ordinary differential equations incorporating K-P and Caputo-type K-P derivatives, expressed through bivariate Mittag-Leffler functions $E_2(x, y)$. The study further advances operational calculus for these operators by deriving m_p-Laplace transform relations applicable to: (i) infinite series representations, (ii) K-P integrals, (iii) left-sided K-P derivatives, and (iv) Caputo-type K-P operators. As a key application, we analyze an initial-boundary value problem for a subdiffusion equation governed by Caputo-type K-P derivatives, demonstrating the practical utility of these theoretical developments. The obtained results provide new analytical methods for studying fractional differential equations with K-P operators and their applications to subdiffusion processes. The developed techniques significantly expand the possibilities for solving boundary value problems involving these generalized fractional operators.

1. Introduction

The field of fractional calculus has experienced significant growth in recent years, with researchers developing increasingly sophisticated operators to model complex phenomena across scientific disciplines [39]. This expansion comes from fractional models' ability to capture non-local effects and memory properties that classical integer-order models cannot represent [37].

1.1. **Theoretical Foundations and Applications Across Disciplines.**

The mathematical theory of fractional operators has been established through several landmark works:

- Fundamental theories of fractional integrals and derivatives were presented in [39]
- Comprehensive treatments of fractional differential equations appear in [37] and [21]

Date: Received: Apr 11, 2025; Accepted: May 7, 2025.

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2020 *Mathematics Subject Classification.* 26A33; 34A08; 35R11.

Key words and phrases. Katugampola-Prabhakar fractional integral, Katugampola-Prabhakar fractional derivatives, Mittag-Leffler type function, Generalized fractional integrals, Generalized fractional derivatives.

Fractional calculus has found applications in diverse fields: Viscoelastic materials [27], Nanotechnology systems [3], Fractional dynamical systems [42],Continuum mechanics [2], Financial modeling [22], Biological processes [6], Anomalous diffusion on hyperspheres [7],Nonlinear fractional equations [29].

1.2. Operator Generalizations.

Researchers have developed numerous extensions to classical fractional operators. Namely, Prabhakar-type operators originally formulated by [20], Hilfer-Prabhakar variants [12], k-parameter extensions [35] (k, s)-generalizations [40], q-analogs [32], Ψ-operators [8, 41, 19, 38].

Katugampola-type operators were introduced in [17], subsequent developments can be found in [18, 1, 31]. For more applications we refer to [28, 5, 25, 23, 24, 4, 30, 34].

1.3. Current Research Challenges and originality of presented outcomes.

Despite progress, significant limitations remain:

- Solutions often require infinite series with poorly characterized special functions;
- Key properties (bounds, zeros, monotonicity) remain unverified for many functions;
- General methods for proving uniqueness are lacking.

Recent particular solutions include: Analysis of bivariate Mittag-Leffler functions [15, 16], Spectral methods for uniqueness proofs [16].

This work makes three primary advances:

- Introduces new K-P fractional operators with complete characterization;
- Develops m_p-Laplace transform methods for series solutions;
- Provides general existence/uniqueness proofs for: Initial value problems for Subdiffusion equations.

2. Preliminaries

Here, we recall some special functions, fractional integrals, and derivatives, a specific class of functions, and rewrite their auxiliary properties.

Definition 2.1. [21] Let α, β be positive real numbers, and $\gamma, x \in \mathbb{R}$. Then the function

$$E_{\alpha,\beta}^{\gamma}(x) := \sum_{k=0}^{+\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k+\beta)} x^k$$

is called the Prabhakar function, where $(\gamma)_k = \dfrac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ is the Pochhammer symbol.

Definition 2.2. [9] The bivariate Mittag-Leffler function $E_2(x, y)$ is defined by

$$E_2(x,y) = E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right) =$$

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m+\beta_2 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_3+\beta_3 n)}.$$

Here $\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{R},\ \min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3\}>0\,.$

Let us give a short description of specific classes of functions as follows:

$$AC^n_{\delta_p}[a,b]:=\left\{f:[a,b]\to R:\quad \delta_p^{n-1}f(x)\in AC[a,b]\ ,\ \delta_p=x^{1-p}\frac{d}{dx}\right\}\ ,$$

$$AC[a,b]:=\left\{f:[a,b]\to R:\ f(x)=c+\int\limits_a^x\varphi(t)dt,\ \varphi(t)\in L_1(a,b)\right\},$$

$$L_p(a,b)=\left\{\|f\|_p=\left(\int\limits_a^b|f(t)|^pdt\right)^{\frac{1}{p}},\ 1\leqslant p\leqslant+\infty\,,\ -\infty\leqslant a,b\leqslant+\infty\right\}.$$

Here $AC[a,b]$ is a class of absolutely continuous functions and $L_p(a,b)$ is a class of Lebesgue measurable functions [21] and $AC^n_{\delta_p}[a,b]$ class of the functions defined in [23].

Let us introduce a new class of functions:

Definition 2.3. A space of the functions $L_1^p(0,b)$ is defined as follows:

$$L_1^p(0,b)=\{u:(0,b)\to R,\ \ u(t)t^{p-1}\in L_1(0,b),\ \ u(t)t^{p-1}=g'(t),\\ g(t)\in AC^1_{\delta_p}(0,b]\},$$

$$\|f(t)\|_{L_1^p(0,b)}:=\int\limits_0^b\left|t^{p-1}f(t)\right|dt\,,\quad(0<p<+\infty,\ 0<b<+\infty)\,.$$

If $p=1$, then $L_1^p(0,b)=L_1(0,b)$.

Definition 2.4. [39] Let $f(x)\in L_1\left(a,b\right)$. The integrals

$${}^{RL}I^\alpha_{a+}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^x\frac{f\left(t\right)}{\left(x-t\right)^{1-\alpha}}dt,\qquad a<x,$$

$${}^{RL}I^\alpha_{b-}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_x^b\frac{f\left(t\right)}{\left(t-x\right)^{1-\alpha}}dt,\qquad x<b$$

are called the left and right-sided Riemann-Liouville fractional integrals of order $\alpha>0$.

Definition 2.5. [39] Let $\alpha>0,\ 0\le a<b\le+\infty$, and $f(x)\in L_1\left(a,b\right)$. The integrals

$${}_HI^\alpha_{a+}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^x\left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t},\qquad a<x,$$

$$_{H}I_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad x < b$$

are called the left and right-sided Hadamard fractional integrals of order α, where $\log \frac{x}{t} = \ln \frac{x}{t}$.

Definition 2.6. [20] Let $\alpha, \beta \in \mathbb{R}^{+}$, $\gamma, \delta \in \mathbb{R}$, $f(x) \in L_1(a,b)$, for $0 \leq a < b \leq +\infty$. The integrals

$$_{P}I_{a+}^{\alpha,\beta,\gamma,\delta}f(x) := \int_{a}^{x} (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta(x-t)^{\alpha}\right) f(t)\, dt, \;\; x > a,$$

$$_{P}I_{b-}^{\alpha,\beta,\gamma,\delta}f(x) := \int_{x}^{b} (t-x)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta(t-x)^{\alpha}\right) f(t)\, dt, \;\; x < b$$

are called the left and right-sided Prabhakar fractional integrals, where β is the order of the integral operators.

Definition 2.7. [23] Let $\alpha, p \in \mathbb{R}^{+}$, $f(x) \in L_1(a,b)$. These integrals

$$_{K}I_{a+}^{\alpha,p}f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} \frac{t^{\rho-1} f(t)}{(x^{p}-t^{p})^{1-\alpha}} dt, \;\; a < x,$$

$$_{K}I_{b-}^{\alpha,p}f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{b} \frac{t^{\rho-1} f(t)}{(t^{p}-x^{p})^{1-\alpha}} dt, \;\; x < b$$

are the left and right-sided Katugampola fractional integrals, and α is the order of the integral operators.

Definition 2.8. [17] Let $p > 0$, $\alpha \in \mathbb{R}$, $n-1 < \alpha < n, n = [\alpha]+1$, and $x \in [a,b]$, $p \in \mathbb{R}$. The derivatives

$$_{K}D_{a+}^{\alpha,p}f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-p}\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{t^{\rho-1} f(t)}{(x^{p}-t^{p})^{\alpha-n+1}} dt, \; x > a,$$

$$_{K}D_{b-}^{\alpha,p}f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-x^{1-p}\frac{d}{dx}\right)^{n} \int_{x}^{b} \frac{t^{\rho-1} f(t)}{(t^{p}-x^{p})^{\alpha-n+1}} dt, \; x < b$$

are called the left and right-sided Katugampola fractional derivatives, and α is the order of the derivatives.

Definition 2.9. [43] The function $f(t)$ is completely monotonic in (a,b), if and only if, $f(-t)$ is absolutely monotonic in $(-a,-b)$ and it satisfies the following inequalities:

$$(-1)^{n} f^{(n)}(t) \geqslant 0 \;, \quad n = 0, 1, 2, \ldots$$

Definition 2.10. [26] Let $f : [a,+\infty) \to \mathbb{R}\,(a \geqslant 0)$ be a real-valuable function for $p > 0$.Then the m_p-Laplace transform of the function $f(t)$ is defined as follows:

$$F(s) = L_{m_p}\{f(t); s\} = \int_a^{+\infty} e^{-s\frac{t^p - a^p}{p}} f(t)t^{p-1}dt.$$

Definition 2.11. [26] For given continuous functions $f(x)$ and $g(x)$ defined on $[a,+\infty)\,(a \geqslant 0)$, the m_p-convolution is defined as

$$(f(x) * g(x))_{m_p} = \int_a^x \left(f(x^p - t^p + a^p)^{\frac{1}{p}} g\,(t)\, t^{p-1}dt\right).$$

Theorem 2.12. [33] *The Prabhakar function* $t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-t^{\alpha})$ *is completely monotonic for* $0 < \alpha \leqslant 1,\quad 1 \geqslant \beta \geqslant \alpha\gamma\ > 0$ *and* $t > 0$.

Theorem 2.13. [23] *If* $n-1 < \alpha < n, n = [a]+1, n \in N, p > 0,\ {}_K I^{n-\alpha,p}_{a+} f(x) \in AC^n_{\delta_p}[a,b]$, *then*

$$_K I^{\alpha,p}_{a+}\, {}_K D^{\alpha,p}_{a+} f(x) = f(x) - \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha - j + 1)} \left(\frac{x^p - a^p}{p}\right)^{\alpha - j} \left[\lim_{x \to a+} \left(x^{1-p}\frac{d}{dx}\right)^{n-j} {}_K I^{n-\alpha,p}_{a+} f(x)\right],$$

$$_K I^{\alpha,p}_{b-}\, {}_K D^{\alpha,p}_{b-} f(x) = f(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}}{\Gamma(\alpha - j + 1)} \left(\frac{b^p - x^p}{p}\right)^{\alpha - j} \left[\lim_{x \to b-} \left(x^{1-p}\frac{d}{dx}\right)^{n-j} {}_K I^{n-\alpha,p}_{b-} f(x)\right].$$

Lemma 2.14. [23] *Let* $0 \leqslant a < b < +\infty,\ p > 0$ *and* $n \in \mathbb{N}$. *The space* $AC^n_{\delta_p}[a,b]$ *consist of those and only those functions* $f(t)$ *which can be represented in the form of*

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{\Gamma\,(k+1)} \left(\frac{x^p - a^p}{p}\right)^k + \frac{1}{\Gamma\,(n)} \int_a^x \left(\frac{x^p - s^p}{p}\right)^{n-1} \varphi(s)ds \tag{2.1}$$

or

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{\Gamma(k+1)} (-1)^k \left(\frac{b^p - x^p}{p}\right)^k + \frac{(-1)^{n-1}}{\Gamma\,(n)} \int_a^x \left(\frac{s^p - x^p}{p}\right)^{n-1} \varphi(s)ds\,, \tag{2.2}$$

where $c_k\,(k = 0, 1, ..., n-1)$ *are arbitrary real constants and* $\varphi \in L_1[a,b]$.

Lemma 2.15. [43] *If $f(t)$ is completely monotonic on $c<t<+\infty$, then for any number $a>c$ the sequences*

$$\left\{(-1)^n f^{(n)}(a)\right\}_{n=0}^{+\infty}, \quad \left\{(-1)^n f^{(n)}(a)\right\}_{n=1}^{+\infty}$$

are positive.

Lemma 2.16. [16] *If $\alpha_3>-\alpha_4,\ \alpha_3>0\,,\ \beta_2>-\beta_3,\ \beta_2>0$, then the following Euler-type integral representation holds true:*

$$E_2\left(\begin{array}{c}\gamma_1,\alpha_1,\beta_1;1,0\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)=$$
$$\frac{1}{\Gamma(\gamma_1)}\int\limits_0^1\int\limits_0^{+\infty}\xi^{\frac{\delta_1}{2}-1}\eta^{\gamma_1-1}(1-\xi)^{\frac{\delta_1}{2}-1}e^{-\eta}e_{\alpha_3,-\alpha_4}^{\frac{\delta_1}{2},\delta_2}\left(x\eta^{\alpha_1}\xi^{\alpha_3}\right)\times$$
$$e_{\beta_2,-\beta_3}^{\frac{\delta_1}{2},\delta_3}\left(y\eta^{\beta_1}(1-\xi)^{\beta_2}\right)d\xi d\eta.$$

Here $e_{\alpha,\beta}^{\mu,\delta}(z)=\sum\limits_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n-\mu)\Gamma(\delta-\beta n)},\ (\alpha>\beta,\ \alpha>0)$ *is the Wright-type function.*

Lemma 2.17. [16] *If all conditions of the Lemma 2.16 and $\frac{\delta_1}{2}>\alpha_3\theta,\ \frac{\delta_1}{2}>\beta_2\theta,\ \gamma_1>\theta\left(\alpha_1+\beta_1\right),\ 0\leqslant\theta<\frac{1}{2}$ are valid, then the following holds true:*

$$\left|E_2\left(\begin{array}{c}\gamma_1,\alpha_1,\beta_1;1,0\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)\right|\leqslant$$
$$\frac{\tilde{C}}{(-x)^{\theta}(-y)^{\theta}}\frac{\Gamma\left(\frac{\delta_1}{2}-\alpha_3\theta\right)\Gamma\left(\frac{\delta_1}{2}-\beta_2\theta\right)\Gamma\left(\gamma_1-\theta\left(\alpha_1+\beta_1\right)\right)}{\Gamma\left(\delta_1-\theta\left(\alpha_3+\beta_2\right)\right)}.$$

In the following sections, we will present the main results related to the K-P fractional integrals and derivatives.

3. Properties of the K-P fractional integral operators

In this section, we define the K-P fractional integral operators and some properties of these operators.

Definition 3.1. Let $\alpha,\beta,p\in\mathbb{R}^+,\ \gamma,\delta\in\mathbb{R}$ and $f(x)\in L_1^p(a,b)$ where $0\leq a<b\leq+\infty$. The integrals

$${}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x):=\int\limits_a^x\left(\frac{x^p-t^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)t^{p-1}f\left(t\right)dt,\ \ x>a,$$

$${}_{KP}I_{b-}^{\alpha,\beta,\gamma,\delta,p}f(x):=\int\limits_x^b\left(\frac{t^p-x^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{t^p-x^p}{p}\right)^{\alpha}\right)t^{p-1}f\left(t\right)dt,\ \ x<b$$

are called the Katugampola-Prabhakar fractional integrals. Here, β is the order of the integrals.

Lemma 3.2. *If $\alpha,\beta,p\in\mathbb{R}^+$ and $\gamma,\delta\in\mathbb{R}$, $f(x)\in L_1^p(a,b)$, then for $x>a$, the following relations are valid:*

$$
\begin{aligned}
{}_{KP}I_{a+}^{\alpha,\beta,0,\delta,1}f(x) &= {}^{RL}I_{a+}^{\beta}f(x)\,,\ where\ p=1,\ \gamma=0,\\
\lim_{p\to+0}{}_{KP}I_{a+}^{\alpha,\beta,0,\delta,p}f(x) &= {}_{H}I_{a+}^{\beta}f(x)\,,\ where\ \gamma=0,\\
{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,1}f(x) &= {}_{P}I_{a+}^{\alpha,\beta,\gamma,\delta}f(x)\,,\ where\, p=1,\\
{}_{KP}I_{a+}^{\alpha,\beta,0,\delta,p}f(x) &= {}_{K}I_{a+}^{\beta,p}f(x)\,,\ where\,\gamma=0,\\
\lim_{p\to+0}{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) &= {}_{HP}I_{a+}^{\alpha,\beta,\gamma,\delta}f(x)\,.
\end{aligned}
$$

Here ${}_{HP}I_{a+}^{\alpha,\beta,\gamma,\delta}f(x)$ is the Hadamard-type Prabhakar fractional integral.

Proof. Here, we show the proof of only the fifth equality based on the locally uniform convergence of the Prabhakar function [20], Definition 3.1, and using the L'Hospital rule. The proofs of the other equalities are easy for the reader to recognize. In addition, the L'Hospital rule is used to calculate the value of the limit when $p\to+0$ in the proof of the second equality.

$$
\lim_{p\to+0}{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) = \lim_{p\to+0}\int_a^x\left(\frac{x^p-t^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)t^{p-1}f(t)\,dt =
$$

$$
\int_a^x\lim_{p\to+0}\left(\frac{x^p-t^p}{p}\right)^{\beta-1}\lim_{p\to+0}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)t^{p-1}f(t)\,dt =
$$

$$
\int_a^x\left(\ln\frac{x}{t}\right)^{\beta-1}\lim_{p\to+0}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\ln\frac{x}{t}\right)^{\alpha}\right)\frac{f(t)}{t}dt = {}_{HP}I_{a+}^{\alpha,\beta,\gamma,\delta}f(x).
$$

□

Lemma 3.3. *If $\alpha,\beta,p,\mu\in\mathbb{R}^+$,$\gamma,\delta\in\mathbb{R}$ and $f(x)\in L_1^p(a,b)$, then the following equality holds between Katugamola and Katugampola-Prabhakar integrals.*

$$
{}_{K}I_{a+}^{\mu,p}\,{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) = {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}\,{}_{K}I_{a+}^{\mu,p}f(x) = {}_{KP}I_{a+}^{\alpha,\beta+\mu,\gamma,\delta,p}f(x)\,. \tag{3.1}
$$

Proof. Using Definition 2.7, Definition 3.1 and Definition 2.1, we rewrite ${}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}\,{}_{K}I_{a+}^{\mu,p}f(x) = {}_{KP}I_{a+}^{\alpha,\beta+\mu,\gamma,\delta,p}f(x)$ as follows:

$$
{}_{K}I_{a+}^{\mu,p}\,{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) =
$$

$$
\frac{1}{\Gamma(\mu)}\int_a^x\left(\frac{x^p-t^p}{p}\right)^{\mu-1}t^{p-1}\int_a^t\left(\frac{t^p-s^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{t^p-s^p}{p}\right)^{\alpha}\right)s^{p-1}f(s)\,ds\,dt =
$$

$$
\frac{1}{\Gamma(\mu)}\int_a^x s^{p-1}f(s)\,ds\int_s^x\left(\frac{t^p-s^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{t^p-s^p}{p}\right)^{\alpha}\right)\left(\frac{x^p-t^p}{p}\right)^{\mu-1}t^{p-1}\,dt.
$$

Now we continue calculating by using the following remarks:
$y=\frac{t^p-s^p}{x^p-s^p}$, $dy=\frac{pt^{p-1}dt}{x^p-s^p}$, $t^p=(x^p-s^p)\,y+s^p$, $t^{p-1}dt=\frac{x^p-s^p}{p}dy$ in the inner

integral of the last line. After certain calculations, we have

$$\int_a^x \left(\frac{x^p-s^p}{p}\right)^{\beta+\mu-1} \sum_{n=0}^{+\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta+\mu)} \frac{\delta^n \left(\frac{x^p-s^p}{p}\right)^{\alpha n}}{n!} f(s)\, s^{p-1} ds =$$

$$\int_a^x \left(\frac{x^p-s^p}{p}\right)^{\beta+\mu-1} E^{\gamma}_{\alpha,(\beta+\mu)} \left(\delta \left(\frac{x^p-s^p}{p}\right)^{\alpha}\right) f(s)\, s^{p-1} ds = {}_{KP}I^{\alpha,\beta+\mu,\gamma,\delta,p}_{a+} f(x).$$

□

Lemma 3.4. *If* $n-1<\beta\leqslant n$, $n=[\eta]+1$, $p>0, \alpha>0$, $\beta>\mu$, $f(x)\in L_1^p(a,b)$, *then*

$$ {}_K D^{\eta,p}_{a+}\, {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = {}_{KP}I^{\alpha,\beta-\eta,\gamma,\delta,p}_{a+} f(x) \ . \tag{3.2}$$

If $\beta>n$, *then*

$$\left(\frac{d}{dx}\right)^n {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = \left(x^{p-1}\right)^n {}_{KP}I^{\alpha,\beta-n,\gamma,\delta,p}_{a+} f(x) \ ,$$

$$\left(x\frac{d}{dx}\right)^n {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = \left(x^{p}\right)^n {}_{KP}I^{\alpha,\beta-n,\gamma,\delta,p}_{a+} f(x) ,$$

$$\left(x^{1-p}\frac{d}{dx}\right)^n {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = {}_{KP}I^{\alpha,\beta-n,\gamma,\delta,p}_{a+} f(x) \ .$$

Proof. Here, we present a proof of Equation (3.2) by using the equality (3.1). The other equalities can be proved similarly.

$$ {}_K D^{\eta,p}_{a+}\, {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = \left(x^{1-p}\frac{d}{dx}\right)^n {}_K I^{n-\mu,p}_{a+}\, {}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) =$$

$$\left(x^{1-p}\frac{d}{dx}\right)^n {}_{KP}I^{\alpha,\beta+n-\mu,\gamma,\delta,p}_{a+} f(x) =$$

$$\left(x^{1-p}\frac{d}{dx}\right)^n \int_a^x \sum_{n=0}^{+\infty} \frac{(\gamma)_n \delta^n \left(\frac{x^p-s^p}{p}\right)^{\alpha n+\beta+n-\mu-1}}{\Gamma(\alpha n+\beta+n-\mu) n!} f(s)\, s^{p-1} ds =$$

$$\left(x^{1-p}\frac{d}{dx}\right)^{n-1}\int\limits_a^x\sum_{n=0}^{+\infty}\frac{(\gamma)_n\delta^n\left(\frac{x^p-s^p}{p}\right)^{\alpha n+\beta+n-\mu-2}}{\Gamma\left(\alpha n+\beta+n-\mu-1\right)n!}f\left(s\right)s^{p-1}ds=$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int\limits_a^x\left(\frac{x^p-s^p}{p}\right)^{\beta-\mu-1}\sum_{n=0}^{+\infty}\frac{(\gamma)_n}{\Gamma\left(\alpha n+\beta-\mu\right)}\frac{\delta^n\left(\frac{x^p-s^p}{p}\right)^{\alpha n}}{n!}f\left(s\right)s^{p-1}ds=$$

$$\int\limits_a^x\left(\frac{x^p-s^p}{p}\right)^{\beta-\mu-1}E^{\gamma}_{\alpha,(\beta-\mu)}\left(\delta\left(\frac{x^p-s^p}{p}\right)^{\alpha}\right)f\left(s\right)s^{p-1}ds={}_{KP}I^{\alpha,\beta-\eta,\gamma,\delta,p}_{a+}f\left(x\right).$$

$\square$

Lemma 3.5. *If $n-1<\beta,\beta_1\leqslant n$, $n=[\beta]+1$, $p>0,\alpha>0,\beta>0$, $\beta>\beta_1,\gamma,\gamma_1\in\mathbb{R}$, then for $f(x)\in L^p_1(a,b)$*

$$_{KP}I^{\alpha,\beta_1,\gamma_1,\delta,p}_{a+}\,{}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+}f(x)={}_{KP}I^{\alpha,\beta+\beta_1,\gamma+\gamma_1,\delta,p}_{a+}f\left(x\right). \tag{3.3}$$

Proof. We present proof of Equation (3.3) by using form of the function $E^{\gamma}_{\alpha,\beta}(x)$ and based on the Definition 3.1, as follows:

$$_{KP}I^{\alpha,\beta_1,\gamma_1,\delta,p}_{a+}\,{}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+}f(x)=\int\limits_a^x\left(\frac{x^p-t^p}{p}\right)^{\beta_1-1}E^{\gamma_1}_{\alpha,\beta_1}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)t^{p-1}\times$$

$$\left[\int\limits_a^t\left(\frac{t^p-s^p}{p}\right)^{\beta-1}E^{\gamma}_{\alpha,\beta}\left(\delta\left(\frac{t^p-s^p}{p}\right)^{\alpha}\right)s^{p-1}f\left(s\right)ds\right]dt$$

$$\int\limits_a^x s^{p-1}f\left(s\right)\left[\int\limits_s^x\left(\frac{x^p-t^p}{p}\right)^{\beta_1-1}\left(\frac{t^p-s^p}{p}\right)^{\beta-1}E^{\gamma_1}_{\alpha,\beta_1}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)\times\right.$$

$$\left.E^{\gamma}_{\alpha,\beta}\left(\delta\left(\frac{t^p-s^p}{p}\right)^{\alpha}\right)t^{p-1}dt\right]ds.$$

If we use the following remarks $y=\frac{t^p-s^p}{x^p-s^p},dy=\frac{pt^{p-1}dt}{x^p-s^p}$, $t^p=\left(x^p-s^p\right)y+s^p$, $t^{p-1}dt=\frac{x^p-s^p}{p}dy$ on the second integral in the last line, and after some simplifications, we obtain

$$\int\limits_a^x\left(\frac{x^p-s^p}{p}\right)^{\beta+\beta_1-1}E^{(\gamma+\gamma_1)}_{\alpha,\beta+\beta_1}\left(\delta\left(\frac{x^p-s^p}{p}\right)^{\alpha}\right)s^{p-1}f\left(s\right)ds=$$

$$_{KP}I^{\alpha,\beta+\beta_1,\gamma+\gamma_1,\delta,p}_{a+}f\left(x\right).$$

$\square$

Lemma 3.6. *If $\alpha,\beta,p,\in\mathbb{R}^{+},\gamma,\delta\in\mathbb{R}$, $0\leq a<b<+\infty$, then the Katugampola-Prabhakar fractional integral operator is bounded on $L^p_1(a,b)$ and the following* *inequality holds:*

$$\left\| {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) \right\|_{L_1^p(a,b)} \le D \| f(x) \|_{L_1^p(a,b)}\,, \tag{3.4}$$

where

$$D = \left(\frac{b^p - a^p}{p} \right)^{\beta} \sum_{k=0}^{+\infty} \frac{|(\gamma)_k|}{|\Gamma(\alpha k + \beta)|\,[\alpha k + \beta]} \frac{\left| \delta \left(\frac{b^p - a^p}{p} \right)^{\alpha} \right|^k}{k!}\,. \tag{3.5}$$

Proof. To prove the inequality (3.4), we check the series (3.5) for convergence. For that we denote by b_k the $k\,th$ term of the series (3.5) as follows

$$b_k = \frac{|(\gamma)_k|}{|\Gamma(\alpha k + \beta)|\,[\alpha k + \beta]} \frac{\left| \delta \left(\frac{b^p - a^p}{p} \right)^{\alpha} \right|^k}{k!}\,,$$

$$b_{k+1} = \frac{\left|(\gamma)_{k+1}\right|}{|\Gamma(\alpha k + \alpha + \beta)|\,[\alpha k + \alpha + \beta]} \frac{\left| \delta \left(\frac{b^p - a^p}{p} \right)^{\alpha} \right|^{k+1}}{(k+1)!},$$

then we use the Ratio (Dalamber) test

$$\lim_{k\to+\infty} \frac{\left|(\gamma)_{k+1}\right| \left| \delta \left(\frac{b^p - a^p}{p} \right)^{\alpha} \right|^{k+1}}{|\Gamma(\alpha k + \alpha + \beta)|\,[\alpha k + \alpha + \beta]\,(k+1)!} \frac{|\Gamma(\alpha k + \beta)|\,[\alpha k + \beta]\ k!}{\left|(\gamma)_k\right| \left| \delta \left(\frac{b^p - a^p}{p} \right)^{\alpha} \right|^{k}} =$$

$$\lim_{k\to+\infty} \frac{|\Gamma(\alpha k + \beta)|}{|\Gamma(\alpha k + \alpha + \beta)|} \left(1 - \frac{\alpha}{[\alpha k + \alpha + \beta]} \right) \frac{|\delta| \left(\frac{b^p - a^p}{p} \right)^{\alpha}}{(k+1)}.$$

We know from [14] that the Gamma function has the following asymptotics:

$$\Gamma(\alpha k + \beta) = \sqrt{2\pi} (\alpha k)^{\alpha k + \beta - \frac{1}{2}} e^{-\alpha k} [1 + o(1)]\,,\ k \to +\infty\,. \tag{3.6}$$

Here $\alpha > 0,\ \beta \in \mathbb{R}$. If we use (3.6), then we obtain

$$\lim_{k\to+\infty} \frac{|\Gamma(\alpha k + \beta)|}{|\Gamma(\alpha k + \alpha + \beta)|} \left(1 - \frac{\alpha}{[\alpha k + \alpha + \beta]} \right) \frac{|\delta| \left(\frac{b^p - a^p}{p} \right)^{\alpha}}{(k+1)} \sim$$

$$\lim_{k\to+\infty} \frac{\sqrt{2\pi} (\alpha k)^{\alpha k + \beta - \frac{1}{2}} e^{-\alpha k} [1 + o(1)]}{\sqrt{2\pi} (\alpha k)^{\alpha k + \alpha + \beta - \frac{1}{2}} e^{-\alpha k} [1 + o(1)]} \left(1 - \frac{\alpha}{[\alpha k + \alpha + \beta]} \right) \frac{|\delta| \left(\frac{b^p - a^p}{p} \right)^{\alpha}}{(k+1)} =$$

$$\lim_{k\to+\infty} \frac{1}{(\alpha k)^{\alpha}} \left(1 - \frac{\alpha}{[\alpha k + \alpha + \beta]} \right) \frac{|\delta| \left(\frac{b^p - a^p}{p} \right)^{\alpha}}{(k+1)} = 0 < 1$$

Hence, the series (3.5) is convergent, and D is finite. That is why, now we show that inequality (3.4) is true, by using the norm of the $L_1^p(a,b)$:

$$\int_a^b \left|x^{p-1}g(x)\right|dx = \int_a^b \left|x^{p-1}{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x)\right|dx =$$

$$\int_a^b \left|x^{p-1}\int_a^x \left(\frac{x^p-t^p}{p}\right)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)t^{p-1}f\left(t\right)dt\right|dx \le$$

$$\int_a^b x^{p-1}\int_a^x \left(\frac{x^p-t^p}{p}\right)^{\beta-1} \left|E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)\right|\left|t^{p-1}f\left(t\right)\right|dtdx =$$

$$\int_a^b \left|t^{p-1}f\left(t\right)\right|\left(\int_t^b \left(\frac{x^p-t^p}{p}\right)^{\beta-1}\left|E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right)\right|d\left(\frac{x^p-t^p}{p}\right)\right)dt \le$$

$$\int_a^b \left|t^{p-1}f\left(t\right)\right|\left(\int_0^{\frac{b^p-a^p}{p}} u^{\beta-1}\left|\sum_{k=0}^{+\infty}\frac{(\gamma)_k}{\Gamma\left(\alpha k+\beta\right)}\frac{\left(\delta u^{\alpha}\right)^k}{k!}\right|du\right)dt \le$$

$$\int_a^b \left|t^{p-1}f\left(t\right)\right|\left(\sum_{k=0}^{+\infty}\frac{\left|(\gamma)_k\right|}{\left|\Gamma\left(\alpha k+\beta\right)\right|}\frac{|\delta|^k}{k!}\int_0^{\frac{b^p-a^p}{p}} u^{\alpha k+\beta-1}du\right)dt =$$

$$\int_a^b \left|t^{p-1}f\left(t\right)\right|\left(\sum_{k=0}^{+\infty}\frac{\left|(\gamma)_k\right|}{\left|\Gamma\left(\alpha k+\beta\right)\right|}\frac{|\delta|^k}{k!}\left(\left.\frac{u^{\alpha k+\beta}}{[\alpha k+\beta]}\right|_{u=0}^{u=\frac{b^p-a^p}{p}}\right)\right)dt =$$

$$\int_a^b \left|t^{p-1}f\left(t\right)\right|\left(\frac{b^p-a^p}{p}\right)^{\beta}\sum_{k=0}^{+\infty}\frac{\left|(\gamma)_k\right|}{\left|\Gamma\left(\alpha k+\beta\right)\right|[\alpha k+\beta]}\frac{\left|\delta\left(\frac{b^p-a^p}{p}\right)^{\alpha}\right|^k}{k!}dt.$$

□

Lemma 3.7. *If $\alpha,\beta,p,\in\mathbb{R}^+$,$\gamma,\delta\in\mathbb{R}$, $0\le a<b<+\infty$, Then the Katugampola-Prabhakar fractional integral operator is bounded on $C[a,b]$ and the following inequality holds*

$$\left\|{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x)\right\|_{C[a,b]} \le D\|f\left(x\right)\|_{C[a,b]}\,.$$

Here $\|f\left(x\right)\|_{C[a,b]} = \max\limits_{a\le x\le b}|f(x)|$.

Proof.

$$\left\| {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) \right\|_{C[a,b]} = \max_{a\le x\le b} \left| {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) \right| =$$

$$\max_{a\le x\le b} \left| \int_a^x \left(\frac{x^p-t^p}{p}\right)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right) t^{p-1} f(t)\,dt \right| \le$$

$$\|f(x)\|_{C[a,b]} \max_{a\le x\le b} \left| \int_a^x \left(\frac{x^p-t^p}{p}\right)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right) t^{p-1} dt \right| =$$

$$\|f(x)\|_{C[a,b]} \max_{a\le x\le b} \left| \int_a^x \left(\frac{x^p-t^p}{p}\right)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right) d\left(\frac{x^p-t^p}{p}\right) \right| =$$

$$\|f(x)\|_{C[a,b]} \max_{a\le x\le b} \left| \int_0^{\frac{x^p-a^p}{p}} u^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\delta u^{\alpha}\right) du \right| \le$$

$$\|f(x)\|_{C[a,b]} \int_0^{\frac{b^p-a^p}{p}} u^{\beta-1} \left| E_{\alpha,\beta}^{\gamma}\left(\delta u^{\alpha}\right) \right| du = D\|f(x)\|_{C[a,b]}.$$

□

4. Definitions and properties of the K-P fractional derivatives

4.1. K-P fractional derivatives.

Definition 4.1. Let $\alpha,\beta,p \in \mathbb{R}^+$ and $\gamma,\delta \in \mathbb{R}$, $n-1<\beta<n, n=[\beta]+1$, and $f(x)$ be a given function on $[a,b]$. The following

$$^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) := \left(x^{1-p}\frac{d}{dx}\right)^n {}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p} f(x) =$$

$$\left(x^{1-p}\frac{d}{dx}\right)^n \int_a^x \left(\frac{x^p-t^p}{p}\right)^{n-1-\beta} E_{\alpha,n-\beta}^{-\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right) t^{p-1} f(t)\,dt,\ x>a\,, \tag{4.1}$$

$$^{KP}D_{b-}^{\alpha,\beta,\gamma,\delta,p} f(x) := \left(-x^{1-p}\frac{d}{dx}\right)^n {}_{KP}I_{b-}^{\alpha,n-\beta,-\gamma,\delta,p} f(x) =$$

$$\left(-x^{1-p}\frac{d}{dx}\right)^n \int_x^b \left(\frac{t^p-x^p}{p}\right)^{n-1-\beta} E_{\alpha,n-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-x^p}{p}\right)^{\alpha}\right) t^{p-1} f(t)\,dt,\ x<b$$

are called the left and right-sided K-P fractional derivatives. Here, β is the order of the derivatives.

Below, we present certain properties of these derivatives.

Lemma 4.2. *If $\alpha, \beta, p \in \mathbb{R}^+$ and $\gamma, \delta \in \mathbb{R}$, $n-1 < \beta \leqslant n, n = [\beta]+1$, then the following equalities hold for $f(x) \in L_1^p[a,b]$:*

$$ {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = \sum_{k=0}^{+\infty} \frac{(\gamma)_k \delta^k}{k!} {}_K I_{a+}^{\alpha k+\beta,p} f(x) \,, \tag{4.2} $$

$$ {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = \sum_{k=0}^{+\infty} \frac{(-\gamma)_k \delta^k}{k!} {}_K I_{a+}^{\alpha k-\beta,p} f(x) \ . \tag{4.3} $$

Proof. Let us prove (4.3). (4.2) can be proved similarly. Using the definition of K-P integral and the Prabhakar function, one can get

$$ {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = \left(x^{1-p} \frac{d}{dx} \right)^n {}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p} f(x) = $$

$$ \left(x^{1-p} \frac{d}{dx} \right)^n \int\limits_a^x \left(\frac{x^p - t^p}{p} \right)^{n-\beta-1} E_{\alpha,n-\beta}^{-\gamma} \left(\delta \left(\frac{x^p - t^p}{p} \right)^{\alpha} \right) t^{p-1} f(t)\, dt = $$

$$ \left(x^{1-p} \frac{d}{dx} \right)^n \int\limits_a^x \sum_{k=0}^{+\infty} \frac{(-\gamma)_k}{\Gamma(ak+n-\beta)} \frac{\delta^k}{k!} \left(\frac{x^p - t^p}{p} \right)^{\alpha k+n-\beta-1} t^{p-1} f(t)\, dt = $$

$$ \sum_{k=0}^{+\infty} \frac{(-\gamma)_k \delta^k}{k!} \frac{1}{\Gamma(ak-\beta)} \int\limits_a^x \left(\frac{x^p - t^p}{p} \right)^{\alpha k-\beta-1} t^{p-1} f(t)\, dt = $$

$$ \sum_{k=0}^{+\infty} \frac{(-\gamma)_k \delta^k}{k!} {}_K I_{a+}^{\alpha k-\beta} f(x) \ . $$

Note that the uniform convergence of the infinite series representing the Prabhakar function and the regularity of the function $f(x)$ allow us to use term-by-term integration. □

Lemma 4.3. *If $\alpha, \beta, p \in \mathbb{R}^+$ and $\gamma, \delta, \sigma, \gamma_1 \in \mathbb{R}, n-1 < \beta, \sigma, \gamma_1 \leqslant n, n = [\beta]+1$, then for $f(x) \in L_1^p[a,b]$ the following equalities are true:*

$$ {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = f(x) \,, $$

$$ {}^{KP}D_{a+}^{\alpha,\sigma,\gamma,\delta,p} {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = {}_K I_{a+}^{\beta-\sigma,p} f(x), \ \sigma < \beta, $$

$$ {}^{KP}D_{a+}^{\alpha,\sigma,\gamma_1,\delta,p} {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = {}_{KP}I_{a+}^{\alpha,\beta-\sigma,\gamma-\gamma_1,\delta,p} f(x), \ \sigma < \beta, \ \gamma_1 < \gamma. $$

Proof. Here, we give the proof of the last equality of this lemma. The other equalities can be proved similarly. Using the results of the Lemma 3.4 and (3.3) one can get easily

$$ {}^{KP}D_{a+}^{\alpha,\sigma,\gamma_1,\delta,p} {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = $$

$$ \left(x^{1-p} \frac{d}{dx} \right)^n {}_{KP}I_{a+}^{\alpha,n-\sigma,-\gamma_1,\delta,p} {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = $$

$$ \left(x^{1-p} \frac{d}{dx} \right)^n {}_{KP}I_{a+}^{\alpha,n+\beta-\sigma,\gamma-\gamma_1,\delta,p} f(x) = {}_{KP}I_{a+}^{\alpha,\beta-\sigma,\gamma-\gamma_1,\delta,p} f(x). $$

□

Lemma 4.4. *If $\alpha,\beta,p\in\mathbb{R}^+$ and $\gamma,\delta,\lambda\in\mathbb{R}$, $n-1<\beta\leqslant n, n=[\beta]+1$, then these equalities hold for $x>a$*

$$
{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}\left(\frac{x^p-a^p}{p}\right)^{\lambda} = \Gamma\left(\lambda+1\right)\left(\frac{x^p-a^p}{p}\right)^{\lambda+\beta}E_{\alpha,(\lambda+\beta+1)}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right), \tag{4.4}
$$

where $\lambda>-1$,

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}\left(\frac{x^p-a^p}{p}\right)^{\lambda} = \Gamma\left(\lambda+1\right)\left(\frac{x^p-a^p}{p}\right)^{\lambda-\beta}E_{\alpha,(\lambda-\beta+1)}^{-\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right), \tag{4.5}
$$

where $\lambda>\beta-1$.

The proof is straightforward. The next statement is connected with the regularity of the function.

Theorem 4.5. *If $\alpha,\beta,p,\in\mathbb{R}^+$,$\gamma,\delta\in\mathbb{R}$, $0<a<b<+\infty$ and $n\in\mathbb{N}$, $n=[\beta]+1$, $n-1<\beta\leqslant n$, $f(x)\in AC_{\delta_p}^n[a,b]$, then Katugampola-Prabhakar fractional derivatives ${}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x)$ and ${}^{KP}D_{b-}^{\alpha,\beta,\gamma,\delta,p}f(x)$ exist almost everywhere on $[a,b]$, and the following equalities are true:*

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) = \sum_{k=0}^{n-1}c_k\left(\frac{x^p-a^p}{p}\right)^{k-\beta}E_{\alpha,(k-\beta+1)}^{-\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right)+ \int\limits_a^x\left(\frac{x^p-s^p}{p}\right)^{n-\beta-1}E_{\alpha,(n-\beta)}^{-\gamma}\left(\delta\left(\frac{x^p-s^p}{p}\right)^{\alpha}\right)\varphi(s)ds\,, \tag{4.6}
$$

$$
{}^{KP}D_{b-}^{\alpha,\beta,\gamma,\delta,p}f(x) = \sum_{k=0}^{n-1}c_k(-1)^k\left(\frac{b^p-x^p}{p}\right)^{k-\beta}E_{\alpha,(k-\beta+1)}^{-\gamma}\left(\delta\left(\frac{b^p-x^p}{p}\right)^{\alpha}\right)+ (-1)^{n-1}\int\limits_x^b\left(\frac{s^p-x^p}{p}\right)^{n-\beta-1}E_{\alpha,(n-\beta)}^{-\gamma}\left(\delta\left(\frac{s^p-x^p}{p}\right)^{\alpha}\right)\varphi(s)ds\,. \tag{4.7}
$$

Proof. Let us first show the proof of (4.6). If we apply the ${}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}$ operator to both-sides of (2.1) equality, then we have

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x) = \sum_{k=0}^{n-1} \frac{c_k}{\Gamma(k+1)} {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} \left(\frac{x^p - a^p}{p}\right)^k +
$$

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} \left[\frac{1}{\Gamma(n)} \int_a^x \left(\frac{x^p - s^p}{p}\right)^{n-1} \varphi(s)ds \right]. \tag{4.8}
$$

We calculate each term on the right side of the equality (4.8). If we calculate the first term by using equality (4.5), then we obtain the following equality:

$$
\sum_{k=0}^{n-1} \frac{c_k}{\Gamma(k+1)} {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} \left(\frac{x^p - a^p}{p}\right)^k =
$$

$$
\sum_{k=0}^{n-1} c_k \left(\frac{x^p - a^p}{p}\right)^{k-\beta} E_{\alpha,(k-\beta+1)}^{-\gamma} \left(\delta \left(\frac{x^p - a^p}{p}\right)^{\alpha}\right). \tag{4.9}
$$

Now, we calculate the second term on the right side of the equality (4.8) by using the equality (4.1), as follows:

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} \left[\frac{1}{\Gamma(n)} \int_a^x \left(\frac{x^p - s^p}{p}\right)^{n-1} \varphi(s)ds \right] =
$$

$$
\left(x^{1-p}\frac{d}{dx}\right)^n {}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p} \left[\frac{1}{\Gamma(n)} \int_a^x \left(\frac{x^p - s^p}{p}\right)^{n-1} \varphi(s)ds \right] =
$$

$$
\left(x^{1-p}\frac{d}{dx}\right)^n \sum_{m=0}^{+\infty} \frac{(-\gamma)_m}{\Gamma(\alpha m + n - \beta)} \frac{\delta^m}{m!\Gamma(n)} \times
$$

$$
\int_a^x \left(\frac{x^p - t^p}{p}\right)^{\alpha m + n - 1 - \beta} t^{p-1}\, dt \int_a^t \left(\frac{t^p - s^p}{p}\right)^{n-1} \varphi(s)ds =
$$

$$
\left(x^{1-p}\frac{d}{dx}\right)^n \sum_{m=0}^{+\infty} \frac{(-\gamma)_m}{\Gamma(\alpha m + n - \beta)} \frac{\delta^m}{m!\Gamma(n)} \times
$$

$$
\int_a^x \varphi(s)ds \int_s^x \left(\frac{x^p - t^p}{p}\right)^{\alpha m + n - 1 - \beta} \left(\frac{t^p - s^p}{p}\right)^{n-1} t^{p-1}dt.
$$

Considering the following $y = \frac{t^p - s^p}{x^p - s^p}$ change of variables in the inner integral of the last line, after some simplifications, we obtain the following result:

$$
{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}\left[\frac{1}{\Gamma(n)}\int_a^x\left(\frac{x^p-s^p}{p}\right)^{n-1}\varphi(s)ds\right]=
$$

$$
\int_a^x\left(\frac{x^p-s^p}{p}\right)^{n-\beta-1}E_{\alpha,(n-\beta)}^{-\gamma}\left(\delta\left(\frac{x^p-s^p}{p}\right)^{\alpha}\right)\varphi(s)ds\;. \tag{4.10}
$$

If we substitute (4.9) and (4.10) into (4.8), then we obtain (4.6). The equality (4.7) can be proved in the same way as the equality (4.6), but here, ${}^{KP}D_{b-}^{\alpha,\beta,\gamma,\delta,p}$ and equality (2.2) will be used instead of the ${}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}$ and (2.1). □

The following statement describes the relation of K-P integral and integral-differential operators.

Theorem 4.6. *If $\alpha,\beta,p,\in\mathbb{R}^+$,$\gamma,\delta\in\mathbb{R}$, $0<a<b<+\infty$ and $n\in\mathbb{N}$, $n=[\beta]+1$, $n-1<\beta<n$, $f(x)\in L_1^p[a,b]$, ${}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)\in AC_{\delta_p}^n[a,b]$, then for $x>a$*

$$
{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x)=
$$

$$
f(x)-\sum_{k=0}^{n-1}c_{n-k}\left(\frac{x^p-a^p}{p}\right)^{k-n+\beta}E_{\alpha,(k-n+\beta+1)}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right) \tag{4.11}
$$

holds. Here $c_{n-k}=\lim\limits_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^{n-k}{}_{KP}I_{a+}^{\alpha,n-\beta,\gamma,\delta,p}f(x)$, $k=0,1,2,...,n-1$. In particular, if $\beta=n$, then we have

$$
{}_{KP}I_{a+}^{\alpha,n,\gamma,\delta,p}{}^{KP}D_{a+}^{\alpha,n,\gamma,\delta,p}f(x)=
$$

$$
f(x)-\sum_{k=0}^{n-1}c_{n-k}\left(\frac{x^p-a^p}{p}\right)^{k}E_{\alpha,(k+1)}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right)\;.
$$

Proof. We know from [23], if ${}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)\in AC_{\delta_p}^n[a,b]$, then $\left(x^{1-p}\frac{d}{dx}\right)^{n-1}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)\in AC[a,b]$. Namely,

$$
\left(x^{1-p}\frac{d}{dx}\right)^{n-1}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)=c_{n-1}+\int_a^x\varphi(s)ds\;, \tag{4.12}
$$

or

$$
{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)=\sum_{k=0}^{n-1}\frac{c_k}{\Gamma(k+1)}\left(\frac{x^p-a^p}{p}\right)^{k}+
$$

$$
\frac{1}{\Gamma(n)}\int_a^x\left(\frac{x^p-s^p}{p}\right)^{n-1}\varphi(s)ds\;. \tag{4.13}
$$

Here $c_{n-k}=\lim\limits_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^{n-k}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)$, $k=0,1,2,...,n-1$. and $\varphi\in L_1[a,b]$. If we apply the $\left(x^{1-p}\frac{d}{dx}\right)$ operator to both-side of equality (4.12), then we obtain the following relation:

$$ {}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = x^{1-p}\varphi(x) \; . \tag{4.14} $$

Now we apply the ${}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}$ to both-sides of equation (4.14) and use (4.2):

$$ {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}\left[x^{1-p}\varphi(x)\right] = \sum_{m=0}^{+\infty} \frac{(\gamma)_m \delta^m}{m!}\, {}_{K}I_{a+}^{\alpha m+\beta,p}\left[x^{1-p}\varphi(x)\right] \; . \tag{4.15} $$

On the other hand, If we apply the ${}^{KP}D_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}$ operator to both side of equality (4.13) and using the first result of the lemma 4.3 and after the some simplifies we have the following equality:

$$ f(x) = \sum_{k=0}^{n-1} c_{n-k} \left(\frac{x^p-a^p}{p}\right)^{k-n+\beta} E_{\alpha,(k-n+\beta+1)}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right) + \sum_{m=0}^{+\infty} \frac{(\gamma)_m \delta^m}{m!}\, {}_{K}I_{a+}^{\alpha m+\beta,p}\left[x^{1-p}\varphi(x)\right] \; . \tag{4.16} $$

If we substitute equation (4.15) into the second term on the right side of equation (4.16), then we obtain equation (4.11). □

Let us now present the definition and certain properties of Caputo-type K-P derivatives.

4.2. **Caputo-type K-P fractional derivatives.**

Definition 4.7. Let $\alpha, \beta, p \in \mathbb{R}^+$ and $\gamma, \delta \in \mathbb{R}$, $n-1 < \beta \le n, n = [\beta]+1$, also $f(x) \in AC_{\delta_p}^n[a,b]$. The operators

$$ {}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) = {}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}\left(x^{1-p}\frac{d}{dx}\right)^n f(x) = \int_a^x \left(\frac{x^p-t^p}{p}\right)^{n-1-\beta} E_{\alpha,n-\beta}^{-\gamma}\left(\delta\left(\frac{x^p-t^p}{p}\right)^{\alpha}\right) t^{p-1}\left(t^{1-p}\frac{d}{dt}\right)^n f(t)\, dt, \; x > a \, , \tag{4.17} $$

$$ {}^{CKP}D_{b-}^{\alpha,\beta,\gamma,\delta,p} f(x) = {}_{KP}I_{b-}^{\alpha,n-\beta,-\gamma,\delta,p}\left(-x^{1-p}\frac{d}{dx}\right)^n f(x) = \int_x^b \left(\frac{t^p-x^p}{p}\right)^{n-1-\beta} E_{\alpha,n-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-x^p}{p}\right)^{\alpha}\right) t^{p-1}\left(-t^{1-p}\frac{d}{dt}\right)^n f(t)\, dt, \; x < b $$

are called left and right-sided Caputo-type K-P fractional derivatives.

Lemma 4.8. *If $\alpha, \beta, p, \in \mathbb{R}^+$,$\gamma, \delta \in \mathbb{R}$,$n-1<\beta \leq n$, $n=[\beta]+1$ and $f(x) \in AC^n_{\delta_p}[a,b]$, then there is the following relation between the K-P fractional derivative and the Caputo-type K-P fractional derivative of the $f(x)$:*

$$
{}^{CKP}D^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = {}^{KP}D^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) - \sum_{k=0}^{n-1} \left(\frac{x^p-a^p}{p}\right)^{k-\beta} E^{(-\gamma)}_{\alpha,(k-\beta+1)}\left[\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right] f^{(k)}(a)\,, \tag{4.18}
$$

where $f^{(k)}(a) = \lim\limits_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^k f(x)$.

Theorem 4.9. *If $\alpha, \beta, p, \in \mathbb{R}^+$,$\gamma, \delta \in \mathbb{R}$,$n-1<\beta \leq n$, $n=[\beta]+1$ and $f(x) \in AC^n_{\delta_p}[a,b]$, then*

$$
{}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+}\ {}^{CKP}D^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)}\left(\frac{x^p-a^p}{p}\right)^{k} \tag{4.19}
$$

is valid. Here $f^{(k)}(a) = \lim\limits_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^k f(x)$.

4.3. **Application of the generalized Laplace transform.**

Theorem 4.10. *Let assume that $\alpha, \beta, p, \in \mathbb{R}^+$,$\gamma, \delta, s \in \mathbb{R}$, $a \geqslant 0$, $F(s) = L_{m_p}\{f(x); s\}$. Then the following equalities are held:*

$$
L_{m_p}\left\{{}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x); s\right\} = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha-\delta)^\gamma} F(s), \ s^\alpha > \delta, \tag{4.20}
$$

$$
L_{m_p}\left\{{}^{KP}D^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x); s\right\} = \frac{s^{\alpha(-\gamma)+\beta}}{(s^\alpha-\delta)^{-\gamma}} F(s) - \sum_{k=0}^{n-1} s^k \left[\lim_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1} {}_{KP}I^{\alpha,n-\beta,-\gamma,\delta,p}_{a+} f(x)\right], \tag{4.21}
$$

$$
L_{m_p}\left\{{}^{CKP}D^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x); s\right\} = \frac{s^{\alpha(-\gamma)+\beta}}{(s^\alpha-\delta)^{-\gamma}} F(s) - \sum_{k=0}^{n-1} \frac{s^{\alpha(-\gamma)-n+k+\beta}}{(s^\alpha-\delta)^{-\gamma}} \left[\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1} f(a)\right], \tag{4.22}
$$

where $\left[\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1} f(a)\right] = \left[\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1} f(x)\right]_{x=a}$.

Proof. To prove (4.20), we use Definitions 2.10 and 2.11, and rewrite the K-P fractional integral as follows:

$$
{}_{KP}I^{\alpha,\beta,\gamma,\delta,p}_{a+} f(x) = \left(f(x) * \left(\frac{x^p-a^p}{p}\right)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right)\right)_{m_p}. \tag{4.23}
$$

Based on the properties of the m_p-convolution [26] and (4.23), we have

$$L_{m_p}\left\{{}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x);s\right\}=$$
$$L_{m_p}\left\{\left(f(x)*\left(\frac{x^p-a^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right)\right)_{m_p};s\right\}=$$
$$L_{m_p}\left\{f(x);s\right\}L_{m_p}\left\{\left(\frac{x^p-a^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right);s\right\}. \tag{4.24}$$

Let us find an image of the second multiplier on the formula (4.24):

$$L_{m_p}\left\{\left(\frac{x^p-a^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right);s\right\}=$$
$$\int_a^{+\infty}e^{-s\frac{t^p-a^p}{p}}\left(\frac{t^p-a^p}{p}\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta\left(\frac{t^p-a^p}{p}\right)^{\alpha}\right)t^{p-1}dt=$$
$$\int_a^{+\infty}e^{-s\frac{t^p-a^p}{p}}\sum_{m=0}^{+\infty}\frac{(\gamma)_m\delta^m}{\Gamma(\alpha m+\beta)}\frac{\left(\frac{t^p-a^p}{p}\right)^{\alpha m+\beta-1}}{\Gamma(m+1)}t^{p-1}dt=$$
$$\sum_{m=0}^{+\infty}\frac{(\gamma)_m\delta^m p^{-\alpha m-\beta+1}}{\Gamma(\alpha m+\beta)\Gamma(m+1)}\int_a^{+\infty}e^{-s\frac{t^p-a^p}{p}}(t^p-a^p)^{\alpha m+\beta-1}t^{p-1}dt=$$
$$\sum_{m=0}^{+\infty}\frac{(\gamma)_m\delta^m p^{-\alpha m-\beta+1}}{\Gamma(\alpha m+\beta)\Gamma(m+1)}\frac{p^{\alpha m+\beta-1}}{s^{1+\alpha m+\beta-1}}\Gamma(1+\alpha m+\beta-1)=$$
$$\frac{1}{s^{\beta}}\sum_{m=0}^{+\infty}\frac{(\gamma)_m}{\Gamma(m+1)}\left(\frac{\delta}{s^{\alpha}}\right)^m=\frac{1}{s^{\beta}}(-1)^m\sum_{m=0}^{+\infty}\frac{(\gamma)_m}{\Gamma(m+1)}\left(-\frac{\delta}{s^{\alpha}}\right)^m.$$

We know from [14] that the following relation holds:

$$(1+z)^{-\gamma}=\sum_{m=0}^{+\infty}(-1)^m\frac{\Gamma(\gamma+m)}{\Gamma(\gamma)\Gamma(m+1)}z^m. \tag{4.25}$$

Based on (4.25), we have

$$\frac{1}{s^{\beta}}(-1)^m\sum_{m=0}^{+\infty}\frac{(\gamma)_m}{\Gamma(m+1)}\left(-\frac{\delta}{s^{\alpha}}\right)^m=\frac{s^{-\beta}}{\left(1+\left(-\frac{\delta}{s^{\alpha}}\right)\right)^{\gamma}}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\delta)^{\gamma}}. \tag{4.26}$$

Due to (4.24),(4.26) we obtain (4.20).
Let us now prove (4.21) as follows:

$$L_{m_p}\left\{{}^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x);s\right\}=L_{m_p}\left\{\left(x^{1-p}\frac{d}{dx}\right)^{n}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x);s\right\}=$$
$$s^{n}\frac{s^{-\alpha\gamma-n+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}F(s)-\sum_{k=0}^{n-1}s^{k}\left[\lim_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)\right]=$$
$$\frac{s^{\alpha(-\gamma)+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}F(s)-\sum_{k=0}^{n-1}s^{k}\left[\lim_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1}{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)\right].$$

Now we show proof of the (4.22) in an analytic way:

$$L_{m_p}\left\{{}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}f(x);s\right\}=$$
$$L_{m_p}\left\{{}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}\left(x^{1-p}\frac{d}{dx}\right)^{n}f(x);s\right\}=$$
$$\frac{s^{\alpha(-\gamma)-n+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}L_{m_p}\left\{\left(x^{1-p}\frac{d}{dx}\right)^{n}f(x);s\right\}=$$
$$\frac{s^{\alpha(-\gamma)-n+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}\left[s^{n}L_{m_p}\{f(x);s\}-\sum_{k=0}^{n-1}s^{k}\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1}f(a)\right]=$$
$$\frac{s^{\alpha(-\gamma)+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}F(s)-\sum_{k=0}^{n-1}\frac{s^{\alpha(-\gamma)-n+k+\beta}}{(s^{\alpha}-\delta)^{-\gamma}}\left(x^{1-p}\frac{d}{dx}\right)^{n-k-1}f(a).$$

Here we have used some properties presented in [26]. □

Lemma 4.11. *Assume that,* $\alpha,\beta,p,\in\mathbb{R}^{+},\gamma,\delta,s\in\mathbb{R},\ a\geqslant 0,\ k=0,1,1...,n-1.$ *If* $\left|\left(\frac{\lambda}{s^{\beta}}\right)\left(1-\frac{\delta}{s^{\alpha}}\right)^{-\gamma}\right|<1$. *Then the following equalities hold:*

$$L_{m_p}\left\{\left(\frac{x^{p}-a^{p}}{p}\right)^{\beta-1}\sum_{m=0}^{+\infty}\left[\lambda\left(\frac{x^{p}-a^{p}}{p}\right)^{\beta}\right]^{m}\times E_{\alpha,(\beta+m\beta)}^{\gamma+m\gamma}\left(\delta\left(\frac{x^{p}-a^{p}}{p}\right)^{\alpha}\right);s\right\}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\delta)^{\gamma}-\lambda s^{\alpha\gamma-\beta}},\tag{4.27}$$

$$L_{m_p}\left\{\left(\frac{x^{p}-a^{p}}{p}\right)^{k-n+\beta}\sum_{m=0}^{+\infty}\left[\lambda\left(\frac{x^{p}-a^{p}}{p}\right)^{\beta}\right]^{m}\times E_{\alpha,(k-n+\beta+m\beta+1)}^{\gamma+m\gamma}\left(\delta\left(\frac{x^{p}-a^{p}}{p}\right)^{\alpha}\right);s\right\}=\frac{s^{\alpha\gamma-(k-n+\beta+1)}}{(s^{\alpha}-\delta)^{\gamma}-\lambda s^{\alpha\gamma-\beta}}.\tag{4.28}$$

Proof. Here we show the proof of the equality 4.27; the equality 4.28 has been proved similarly. We know from [26] that the following formula holds:

$$L_{m_p}\left\{(t^p-a^p)^{\mu};s\right\}=\frac{p^{\mu}}{s^{1+\mu}}\Gamma\left(1+\mu\right). \tag{4.29}$$

Therefore,

$$L_{m_p}\left\{\left(\frac{x^p-a^p}{p}\right)^{\beta-1}\sum_{m=0}^{+\infty}\left[\lambda\left(\frac{x^p-a^p}{p}\right)^{\beta}\right]^m E_{\alpha,(\beta+m\beta)}^{\gamma+m\gamma}\left(\delta\left(\frac{x^p-a^p}{p}\right)^{\alpha}\right);s\right\}=$$

$$L_{m_p}\left\{\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma\left(\gamma+m\gamma+j\right)\lambda^m\delta^j p^{-\beta m-\alpha j-\beta+1}}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(\alpha j+\beta m+\beta\right)\Gamma\left(1+j\right)}(x^p-a^p)^{\beta m+\alpha j+\beta-1}\right\}=$$

$$\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma\left(\gamma+m\gamma+j\right)\lambda^m\delta^j p^{-\beta m-\alpha j-\beta+1}}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(\alpha j+\beta m+\beta\right)\Gamma\left(1+j\right)}L_{m_p}\left\{(x^p-a^p)^{\beta m+\alpha j+\beta-1}\right\}=$$

$$\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma\left(\gamma+m\gamma+j\right)\lambda^m\delta^j p^{-\beta m-\alpha j-\beta+1}}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(\alpha j+\beta m+\beta\right)\Gamma\left(1+j\right)}\frac{p^{\beta m+\alpha j+\beta-1}}{s^{\beta m+\alpha j+\beta}}\Gamma\left(\beta m+\alpha j+\beta\right)=$$

$$\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma\left(\gamma+m\gamma+j\right)\lambda^m\delta^j}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(1+j\right)}\frac{1}{s^{\beta m+\alpha j+\beta}}=$$

$$\frac{1}{s^{\beta}}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{s^{\beta}}\right)^m\sum_{j=0}^{+\infty}(-1)^j\frac{\Gamma\left(\gamma+m\gamma+j\right)}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(1+j\right)}\left(-\frac{\delta}{s^{\alpha}}\right)^j.$$

Based on the formula (4.25), one can get

$$\frac{1}{s^{\beta}}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{s^{\beta}}\right)^m\sum_{j=0}^{+\infty}(-1)^j\frac{\Gamma\left(\gamma+m\gamma+j\right)}{\Gamma\left(\gamma+m\gamma\right)\Gamma\left(1+j\right)}\left(-\frac{\delta}{s^{\alpha}}\right)^j=$$

$$\frac{1}{s^{\beta}}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{s^{\beta}}\right)^m\left(1-\frac{\delta}{s^{\alpha}}\right)^{-\gamma-m\gamma}=$$

$$\frac{1}{s^{\beta}}\left(1-\frac{\delta}{s^{\alpha}}\right)^{-\gamma}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{s^{\beta}}\right)^m\left(1-\frac{\delta}{s^{\alpha}}\right)^{-m\gamma}=$$

$$\frac{1}{s^{\beta}}\left(1-\frac{\delta}{s^{\alpha}}\right)^{-\gamma}\sum_{m=0}^{+\infty}\left[\left(\frac{\lambda}{s^{\beta}}\right)\left(1-\frac{\delta}{s^{\alpha}}\right)^{-\gamma}\right]^m.$$

If $\left|\left(\frac{\lambda}{s^\beta}\right)\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}\right|<1$ is valid, then

$$\frac{1}{s^\beta}\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}\sum_{m=0}^{+\infty}\left[\left(\frac{\lambda}{s^\beta}\right)\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}\right]^m =$$

$$\frac{1}{s^\beta}\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}\left[\frac{1}{1-\left(\frac{\lambda}{s^\beta}\right)\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}}\right] =$$

$$\frac{1}{s^\beta}\left(1-\frac{\delta}{s^\alpha}\right)^{-\gamma}\left[\frac{s^{-\alpha\gamma+\beta}}{s^{-\alpha\gamma+\beta}-\lambda(s^\alpha-\delta)^{-\gamma}}\right] = \frac{(s^\alpha-\delta)^{-\gamma}}{s^{-\alpha\gamma+\beta}}\frac{s^{-\alpha\gamma+\beta}}{s^{-\alpha\gamma+\beta}-\lambda(s^\alpha-\delta)^{-\gamma}} =$$

$$\frac{(s^\alpha-\delta)^{-\gamma}}{s^{-\alpha\gamma+\beta}-\lambda(s^\alpha-\delta)^{-\gamma}} = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha-\delta)^{\gamma}-\lambda s^{\alpha\gamma-\beta}}.$$

□

5. Cauchy problems for ordinary differential equations of fractional order

In this section, we will present the results of our research using the K-P and Caputo-type K-P fractional derivatives to consider the Cauchy problems for fractional ordinary differential equations.

Let us find a solution that satisfies the following ordinary fractional differential equation.

$$^{KP}D_{a+}^{\alpha,\beta,\gamma,\delta,p}u(x)-\lambda\,u(x)=f(x)\,,\,a<x<b, \tag{5.1}$$

and the following initial conditions:

$$\lim_{x\to+a}\left(x^{1-p}\frac{d}{dx}\right)^k {}_{KP}I_{a+}^{\alpha,n-\beta,-\gamma,\delta,p}f(x)=B_k\,,\quad k=0,1,2,...,n-1\,. \tag{5.2}$$

Here $\alpha,\beta,p,\in\mathbb{R}^+$,$\gamma,\delta,\lambda,a,B_k\in\mathbb{R}$, $n-1<\beta\le n$, $a\geqslant 0$ and $f(x)$ is a given function.

Theorem 5.1. *If $f(t)\in C^1(a,b]$, then the Cauchy problem (5.1)-(5.2) has a unique solution represented as follows:*

$$u(x)=\sum_{k=0}^{n-1}B_k\left(\frac{x^p-a^p}{p}\right)^{k-n+\beta}\Gamma(\gamma)\ \times$$

$$E_2\left(\begin{matrix}\gamma,\gamma,1\,;1,0\\ k-n+\beta+1,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\left|\begin{matrix}\lambda p^{-\beta}(x^p-a^p)^\beta\\ \delta p^{-\alpha}(x^p-a^p)^\alpha\end{matrix}\right.\right)+$$

$$\Gamma(\gamma)\int_a^x\left(\frac{x^p-t^p}{p}\right)^{\beta-1}E_2\left(\begin{matrix}\gamma,\gamma,1\,;1,0\\ \beta,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\left|\begin{matrix}\lambda p^{-\beta}(x^p-t^p)^\beta\\ \delta p^{-\alpha}(x^p-t^p)^\alpha\end{matrix}\right.\right)t^{p-1}f(t)\,dt\,. \tag{5.3}$$

Proof. To prove the theorem we apply the ${}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p}$ operator to both sides of the equation (5.1), after that we use the equality (4.11) and initial conditions

(5.2), then we obtain the following second kind Volterra integral equation:

$$u(x) - \lambda\, {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} u(x) = g(x)\,. \tag{5.4}$$

Here

$$g(x) = {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} f(x) + \sum_{k=0}^{n-1} B_k \left(\frac{x^p - a^p}{p}\right)^{k-n+\beta} E_{\alpha,(k-n+\beta+1)}^{\gamma}\left(\delta\left(\frac{x^p - a^p}{p}\right)^{\alpha}\right). \tag{5.5}$$

To find a solution to the integral equation (5.4), we use the successive approximation method as follows:

$$u_0(x) = g(x)\,,$$
$$u_n(x) = g(x) + \lambda\, {}_{KP}I_{a+}^{\alpha,\beta,\gamma,\delta,p} u_{n-1}(x).$$

After some calculations, we obtain

$$u_n(x) = g(x) + \sum_{m=1}^{n} \lambda^m\, {}_{KP}I_{a+}^{\alpha,m\beta,m\gamma,\delta,p} g(x),$$

or

$$u(x) = \lim_{x\to+\infty} u_n(x) = g(x) + \sum_{m=1}^{+\infty} \lambda^m\, {}_{KP}I_{a+}^{\alpha,m\beta,m\gamma,\delta,p} g(x).$$

If we change $g(x)$ by using the equality (5.5), then after the some calculations we obtain the $u(x)$ function in the following form:

$$u(x) = \sum_{k=0}^{n-1} B_k \left(\frac{x^p - a^p}{p}\right)^{k-n+\beta} \times \sum_{m=0}^{+\infty} \left[\lambda\left(\frac{x^p - a^p}{p}\right)^{\beta}\right]^m E_{\alpha,(k-n+\beta+m\beta+1)}^{\gamma+m\gamma}\left(\delta\left(\frac{x^p - a^p}{p}\right)^{\alpha}\right) + \int_a^x \left(\frac{x^p - t^p}{p}\right)^{\beta-1} \sum_{m=0}^{+\infty} \left[\lambda\left(\frac{x^p - t^p}{p}\right)^{\beta}\right]^m E_{\alpha,(\beta+m\beta)}^{\gamma+m\gamma}\left(\delta\left(\frac{x^p - t^p}{p}\right)^{\alpha}\right) t^{p-1} f(t)\, dt\,. \tag{5.6}$$

If we apply the function $E_2(x,y)$ to (5.6), then we obtain (5.3). If we substitute the function $u(x)$ into (5.1) and (5.2), we know that the $f(x)$ must satisfy $f(x) \in C^1(a,b]$. □

Let us consider another Cauchy problem.

Let us find a solution that satisfies the following ordinary fractional differential equation:

$$^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta,p} u(x) - \lambda\, u(x) = f(x)\,,\ a < x < b \tag{5.7}$$

together with initial conditions:

$$u^{(k)}(a) = \left(x^{1-p}\frac{d}{dx}\right)^k u(x)\Bigg|_{x=a} = A_k\ ,\ \ k = 0, 1, 2, ..., n-1\ , \tag{5.8}$$

where $\alpha, \beta, p, \in \mathbb{R}^+, \gamma, \delta, \lambda, a, B_k \in \mathbb{R}$, $n-1 < \beta \leq n$, $a \geqslant 0$ and $f(x)$ is a given function.

The following statement is valid:

Theorem 5.2. *If $f(t) \in C^1[a,b]$, then the 5.7,5.8 Cauchy-type problem has a unique solution represented as follows:*

$$u(x) = \sum_{k=0}^{n-1}\frac{A_k}{\Gamma(k+1)}\left(\frac{x^p-a^p}{p}\right)^k + \lambda\Gamma(\gamma)\sum_{k=0}^{n-1}A_k\left(\frac{x^p-a^p}{p}\right)^k\left(\frac{x^p-a^p}{p}\right)^\beta \times$$
$$E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta+k+1,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\middle|\begin{matrix}\lambda p^{-\beta}(x^p-a^p)^\beta\\ \delta p^{-\alpha}(x^p-a^p)^\alpha\end{matrix}\right) +$$
$$\int_a^x\left(\frac{x^p-t^p}{p}\right)^{\beta-1}\Gamma(\gamma)\,E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\middle|\begin{matrix}\lambda p^{-\beta}(x^p-t^p)^\beta\\ \delta p^{-\alpha}(x^p-t^p)^\alpha\end{matrix}\right)t^{p-1}f(t)dt. \tag{5.9}$$

6. An initial-boundary problem for sub-diffusion equation with the Caputo-type K-P fractional derivative.

Let us investigate the following time-fractional diffusion equation

$${}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}u(t,x) - u_{xx}(t,x) = f(t,x) \tag{6.1}$$

in a domain $\Omega = \{(t,x):\ 0 < t < T\,,\ 0 < x < 1\}$.
Here, $0 < \beta < 1$, $\gamma, \delta, \in \mathbb{R}$, $\alpha > 0$ and $f(t,x)$- is a given function.

Problem. To find a solution to the equation (6.1) in Ω, satisfying the following

- regularity conditions:
  $u(t,x) \in C\left(\overline{\Omega}\right)$, $u(\cdot,x) \in C^2(0,1)$, ${}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}u(t,\cdot) \in C(0,T)$;
- initial condition: $u(0,x) = B(x)$,$0 \leqslant x \leqslant 1$;
- boundary conditions: $u(t,0) = u(t,1) = 0$, $0 \leqslant t \leqslant T$.

Here, the function $B(x)$ is a given function such that $B(0) = B(1) = 0$.

6.1. The uniqueness of the problem.

Lemma 6.1. *If the function $u(x)$ is absolutely continuous on $[a,b]$, $a \geqslant 0$ and $0 < \alpha \leqslant 1$, $\delta < 0$, $0 < \beta < 1$, $\gamma > 0$, $p > 0, T > t > z > a$, then the following inequality holds:*

$$u(t)\,{}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u(t) \geqslant \frac{1}{2}\,{}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u^2(t)\ ,\ \ 0 < \beta < 1\,. \tag{6.2}$$

Proof. To prove (6.2), we need to prove that

$$u(t)^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u(t)-\frac{1}{2}{}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u^2(t)\geqslant 0$$

is valid. Let us show the main steps:

$$\begin{aligned}
&u(t)^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u(t)-\frac{1}{2}{}^{CKP}D_{a+}^{\alpha,\beta,\gamma,\delta}u^2(t)=\\
&u(t)\int_a^t\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)z^{p-1}\left(z^{1-p}\frac{d}{dz}\right)u(z)dz-\\
&\frac{1}{2}\int_a^t\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)z^{p-1}\left(z^{1-p}\frac{d}{dz}\right)u^2(z)dz=\\
&u(t)\int_a^t\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz-\\
&\int_a^t\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u(z)u_z(z)dz=\\
&\int_a^t(u(t)-u(z))\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz=\\
&\int_a^t\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz\int_z^t u_\tau(\tau)\,d\tau=\\
&\int_a^t u_\tau(\tau)\,d\tau\int_a^\tau\left(\frac{t^p-z^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz=\\
&\int_a^t\frac{\left(\frac{t^p-\tau^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right)u_\tau(\tau)}{\left(\frac{t^p-\tau^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right)}d\tau\int_a^\tau\frac{E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)}{\left(\frac{t^p-z^p}{p}\right)^{\beta}}dz=\\
&\frac{1}{2}\int_a^t\frac{\left(\frac{t^p-\tau^p}{p}\right)^{\beta}}{E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right)}\frac{\partial}{\partial\tau}\left(\int_a^\tau\frac{E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz}{\left(\frac{t^p-z^p}{p}\right)^{\beta}}\right)^2d\tau=\\
&-\frac{1}{2}\int_a^t\frac{\tau^{p-1}\left(\frac{t^p-\tau^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right)}{\left[\left(\frac{t^p-\tau^p}{p}\right)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right)\right]^2}\left(\int_a^\tau\frac{E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-z^p}{p}\right)^{\alpha}\right)u_z(z)dz}{\left(\frac{t^p-z^p}{p}\right)^{\beta}}\right)^2d\tau.
\end{aligned}$$

Based on the Definition 2.9, Lemma 2.15, and Theorem 2.12, we obtain following inequalities:

$$\left(\frac{t^p-\tau^p}{p}\right)^{-\beta} E_{\alpha,1-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right) > 0\,,$$

$$\tau^{p-1}\left(\frac{t^p-\tau^p}{p}\right)^{-\beta-1} E_{\alpha,-\beta}^{-\gamma}\left(\delta\left(\frac{t^p-\tau^p}{p}\right)^{\alpha}\right) < 0.$$

Considering these inequalities, based on the last term, one can easily conclude that (6.2) is valid. □

Theorem 6.2. *Let all conditions of the Lemma 6.1 be valid. If there exists a solution to the Problem, then it is unique.*

Proof. Let us assume that the Problem has two solutions. Namely, $u_1(t,x)$ and $u_2(t,x)$. According to the properties of the linear operators, the function $u(t,x) = u_1\,(t,x) - u_2\,(t,x)$ will also be the solution to the Problem. Consequently, we have the following auxiliary boundary problem:

$$^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}u(t,x) - u_{xx}(t,x) = 0\,; \tag{6.3}$$

$$u(0,x) = 0\,, 0 \leqslant x \leqslant 1\,, \tag{6.4}$$

$$u(t,0) = u(t,1) = 0\,, 0 \leqslant t \leqslant T. \tag{6.5}$$

At first, we multiply the $u(t,x)$ function by both sides of the equation (6.3) and integrate the resulting concerning from 0 to 1, then we have

$$\int_0^1 u(t,x)^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}u(t,x)\,dx - \int_0^1 u(t,x)u_{xx}(t,x)\,dx = 0\,. \tag{6.6}$$

If we apply the technique of integration by parts to the second term on the left side of (6.6), using the conditions (6.5), then we get the following equality:

$$\int_0^1 u(t,x)u_{xx}(t,x)\,dx = -\int_0^1 u_x^2(t,x)\,dx\,. \tag{6.7}$$

Applying the result of the Lemma 6.1 to the first term of the left side of (6.6), we obtain the following inequality:

$$\int_0^1 u(t,x)^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta}u(t,x)\,dx \geqslant \frac{1}{2}{}^{PC}D_{at;\psi}^{\alpha,\beta,\gamma,\delta}\int_0^1 u^2(t,x)dx\,. \tag{6.8}$$

If we substitute (6.7) and (6.8) into 6.6, and using the L_2 space norm , we obtain

$$\frac{1}{2}{}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}\left\|u\left(t,x\right)\right\|_{L_2}^2+\left\|u_x\left(t,x\right)\right\|_{L_2}^2\leqslant 0\,. \tag{6.9}$$

Let us assume that $\left\|u_x\left(t,x\right)\right\|_{L_2}^2\leqslant 0$. According to the properties of the norm, we have $u_x\left(t,x\right)=0$ or $u\left(t,x\right)=g(t)$. Based on the conditions (6.5), we obtain that $u\left(t,x\right)=0$. On the other hand, let us assume that

$${}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta}\left\|u\left(t,x\right)\right\|_{L_2}^2\leqslant 0\,. \tag{6.10}$$

If we apply the integral ${}^{P}I_{ax;\psi}^{\alpha,\beta,\gamma,\delta}$ to both sides of the inequality (6.10), then we have

$$\left\|u\left(t,x\right)\right\|_{L_2}^2-\left\|u\left(0,x\right)\right\|_{L_2}^2\leqslant 0.$$

Since the condition (6.4) and the properties of the norm, we get $u(t,x)\equiv 0$. Therefore, $u_1\left(t,x\right)\equiv u_2\left(t,x\right)$. This means that the solution to the Problem is unique. ☐

Now we prove the existence of the solution to the Problem.

6.2. **The existence of the problem.**

Theorem 6.3. *If* $0\leqslant\theta<\frac{1}{2},\frac{\beta}{2}>\alpha\theta,\gamma>\frac{\theta}{1-\theta},\beta>\theta(\beta-\alpha),\delta<0,\frac{\partial^3 f(t,x)}{\partial x^3}\in C(\overline{\Omega}),f(t,1)=f(t,0)=\frac{\partial^2 f(t,1)}{\partial x^2}=\frac{\partial^2 f(t,0)}{\partial x^2}=0,B'(x)\in L_2(0,1),B^{(3)}(x)\in L_2(0,1),B^{(5)}(x)\in L_2(0,1),B''(0)=B''(1)=B^{(4)}(1)=B^{(4)}(0)=0,f(t,x)\in C(\overline{\Omega}),f(\cdot,x)\in C^{(1)}[0,T]$, *then there exists a solution to the Problem, represented as*

$$u(t,x)=\sum_{m=1}^{+\infty}\left[B_m-(\pi m)^2\Gamma\left(\gamma\right)B_m\left(\frac{t^p}{p}\right)^{\beta}\times\right.$$

$$E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\left|\begin{matrix}-(\pi m)^2\left(p^{-1}t^p\right)^{\beta}\\ \delta\left(p^{-1}t^p\right)^{\alpha}\end{matrix}\right.\right)+\int\limits_0^t\left(\frac{t^p-z^p}{p}\right)^{\beta-1}\Gamma\left(\gamma\right)\times$$

$$\left.E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\left|\begin{matrix}-(\pi m)^2p^{-\beta}(t^p-z^p)^{\beta}\\ \delta p^{-\alpha}(t^p-z^p)^{\alpha}\end{matrix}\right.\right)z^{p-1}f_m(z)dz\right]\sin\pi mx\,. \tag{6.11}$$

Proof. We search solution to the Problem as follows:

$$\begin{aligned}u(t,x)&=\sum_{m=1}^{+\infty}u_m(t)\sin\pi mx\,,\\ f(t,x)&=\sum_{m=1}^{+\infty}f_m(t)\sin\pi mx,\end{aligned} \tag{6.12}$$

where $u_m(t)$ are unknown functions to be found and $f_m(t)=2\int\limits_0^1 f(t,x)\sin m\pi x dx$.

Substituting (6.12) into (6.1) and considering the initial condition, we obtain the following Cauchy problem:

$$\begin{cases} {}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta} u_m(t) - \left[-(\pi m)^2\right] u_m(t) = f_m(t) \,, \\ u_m(0) = B_m \,. \end{cases} \tag{6.13}$$

where B_m are the Fourier coefficients of the given function $B(x)$, defined as

$$B_m = 2\int_0^1 B(x)\sin\pi m x\, dx \,.$$

Based on the solution of the problem (5.7)-(5.8), we can represent the solution to the problem (6.13) as follows:

$$u_m(t) = B_m - (\pi m)^2 \Gamma(\gamma) B_m \left(\frac{t^p}{p}\right)^{\beta} \times$$
$$E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 \\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1 \end{matrix}\middle| \begin{matrix} -(\pi m)^2\left(p^{-1}t^p\right)^{\beta} \\ \delta\left(p^{-1}t^p\right)^{\alpha} \end{matrix}\right) + \int_0^t \left(\frac{t^p - z^p}{p}\right)^{\beta-1} \Gamma(\gamma) \times$$
$$E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 \\ \beta,\beta,\alpha;\gamma,\gamma;1,1 \end{matrix}\middle| \begin{matrix} -(\pi m)^2 p^{-\beta}(t^p - z^p)^{\beta} \\ \delta p^{-\alpha}(t^p - z^p)^{\alpha} \end{matrix}\right) z^{p-1} f_m(z) dz. \tag{6.14}$$

If we substitute (6.14) into (6.12), then we obtain (6.11).

To prove the uniform convergence of infinite series corresponding to $u(t,x)$,$u_{xx}(t,x)$, ${}^{CKP}D_{0+}^{\alpha,\beta,\gamma,\delta} u(t,x)$, we have used the Lemma 2.17 and the following well-known facts:

$$2ab \leqslant a^2 + b^2,\ \sum_{m=1}^{+\infty} |g_m|^2 \leqslant \|g(x)\|^2_{L_2(a,b)}\,,\ \sum_{m=1}^{+\infty} \frac{1}{(\pi m)^2} = \frac{1}{6}.$$

Consequently, we have obtained the following estimates:

$$|u(t,x)| \leqslant \frac{1}{6} + \|B'(x)\|^2_{L_2(0,1)} +$$
$$C_1\left(\sum_{m=1}^{+\infty} \frac{1}{(\pi m)^{4+4\theta}} + \|B'''(x)\|^2_{L_2(0,1)}\right) + C_3 \sum_{m=1}^{+\infty} \frac{1}{(\pi m)^{1+2\theta}}\,,$$

$$|u_{xx}(t,x)| \leqslant \frac{1}{6} + \left\|B^{(3)}(x)\right\|^2_{L_2(0,1)} +$$
$$C_1\left(\sum_{m=1}^{+\infty} \frac{1}{(\pi m)^{4+4\theta}} + \left\|B^{(5)}(x)\right\|^2_{L_2(0,1)}\right) + C_4 \sum_{m=1}^{+\infty} \frac{1}{(\pi m)^{1+2\theta}}\,,$$

$$\left|{}^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta} u(t,x)\right| \leqslant |u_{xx}(t,x)| + |f(t,x)|\,.$$

Here

$$C_1 = \frac{\tilde{C}}{(-\delta)^{\theta}} \frac{\Gamma\left(\frac{\beta+1}{2} - \beta\theta\right)\Gamma\left(\frac{\beta+1}{2} - \alpha\theta\right)\Gamma\left(\gamma - \theta(\gamma+1)\right)}{\Gamma\left(\beta + 1 - \theta(\beta+\alpha)\right)}\Gamma(\gamma)\,,$$

$$C_2 = \frac{\tilde{C}}{(-\delta)^{\theta}} \frac{\Gamma\left(\frac{\beta}{2} - \beta\theta\right)\Gamma\left(\frac{\beta}{2} - \alpha\theta\right)\Gamma\left(\gamma - \theta\left(\gamma + 1\right)\right)}{\Gamma\left(\beta - \theta\left(\beta + \alpha\right)\right)} \Gamma\left(\gamma\right),$$

$C_3 = C_2 M_1 \left[\frac{\left(\frac{T^p}{p}\right)^{\beta-\beta\theta-\alpha\theta}}{\beta-\beta\theta-\alpha\theta}\right]$, $C_4 = C_2 M_2 \left[\frac{\left(\frac{T^p}{p}\right)^{\beta-\beta\theta-\alpha\theta}}{\beta-\beta\theta-\alpha\theta}\right]$,

$\left|f_m^{(1)}(t)\right| = 2\left|\int\limits_0^1 \frac{\partial f(t,x)}{\partial x} \cos\pi m x\, dx\right| \leqslant M_1$,

$\left|f_m^{(3)}(t)\right| = 2\left|\int\limits_0^1 \frac{\partial^3 f(t,x)}{\partial x^3} \cos\pi m x\, dx\right| \leqslant M_2$, $\frac{\partial^3 f(t,x)}{\partial x^3} \in C\left(\overline{\Omega}\right)$, $\frac{\partial f(t,x)}{\partial x} \in C\left(\overline{\Omega}\right)$, $f(t,1) = f(t,0) = \frac{\partial^2 f(t,1)}{\partial x^2} = \frac{\partial^2 f(t,0)}{\partial x^2} = 0$, $\beta > \theta\left(\beta - \alpha\right)$, $B'(x) \in L_2\left(0,1\right)$, $B^{(3)}(x) \in L_2\left(0,1\right)$,$B^{(5)}(x) \in L_2\left(0,1\right)$,$B''(0) = B''(1) = B^{(4)}(1) = B^{(4)}(0) = 0$, $f(t,x) \in C\left(\overline{\Omega}\right)$, $f(t,\cdot) \in C^{(1)}[0,1]$, $\left|B_m^{(3)}\right| = 2\int\limits_0^1 B'''(x)\cos\pi m x dx$,

$\left|B_m^{(1)}\right| = 2\int\limits_0^1 B'(x)\cos\pi m x dx$, $\left|B_m^{(5)}\right| = 2\left|\int\limits_0^1 B^{(5)}(x)\cos\pi m x dx\right|$. Hence, the theorem has been proved. □

7. Example

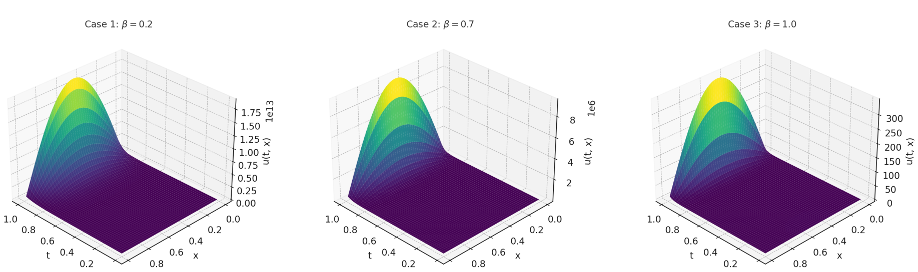
In this section, we consider the particular case, present an explicit solution form, and revise the influence of fractional order compared with the classical case.

Let, $f(t,x) = \frac{z^p}{p}\sin\pi x$, $B(x) = \sin\pi x$. Then solution (6.11) will have the form

$$u(t,x) = \left[1 - (\pi)^2\Gamma\left(\gamma\right)\left(\frac{t^p}{p}\right)^{\beta} E_2\left(\begin{array}{c} \gamma,\gamma,1;1,0 \\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1 \end{array} \middle| \begin{array}{l} -(\pi)^2(p^{-1}t^p)^{\beta} \\ \delta(p^{-1}t^p)^{\alpha} \end{array}\right) + \Gamma\left(\gamma\right)\left(\frac{t^p}{p}\right)^{\beta+1} E_2\left(\begin{array}{c} \gamma,\gamma,1;1,0 \\ \beta+2,\beta,\alpha;\gamma,\gamma;1,1 \end{array} \middle| \begin{array}{l} -(\pi)^2 p^{-\beta}(t^p - z^p)^{\beta} \\ \delta p^{-\beta}(t^p - z^p)^{\alpha} \end{array}\right)\right]\sin\pi x\,,$$

and $f_1(t) = \frac{t^p}{p}$; $B_1 = 1$, $f_m(t) = 0$; $B_m = 0$ for all other m. Let us consider the following cases:

1) p=2 ; $\beta = 0.2$; $\gamma = 1.3$; $\delta = 2$; $0 < x < 1$; $0 < t < 1$; $\alpha = 0.5$;
2) p=2 ; $\beta = 0.7$; $\gamma = 1.3$; $\delta = 2$; $0 < x < 1$; $0 < t < 1$; $\alpha = 0.5$;
3) p=2 ; $\beta = 1$; $\gamma = 1.3$; $\delta = 2$; $0 < x < 1$; $0 < t < 1$; $\alpha = 0.5$;



3D surface plots of the function $u(t,x)$ for three cases with $p=2$, $\gamma=1.3$, $\delta=2$, $\alpha=0.5$, and $\beta=0.2$, 0.7, 1, over the domain $0<x<1$, $0<t<1$.
Each plot highlights the influence of β on the solution profile.

Acknowledgment. The authors express their sincere gratitude to Professor E.T. Karimov for his invaluable insights, constructive feedback, and expert guidance, which greatly enhanced the quality and rigor of this work. The First author acknowledges financial support from Sultan Qaboos University through internal grant no. IG/SCI/MATH/25/08

References

1. Almeida, R., Malinowska, A. B., & Odzijewicz, T. (2016). Fractional differential equations with dependence on the Caputo–Katugampola derivative. arXiv preprint arXiv:1607.06913. https://doi.org/10.1115/1.4034432
2. Atanackovic, T., Pilipovic, S., Stankovic, B., & Zorica, D. (2014). Fractional calculus with applications in mechanics: Vibrations and diffusion processes. Wiley-ISTE. https://doi.org/10.1002/9781118577530
3. Baleanu, D., Guvenc, Z. B., & Machado, J. T. (2010). New trends in nanotechnology and fractional calculus applications. Springer. https://doi.org/10.1007/978-90-481-3293-5
4. Berhail, A., Tabouche, N., Alzabut, J., & Samei, M. E. (2022). Using the Hilfer–Katugampola fractional derivative in initial-value Mathieu fractional differential equations with application to a particle in the plane. Advances in Continuous and Discrete Models, 44. https://doi.org/10.1186/s13662-022-03716-6
5. Boucenna, D., Makhlouf, A. B., & Hammami, M. A. (2020). On Katugampola fractional order derivatives and Darboux problem for differential equations. CUBO A Mathematical Journal, 22, 125–136. https://doi.org/10.4067/S0719-06462020000100125
6. Copot, D., Tenreiro Machado, J., & Bates, J. (2017). The role of fractional calculus in modeling biological phenomena: A review. Communications in Nonlinear Science and Numerical Simulation, 51, 141–159. https://doi.org/10.1016/j.cnsns.2017.04.001
7. Daoudi, A., Benhamou, M., & El Kinani, E. H. (2025). The study of fractional anomalous diffusion laws on the hypersphere. Gulf Journal of Mathematics, 19(2), 121–141. https://doi.org/10.56947/gjom.v19i2.2772
8. Fernandes, A., Restrepo, J. E., & Suragan, D. (2022). Prabhakar-type linear differential equations with variable coefficients. arXiv preprint arXiv:2205.13062. https://doi.org/10.48550/arXiv.2205.13062
9. Garg, M., Manohar, P., & Kalla, S. L. (2013). A Mittag-Leffler function in two variables. Integral Transforms and Special Functions, 24, 934–944. https://doi.org/10.1080/10652469.2013.789872
10. Garrapa, R., & Maione, G. (2017). Fractional Prabhakar derivative and applications in anomalous dielectrics: A numerical approach. Springer Proceedings in Mathematics & Statistics, 407, 429–439. https://doi.org/10.1007/978-3-319-45474-0_38
11. Garra, R., & Garrappa, R. (2018). The Prabhakar or three-parameter Mittag-Leffler function: Theory and application. Communications in Nonlinear Science and Numerical Simulation, 56, 314–329. https://doi.org/10.1016/j.cnsns.2017.08.018
12. Garra, R., Gorenflo, R., Polito, F., & Tomovski, Z. (2014). Hilfer-Prabhakar derivatives and some applications. Applied Mathematics and Computation, 242, 576–589. https://doi.org/10.1016/j.amc.2014.05.129
13. Giusti, A., Colombaro, I., Garra, R., Garrappa, R., Polito, F., Popolizio, M., & Mainardi, F. (2020). A practical guide to Prabhakar fractional calculus. Fractional Calculus and Applied Analysis, 23, 9–54. https://doi.org/10.1515/fca-2020-0002

14. Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. (2014). Mittag-Leffler functions, related topics, and applications. Springer. https://doi.org/10.1007/978-3-662-43930-2
15. Hasanov, A., & Karimov, E. (2023). On a boundary-value problem in a bounded domain for a time-fractional diffusion equation with the Prabhakar fractional derivative. Bulletin of the Karaganda University. Mathematics Series, 111, 39–46. https://doi.org/10.31489/2023M3/39-46
16. Karimov, E., Tokmagambetov, N., & Toshpulatov, M. (2024). On a mixed equation involving Prabhakar fractional order integral-differential operators. Trends in Mathematics, Extended Abstracts, 2, 221–230. https://doi.org/10.1007/978-3-031-42539-4_25
17. Katugampola, U. N. (2014). A new approach to a generalized fractional integral. arXiv preprint arXiv:1106.0965. https://doi.org/10.48550/arXiv.1106.0965
18. Katugampola, U. N. (2015). Mellin transforms of generalized fractional integrals and derivatives. arXiv preprint arXiv:1112.6031. https://doi.org/10.48550/arXiv.1112.6031
19. Khalid, M., & Alha, S. (2023). Generalized formal integral transform on Ψ-Hilfer-Prabhakar fractional derivatives and its applications. Authorea preprint. https://doi.org/10.22541/au.167407978.87072452/v1
20. Kilbas, A. A., Saygo, M., & Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms and Special Functions, 15, 31–49. https://doi.org/10.1080/10652460310001600717
21. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations. Elsevier. DOI:10.1016/S0304-0208(06)80001-0
22. Korbel, J., & Luchko, Y. (2016). Modeling financial processes with a space-time fractional diffusion equation of varying order. Fractional Calculus and Applied Analysis, 19, 1414–1433. https://doi.org/10.1515/fca-2016-0073
23. Lupińska, B. (2021). Properties of the Katugampola fractional operators. Tatra Mountains Mathematical Publications, 79, 135–148. https://doi.org/10.2478/tmmp-2021-0024
24. Lupińska, B., & Schmeidel, E. (2021). Analysis of some Katugampola fractional differential equations with fractional boundary conditions. Mathematical Biosciences and Engineering, 18, 7269–7279. https://doi.org/10.3934/mbe.2021359
25. Lupińska, B., Odzijewicz, T., & Schmeidel, E. (2016). On the solutions to a generalized fractional Cauchy problem. Applicable Analysis and Discrete Mathematics, 10, 332–344. https://doi.org/10.2298/AADM161005023L
26. Ma, L., & Chen, Y. (2024). Analysis of Caputo–Katugampola fractional differential system. The European Physical Journal Plus, 139. DOI:10.1140/epjp/s13360-024-04949-y
27. Mainardi, F. (2010). Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models. World Scientific. https://doi.org/10.1142/p614
28. Nazeer, N., Asjad, M. I., Azam, M. KH., & Akgul, A. (2022). Study of results of Katugampola fractional derivative and Chebyshev inequalities. International Journal of Applied and Computational Mathematics, 8. https://doi.org/10.1007/s40819-022-01426-x
29. Njeddi, N., Batiha, I. M., Harrouche, N., Al-Samadi, M., & Momani, S. (2025). Advanced solutions for nonlinear fractional equations. Gulf Journal of Mathematics, 19(2), 93–110. https://doi.org/10.56947/gjom.v19i2.2575
30. Ntauyas, K., Ahmad, B., & Tariboon, J. (2024). Recent results on fractional Lyapunov-type inequalities: A survey. Fractional Differential Calculus, 14, 109–169. https://doi.org/10.7153/fdc-2024-14-07
31. Oliveira, D. S., & Oliveira, E. C. (2017). Hilfer-Katugampola fractional derivative. arXiv preprint arXiv:1705.07733. https://doi.org/10.48550/arXiv.1705.07733
32. Oliveira, D. S. (2021). Properties of Ψ-Mittag-Leffler fractional integrals. Rendiconti del Circolo Matematico di Palermo, 71, 233–246. https://doi.org/10.1007/s12215-021-00605-x
33. Oliveira, E. C., Mainardi, F., & Vaz, J. (2011). Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. The European Physical Journal Special Topics, 193, 161–171. https://doi.org/10.1140/epjst/e2011-01388-0

34. Ozarslan, R., Sekerci, Y., & Bas, E. (2023). Ψ–Katugampola fractional derivatives and integrals application to mass–spring damper system. Authorea preprint. https://doi.org/10.22541/au.159078077.78858403
35. Panchal, S. K., Khandagale, A. D., & Pravinkumar, V. D. (2016). k-Hilfer-Prabhakar fractional derivatives and applications. arXiv preprint arXiv:1609.05696. https://doi.org/10.48550/arXiv.1609.05696
36. Panchal, S. K., Khandagale, A. D., & Pravinkumar, V. D. (2016). Sumudu transform of Hilfer-Prabhakar fractional derivatives and applications. arXiv preprint arXiv:1608.08017. https://doi.org/10.48550/arXiv.1609.05696
37. Podlubny, I. (1999). Fractional differential equations. Academic Press. https://lib.ugent.be/catalog/rug01:002178612
38. Sachin, M. K., Pravinkumar, V. D., & Kirtiwant, P. G. (2024). Prabhakar and Hilfer-Prabhakar fractional derivatives in the setting of Ψ-fractional calculus and its applications. Journal of Fractional Calculus and Applications, 48, 515–533. https://doi.org/10.46793/KgJMat2404.515M
39. Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). Fractional integrals and derivatives: Theory and applications. Gordon and Breach Science Publishers. https://search.worldcat.org/title/fractional-integrals-and-derivatives-theory-and-applications/oclc/26401805
40. Samraiz, M. T., Perveen, Z., Rahman, G., Nisar, K. S., & Kumar, D. (2020). On the (k, s)-Hilfer-Prabhakar fractional derivative with applications to mathematical physics. Frontiers in Physics, 8, 1–9. https://doi.org/10.3389/fphy.2020.00309
41. Shaimardan, S., Karimov, E., Ruzhanskiy, M., & Mamanazarov, A. (2023). The Prabhakar fractional q-integral and q-differential operators, and their properties. arXiv preprint arXiv:2212.08843. https://doi.org/10.48550/arXiv.2212.08843
42. Tarasov, V. E. (2010). Fractional dynamics: Applications of fractional calculus to dynamics of particles. Springer. https://link.springer.com/book/10.1007/978-3-642-14003-7
43. Widder, D. V. (1946). The Laplace transform. Princeton Mathematical Series. https://archive.org/details/dli.ernet.206074
44. Zhao, D., & Sun, H. G. (2019). Anomalous relaxation model based on the fractional derivative with a Prabhakar-like kernel. Zeitschrift für Angewandte Mathematik und Physik, 70. https://doi.org/10.1007/s00033-019-1086-2

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