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PERIODIC DYNAMICS FOR SEMILINEAR EQUATION WITH BIOLOGICAL APPLICATIONS

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Abstract. This work investigates the existence of periodic solutions in a class of reaction diffusion systems arising in population dynamics. By leveraging the properties of semiFredholm operators and the contraction mapping principle, we establish sufficient conditions for the existence and uniqueness of periodic solutions for the following equation $\frac{d}{dt}v(t) - \Delta v(t) = \mathscr{B}v(t) + \mathscr{N}(t, v(t))$. Specifically, we demonstrate that a small perturbation of the linear part and the Lipschitz constant of the nonlinear part guarantees the periodicity of solutions. As an application of our theoretical framework, we analyze a diffusive logistic model and establish the existence of biologically meaningful periodic oscillations.

1. Introduction and preliminary results

Differential equations play a central role in various scientific fields, such as epidemiology or the dynamics of energy systems [2, 15, 16]. In the study of complex systems, key concepts such as stability and periodicity of solutions have been widely explored, providing essential tools for understanding their behavior [1, 3, 10]. In this context, our paper aims to establish sufficient conditions for the existence of periodic solutions in the following abstract reaction-diffusion equations:

$$\frac{d}{dt}v(t) = \Delta v(t) + \mathscr{B}v(t) + \mathscr{N}(t, v(t)), \quad t \geq 0. \tag{1.1}$$

where $(\mathscr{D}(\Delta), \Delta)$ is the Laplacian operator mapping from $\mathscr{D}(\Delta) \subset (\mathscr{E}, \|.\|)$ to $(\mathscr{E}, \|.\|)$ such that $\overline{\mathscr{D}(\Delta)} \neq \mathscr{E}$, $\mathscr{B}$ is a bounded linear operator on the Banach space $\mathscr{E}$ and $\mathscr{N}$ is a nonlinear continuous function defined from $\mathbb{R}^+ \times \mathscr{E}$ to $\mathscr{E}$.

Periodic solutions of Eq. (1.1) have long been a focal point in the study of dynamical systems, offering critical insights into oscillatory phenomena across applications. While classical approaches, such as those in [4, 5, 12, 14, 20], rely heavily on compactness of the Poincaré map or stringent $\mathscr{N}$-Lipschitz conditions, these methods often prove inadequate for systems arising in biology, ecology, or

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physics, where nonlinearities may lack smoothness or exhibit nonlocal behavior. Such limitations underscore the need for more flexible analytical tools to address periodicity in non-idealized settings. Recently, in [9, 10, 11], the authors have extensively studied the existence of periodic solutions for certain classes of linear and neutral partial differential equations lacking compactness. Their approach relies on the perturbation of semi-Fredholm operators, the well-known Chow and Hale fixed point theorem and the Poincaré map. In this work, we extend previous studies on nonlinear partial differential equation. Our Our methodology circumvents the traditional dependence on Laplacian-generated C_0-semigroup compactness and imposes significantly relaxed assumptions on the nonlinear component $\mathscr{N}$, thereby extending the scope of existing theory to broader classes of reaction-diffusion systems.

Before proceeding, we first introduce the key definitions and present the fundamental theorem on semi-Fredholm operator perturbations that will serve as the foundation for our analysis.

Definition 1.1. An operator $T \in \mathscr{L}(\mathscr{E}, \mathbb{Y})$ is semi-Fredholm if $\dim \mathscr{K}(T) < \infty$ and $\mathscr{R}(T)$ is closed in $\mathbb{Y}$, where $\mathscr{R}(T)$ and $\mathscr{K}(T)$ signify the range and the kernel of T, respectively.

In this study, we will use $\mathcal{SF}_+(\mathscr{E})$ to represent the set of all semi-Fredholm operators defined from the space $\mathscr{E}$ to $\mathscr{E}$.

Now, we announce the following results taken from [18, 14]. Let $T \in \mathscr{L}(\mathscr{E})$. Then, $(\mathscr{E}/\mathscr{K}(T), |\cdot|)$ is a Banach quotient space such that

$$|\overline{z}| = dist(z, \mathscr{K}(T)),$$

where $\overline{z} = z + \mathscr{K}(T)$ belongs to $\mathscr{E}/\mathscr{K}(T)$.

Proposition 1.2. [18] *Let $T \in \mathscr{L}(\mathscr{E})$. So, $\mathscr{R}(T)$ is a closed subspace if and only if for $\delta > 0$, the inequality below holds:*

$$|\overline{z}| \leq \delta \, \|Tz\| \qquad for\, all \qquad z \in \mathscr{E}.$$

Theorem 1.3. [14] *Let $T \in \mathcal{SF}_+(\mathscr{E})$ and $G \in \mathscr{L}(\mathscr{E})$. If $|G| < \dfrac{1}{2\delta(\sqrt{dim\mathscr{K}(T)}+1)}$, where δ is the constant given in Proposition (1.2). Then,*

$$T + G \in \mathcal{SF}_+(\mathscr{E}).$$

Furthermore, $\dim \mathscr{K}(T+G) \leq \dim \mathscr{K}(T)$.

The paper is structured as follows: Section 2 establishes the existence and uniqueness of periodic solutions in the linear case. Building upon the linear analysis developed in Section 2, we generalize these results to the nonlinear case described by Eq. (1.1), establishing the existence of corresponding periodic solutions. Section 4 demonstrates the practical applicability of our theoretical results by investigating a biological system governed by the dynamics established in Section 3.

To develop a rigorous analytical framework for studying the dynamics of Eq. (1.1), we introduce the following assumptions, which are crucial for ensuring the existence and uniqueness of a periodic solution.

(**A**): There are three constants $p \in \mathbb{N}$, $\widehat{K} \in [1, +\infty)$ and $\widehat{\nu} \in \mathbb{R}$ such that $\rho(\Delta) \supset (\widehat{\nu}, +\infty)$ and

$$|(\varsigma I - \Delta)^{-p}| \quad \leq \frac{\widehat{K}}{(\varsigma - \widehat{\nu})^p}, \quad \widehat{\nu} < \varsigma.$$

(**B**): f is T-periodic.

(**C**):

- $\mathscr{N}$ is T-periodic in t. Moreover, there is $N_{R,\mathscr{N}} \geq 0$ such that $\|\mathscr{N}(t,z)\| \leq N_{R,\mathscr{N}}$ for $t \leq T$ and $\|z\| \leq R$.
- $\mathscr{N}$ is locally Lipschitz, which means that for every $H > 0$, there exists $K_{H,F} > 0$ such that for all functions $z, \overline{z} \in \mathscr{E}$ satisfying $\|z\| \leq H$ and $\|\overline{z}\| \leq H$, the inequality below is valid:

$$\|\mathscr{N}(t,\overline{z}) - \mathscr{N}(t,z)\| \ \leq \ K_{H,\mathscr{N}} \ \|\overline{z} - z\|.$$

  $K_{H,\mathscr{N}}$ signifies the local Lipschitz constant of $\mathscr{N}$.

2. Existence of periodic solutions for linear equation

This section establishes the existence and uniqueness of periodic solutions for the linear equation stated below:

$$\frac{d}{dt}v(t) - \Delta v(t) = \mathscr{B}v(t) + f(t) \quad \text{for} \ \ t \geq 0, \tag{2.1}$$

To properly analyze Eq. (2.1) with initial condition $v(0) = v_0$, we must first introduce several key definitions and preliminary results concerning its solutions.

Let $(\mathscr{D}(\Delta_0), \Delta_0)$ be the part of $(\mathscr{D}(\Delta), \Delta)$ defined by:

$$\mathscr{D}(\Delta_0) = \{z \in \mathscr{D}(\Delta) : \Delta z \in \overline{\mathscr{D}(\Delta)}\} \ \text{ and } \ \Delta_0 z = \Delta z \ \text{ for } \ z \in \mathscr{D}(\Delta_0).$$

It is established that Δ_0 is the generator of a C_0-semigroup $\{\mathscr{O}(t), t \geq 0\}$ on $\overline{\mathscr{D}(\Delta)}$ which satisfies:

$$|\mathscr{O}(t)| \leq K e^{\nu t}, \ t \geq 0,$$

where $K \geq 1$ and $\nu \in \mathbb{R}$.

We henceforth assume $K = 1$, without loss of generality.

Theorem 2.1. [13] *Under assumption* (**A**). *If v_0 belongs to $\overline{\mathscr{D}(\Delta)}$, then, $v : (0, +\infty) \to \mathscr{E}$, the integral solution of Eq.* (2.1)*, is unique and satisfies:*

$$v(t) = \mathscr{O}(t)v(0) + \lim_{\varsigma \to +\infty} \int_0^t \mathscr{O}(t-s)\varsigma(\varsigma I - \Delta)^{-1} \left(\mathscr{B}v(s) + f(s)\right) ds, \quad t \geq 0.$$

Through this work, we refer to the integral solution of Eq. (2.1) as the solution. Moreover, the solution of Eq. (2.1) such that $f = 0$, denoted for $t \geq 0$ by $\mathscr{W}(t)v_0 = v(., v_0, \mathscr{B}, 0)$, is expressed as:

$$\mathscr{W}(t)v_0 = \mathscr{O}(t)v_0 + \mathcal{R}(t)v_0, \tag{2.2}$$

where $\mathcal{R}(t)$, $t \geq 0$ is given by:

$$\mathcal{R}(t)v_0 \begin{cases} \lim_{\varsigma \to +\infty} \int_0^{t+\theta} \mathscr{O}(t-s)\varsigma(\varsigma I - \Delta)^{-1}\mathscr{B}v(s)\, ds, & for \quad t \geq 0, \\ 0 & for \quad t = 0. \end{cases}$$

Proposition 2.2. *Under assumption* $(\mathbf{A})$. *Let* $|\mathscr{O}(t)| \leq e^{\nu t}$, $t \geq 0$. *Then*

$$|\mathcal{R}(t)| \leq e^{\nu^+ t}\left(e^{|\mathscr{B}|t} - 1\right),$$

such that $\nu^+ = \sup\{\nu, 0\}$.

Proof. We denote by $v(t, v_0, \mathscr{B}, 0)$ the solution of Eq. (2.1) in the case where $f = 0$. Then

$$\|v(t, v_0, \mathscr{B}, 0)\| \leq e^{\nu^+ t}\|v_0\| + |\mathscr{B}| \int_0^t e^{\nu^+(t-\sigma)}\|v(\sigma, v_0, \mathscr{B}, 0)\|\, d\sigma,$$

and

$$e^{-\nu^+ t}\|v(t, w_0, \mathscr{B}, 0)\| \leq \|v_0\| + |\mathscr{B}| \int_0^t e^{-\nu^+ \sigma}\|v(\sigma, v_0, \mathscr{B}, 0)\|\, d\sigma.$$

From Gronwall-Bellman Inequality, we get

$$\|v(t, v_0, \mathscr{B}, 0)\| \leq e^{(\nu^+ + |\mathscr{B}|)t}\|v_0\|.$$

So,

$$\begin{aligned} \|\mathcal{R}(t)v_0\| &\leq |\mathscr{B}| \int_0^t e^{\nu^+(t-\sigma)}\|v(\sigma, v_0, \mathscr{B}, 0)\|\, d\sigma \leq |\mathscr{B}\ |e^{\nu^+ t} \int_0^t e^{|\mathscr{B}|\sigma}\, d\sigma\ \|v_0\| \\ &\leq e^{\nu^+ t}\left(e^{|\mathscr{B}|t} - 1\right)\|v_0\|. \end{aligned}$$

□

Proposition 2.3. *Assume that* $I - \mathscr{O}(T)$, $T > 0$ *is a semi-Fredholm operator in* $\overline{\mathscr{D}(\Delta)}$ *such that* $n = \dim \mathscr{K}(I - \mathscr{O}(T))$. *Moreover, let* δ *be a positive constant such that*

$$|\overline{z}| \leq \delta\|(I - \mathscr{O}(T))z\|, \quad z \in \overline{\mathscr{D}(\Delta)}.$$

If

$$|\mathscr{B}| < \frac{1}{T}\ln\left(\frac{e^{-\nu^+ T}}{2\delta(1+\sqrt{n})} + 1\right), \tag{2.3}$$

then, $I - \mathcal{W}(T)$ *is a semi-Fredholm operator in* $\overline{\mathscr{D}(\Delta)}$ *with* $\dim \mathscr{K}(I - \mathcal{W}(T)) \leq n$.

Proof. By estimation (2.3), it follows that

$$e^{\nu^+T}\left(e^{|\mathscr{B}|T}-1\right)<\frac{1}{\delta(1+\sqrt{n})},$$

and by Proposition (2.2), one has

$$|\mathcal{R}(T)|<\frac{1}{\delta(1+\sqrt{n})}.$$

Theorem (1.3) and the fact that $I-\mathscr{O}(T)$, $T>0$ is a semi-Fredholm operator in $\overline{\mathscr{D}(\Delta)}$ give that $I-\mathcal{W}(T)$ is a semi-Fredholm operator in $\overline{\mathscr{D}(\Delta)}$ with $\dim\mathscr{K}(I-\mathcal{W}(T))\leq\mathscr{K}(I-\mathscr{O}(T))=n$. $\square$

Theorem 2.4. [14] *Define the linear affine map $\mathcal{L}$ by $\mathcal{L}x=\overline{\mathcal{L}}x+y$ for $x\in\mathscr{E}$. Let $\left(\mathcal{L}^k x_0\right)_{k\in\mathbb{N}}$ be a bounded set in $\mathscr{E}$, $x_0\in\mathscr{E}$. Moreover, assume that $I-\overline{\mathcal{L}}\in\mathcal{SF}_+(\mathscr{E})$. Then, $\mathcal{L}$ has a fixed point.*

Additionally, for the remainder of this work, we say that an equation satisfies property $(\mathcal{P})$ if it has a T-periodic solution, and satisfies property $(\mathcal{UP})$ if it has a unique T-periodic solution.

Theorem 2.5. *Under assumptions* (**A**) *and* (**B**). *Assume that $I-\mathscr{O}(T)$, $T>0$ is a semi-Fredholm operator in $\overline{\mathscr{D}(\Delta)}$. If $\mathscr{B}$ satisfies the estimate:*

$$|\mathscr{B}|<\frac{1}{T}\ln\left(\frac{e^{-\nu^+T}}{2\delta(1+\sqrt{n})}+1\right),\tag{2.4}$$

then, the boundedness of solution for Eq. (2.1) *over the interval* $[0,+\infty)$ *implies that Eq.* (2.1) *verifies the property* $(\mathcal{P})$.

Proof. It is sufficient to demonstrate that the Poincaré map $\mathscr{P}_T$, defined by $\mathscr{P}_T(v_0)=v(T,v_0,\mathscr{B},f)$ on $\overline{\mathscr{D}(\Delta)}$ has a fixed point, where $v(T,v_0,\mathscr{B},f)$ denotes the solution of Eq. (2.1). Since Eq. (2.1) has a unique solution, the Poincaré map $\mathscr{P}_T$ is decomposed as

$$\mathscr{P}_T(v_0)=v(T,v_0,\mathscr{B},0)+v(T,0,\mathscr{B},f),$$

where $v(T,v_0,\mathscr{B},0)$ and $v(T,0,\mathscr{B},f)$ indicate the solutions of Eq. (2.1) with $f=0$ and $v_0=0$, respectively, then $\mathscr{P}_T$ is given by $\mathscr{P}_T(v_0)=\widehat{\mathscr{P}_T}(v_0)+w_0$, where $\widehat{\mathscr{P}_T}(v_0)=v(T,v_0,\mathscr{B},0)$ and $w_0=v(T,0,,f)$. From (2.2), $\widehat{\mathscr{P}_T}$ is expressed as $\widehat{\mathscr{P}_T}=\mathscr{W}(T)=\mathscr{O}(T)+\mathcal{R}(T)$. In addition, Proposition (2.3) and estimation (2.4) imply that $I-\widehat{\mathscr{P}_T}\in\mathcal{SF}_+(\overline{\mathscr{D}(\Delta)})$. Now, let $v(T,v_0,\mathscr{B},f)$ be a bounded solution of Eq. (2.1). Since $\mathscr{P}_T(v_0)=v(T,v_0,\mathscr{B},f)$, it folows that $\mathscr{P}_T^2(v_0)=v(2T,v_0,\mathscr{B},f)$ and $\mathscr{P}_T^3(v_0)=v(3T,v_0,\mathscr{B},f)$. Thus, by induction on n, one obtains

$$\{\mathscr{P}_T^n v_0:n\in\mathbb{N}\}=\{v(nT,v_0,\mathscr{B},f):n\in\mathbb{N}\}.$$

Therefore, since $v(T,v_0,\mathscr{B},f)$ is bounded, the boundedness of $v(nT,v_0,\mathscr{B},f)$ follows from its variation of constants formula. Which implies the boundedness of $(\mathscr{P}_T^n v_0)_{n\geq 0}$ in $(\overline{\mathscr{D}(\Delta)})$. From Theorem (2.4), we deduce that the fixed point set $\mathcal{F}_{\mathscr{P}_T}\neq\varnothing$. Consequently, Eq. (2.1) satisfies the property $(\mathcal{P})$. $\square$

Corollary 2.6. *Under assumptions* (**A**) *and* (**B**). *Let* $|\mathscr{O}(t)| \leq e^{\nu t}$ *for* $t \geq 0$ *and* $\nu < 0$. *If*

$$|\mathscr{B}| < \frac{1}{T} \ln \left(\frac{1 - e^{\nu T}}{2} + 1 \right), \tag{2.5}$$

then, the boundedness of solution for Eq. (2.1) *over the interval* $[0, +\infty)$ *implies that Eq.* (2.1) *verifies the property* $(\mathcal{P})$.

Proof. To establish the desired outcome, we need to determine the explicit values of the parameters δ and n in inequality (2.4).Furthermore, the exponential stability of $\{\mathscr{O}(t), t \geq 0\}$ guarantees that the operator $I - \mathscr{O}(T)$ is invertible for $T > 0$, so, $\mathscr{K}(I - \mathscr{O}(T)) = \overline{\mathscr{D}(\Delta)}$, which implies that $I - \mathscr{O}(T) \in \mathcal{SF}_+(\overline{\mathscr{D}(\cdot)})$ with $n = \dim \mathscr{K}(I - \mathscr{O}(T)) = 0$. Now, for $x \in \overline{\mathscr{D}(\cdot)}$, we get

$$\|x\| \leq \|x - \mathscr{O}(T)x\| + \|\mathscr{O}(T)x\| \leq \|x - \mathscr{O}(T)x\| + e^{\nu T}\|x\|.$$

and

$$(1 - e^{\nu T})\|x\| \leq \|x - \mathscr{O}(T)x\|, \quad x \in \overline{\mathscr{D}(\cdot)}.$$

Thus

$$|\overline{x}| = \|x\| \leq \frac{1}{(1 - e^{\nu T})} \|x - \mathscr{O}(T)x\|.$$

Consequently, From Proposition (1.2), we select δ such that

$$\delta \leq \frac{1}{(1 - e^{\nu T})}.$$

Thus, for δ satisfying $\delta \leq \dfrac{1}{1 - e^{\nu T}}$, the bound in (2.5) ensures the validity of inequality (2.4) and Theorem (2.5) guarantees that Eq. (2.1) verifies the property $(\mathcal{P})$. □

3. Uniqueness of periodic solutions for linear equation

To prove the existence of unique solution for Eq. (2.1), let us introduce the following perturbed operator $\Delta_{\mathscr{B}} : \overline{\mathscr{D}(\Delta_{\mathscr{B}})} = \overline{\mathscr{D}(\Delta)} \to \mathscr{E}$, where $\mathscr{B} + \Delta = \Delta_{\mathscr{B}}$. Under this construction $\Delta_{\mathscr{B}}$ is a Hille-Yosida operator, as established in [8], such that $(\varsigma, +\infty) \subset \rho(\Delta_{\mathscr{B}})$ and

$$|\ (\varsigma I - \Delta_{\mathscr{B}})^{-p}| \quad \leq \frac{K}{(\varsigma - \nu - K|\mathscr{B}|)^p} \quad \text{for} \ \ p \in \mathbb{N} \ \ \text{and} \ \ \varsigma > \nu + K|\mathscr{B}|.$$

We now reformulate Eq. (2.1) in the following abstract form:

$$\frac{d}{dt} v(t) - \Delta_{\mathscr{B}}\ v(t) = f(t) \quad \mathit{for} \ \ t \geq 0, \tag{3.1}$$

with initial data $v(0) = v_0 \in \overline{\mathscr{D}(\Delta)}$.

Consider $\overline{\Delta_{\mathscr{B}}}$ as the part of the operator $\Delta_{\mathscr{B}}$ on $\overline{\mathscr{D}(\Delta_{\mathscr{B}})}$. Then, $\overline{\Delta_{\mathscr{B}}}$ is the generator of a C_0-semigroup $\{\mathscr{O}_{\mathscr{B}}(t)), t \geq 0\}$ such that

$$|\mathscr{O}_{\mathscr{B}}(t)| \leq e^{(|\mathscr{B}| + \nu)t} \quad \text{for} \quad t \geq 0.$$

Now, let $v_0 \in \overline{\mathscr{D}(\Delta)}$, then, Eq. (3.1) has a unique solution $v : [0, +\infty) \to \mathscr{E}$ such that:

$$v(t, v_0) = \mathscr{O}_{\mathscr{B}}(t)v_0 + \lim_{\varsigma \to +\infty} \int_0^t \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma \quad for \quad t \geq 0. \tag{3.2}$$

Theorem 3.1. *Under assumptions* (**A**) *and* (**B**). *Let* $|\mathscr{O}(t)| \leq e^{\nu t}$ *for* $t \geq 0$ *and* $\nu < 0$. *If*

$$|\mathscr{B}| < \min\left\{\frac{1}{T}\ln\left(\frac{1-e^{\nu T}}{2}+1\right), -\nu\right\}, \tag{3.3}$$

then, Eq. (2.1) *verifies the property* $(\mathcal{UP})$. *Additionally, one has*

$$\|v\|_\infty = \sup_{0 \leq \sigma \leq T} \|v(t)\| \leq \frac{T}{1 - e^{(|\mathscr{B}|+\nu)T}} \sup_{0 \leq \sigma \leq T} \|f(\sigma)\|. \tag{3.4}$$

Proof. Let

$$v(t) = \mathscr{O}_{\mathscr{B}}(t)v_0 + \lim_{\varsigma \to +\infty} \int_0^t \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma, \quad t \geq 0.$$

Then,

$$\|v(t)\| \leq e^{(|\mathscr{B}|+\nu)t}\|v_0\| + \int_0^t e^{(|\mathscr{B}|+\nu)(t-\sigma)} \sup_{0 \leq s \leq T} \|f(s)\|\, d\sigma.$$

From the estimation (3.3), we deduce the existence of a strictly positive constant M_0 satisfying:

$$\begin{aligned} \|v(t)\| &\leq \|v_0\| - \frac{\sup_{0 \leq s \leq T}\|f(s)\|}{|\mathscr{B}|+\nu}\left(1 - e^{(|\mathscr{B}|+\nu)t}\right) \\ &\leq M_0\,, \end{aligned}$$

which guarantees the boundedness of solutions for Eq. (2.1) over $[0, +\infty)$ and Corollary (2.6) confirms the existence of T-periodic solution for Eq. (2.1). On the other hand, define v_1 and v_2 as T-periodic solutions of Eq. (2.1), take $x = v_2 - v_1$, then, x is a T-periodic solution of Eq. (2.1). Using (3.2), we get $x(t) = \mathscr{O}_{\mathscr{B}}(t)\,(v_2(0) - v_1(0))$. Hence, estimation (3.3) confirms that $\lim_{t \to +\infty} \|x(t)\| = 0$, from the T-periodicity of x, we get $x(t) = 0$, which implies that Eq. (2.1) verifies the property $(\mathcal{UP})$. On the other hand, let v be the unique T-periodic solution of Eq. (2.1), using (3.2), we get

$$\begin{aligned} v(t) &= v(t+T) = \mathscr{O}_{\mathscr{B}}(t+T)v_0 + \lim_{\varsigma \to +\infty} \int_0^{t+T} \mathscr{O}_{\mathscr{B}}(t+T-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma \\ &= \mathscr{O}_{\mathscr{B}}(T)\left(\mathscr{O}_{\mathscr{B}}(t)v_0 + \lim_{\varsigma \to +\infty} \int_0^t \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma\right) \\ &\quad + \lim_{\varsigma \to +\infty} \int_t^{t+T} \mathscr{O}_{\mathscr{B}}(t-\sigma+T)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma. \end{aligned}$$

Which means that

$$v(t) = (I - \mathscr{O}_{\mathscr{B}}(T))^{-1} \lim_{\varsigma\to+\infty} \int_{t-T}^{t} \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1} f(\sigma)\, d\sigma.$$

Thus

$$\|v\|_\infty \leq T|\,(I - \mathscr{O}_{\mathscr{B}}(T))^{-1}\,| \sup_{0\leq s\leq T} \|f(s)\|.$$

Consequently, we get

$$\|v\|_\infty \leq \frac{T}{1 - e^{(|\mathscr{B}|+\nu)T}} \sup_{0\leq s\leq T} \|f(s)\|,$$

□

4. Existence and uniqueness of periodic solution for Eq. (1.1)

Let us investigate the periodic behavior for Eq. (1.1) with initial condition $v(0) = v_0 \in \overline{\mathscr{D}(\Delta)}$. To do this, we consider $(C_T(\mathbb{R}^+, \mathscr{E}), \|\cdot\|_\infty)$ the set consisting of all continuous T-periodic functions such that $\|\zeta\|_\infty = \sup_{0\leq t\leq T} \|\zeta(t)\|$. Moreover, consider the set $C_R = \{\xi \in C_T(\mathbb{R}^+, \mathscr{E}) : \|\xi\|_\infty \leq R\}$.

Theorem 4.1. *Under assumptions* (**A**) *and* (**C**). *Let* $|\mathscr{O}(t)| \leq e^{\nu t}$ *for* $t \geq 0$ *and* $\nu < 0$. *If* $|\mathscr{B}| < \min\left\{\frac{1}{T}\ln\left(\frac{1-e^{\nu T}}{2}+1\right), -\nu\right\}$ *and if there is* $R > 0$ *such that* $N_{R,\mathscr{N}} \leq \frac{R}{\Upsilon}$ *and* $K_{R,\mathscr{N}} < \frac{1}{\Upsilon}$, *where* $\Upsilon = T\left(1 + \frac{1}{1-e^{(|\mathscr{B}|+\nu)T}}\right)$, *then, Eq.* (1.1) *verifies the property* $(\mathcal{UP})$.

Proof. Let $h \in \mathscr{H}_R = \{v \in C_T : |v|_\infty \leq R\}$ and take $f(t) = \mathscr{N}(t, h_t)$. So, by Theorem (3.1), we conclude that

$$\frac{d}{dt}v(t) - \Delta v(t) = \mathscr{B}v(t) + \mathscr{N}(t, h(t)) \quad for \quad t \geq 0, \tag{4.1}$$

verifies the property $(\mathcal{UP})$ such that:

$$v(t, v_0) = \mathscr{O}_{\mathscr{B}}(t)v_0 + \lim_{\varsigma\to+\infty} \int_0^t \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1}\mathscr{N}(\sigma, h(\sigma))\, d\sigma \quad for \quad t \geq 0.$$

Currently, to demonstrate the result presented in Theorem (4.1), we will confirm that the mapping $\mathcal{Q}$ is a contraction over $\mathscr{H}_R$, where:

$$\mathcal{Q}(h)(t) = \mathscr{O}_{\mathscr{B}}(t)w_0 + \lim_{\varsigma\to+\infty} \int_0^t \mathscr{O}_{\mathscr{B}}(t-\sigma)\varsigma(\varsigma I - \Delta_{\mathscr{B}})^{-1}\mathscr{N}(\sigma, h(\sigma))\, d\sigma.$$

(i) Let $h \in \mathscr{H}_R$, so

$$\|\mathcal{Q}(h)(t)\| \leq e^{(|\mathscr{B}|+\nu)t}\|v_0\| + \int_0^t e^{(|\mathscr{B}|+\nu)(t-\sigma))}\|\mathscr{N}(\sigma, h(\sigma))\|\, d\sigma,$$

from inequality (3.4), we get

$$\|\mathcal{Q}(h)(t)\| \leq \frac{T}{1-e^{(|\mathscr{B}|+\nu)T}} \sup_{0\leq\sigma\leq T} \|\mathscr{N}(\sigma, h(\sigma))\| + T \sup_{0\leq\sigma\leq T} \|\mathscr{N}(\sigma, h(\sigma))\|.$$

So,

$$\begin{aligned}\sup_{0\leq\sigma\leq T}\|\mathcal{Q}(h)(t)\| &\leq \left(1+\frac{1}{1-e^{(|\mathscr{B}|+\nu)T}}\right)T\sup_{0\leq\sigma\leq T}\|\mathscr{N}(\sigma,h(\sigma))\|\\ &\leq \left(1+\frac{1}{1-e^{(|\mathscr{B}|+\nu)T}}\right)T\ N_{R,\mathscr{N}}\\ &\leq R.\end{aligned}$$

so, the operator $\mathcal{Q}$ maps from $\mathscr{H}_R$ to $\mathscr{H}_R$.
ii) Let $h^1, h^2 \in \mathscr{H}_R$, then

$$\begin{aligned}\|\mathcal{Q}(h^1)(t)-\mathcal{Q}(h^2)(t)\| &\leq e^{(|\mathscr{B}|+\nu)t}\|v_1(0)-v_2(0)\|\\ &\quad+\int_0^t e^{(|\mathscr{B}|+\nu)(t-\sigma)}\|\mathscr{N}(\sigma,h^1(\sigma)-\mathscr{N}(\sigma,h^2(\sigma)\|\,d\sigma\\ &\leq \frac{T}{1-e^{(|\mathscr{B}|+\nu)T}}\sup_{0\leq\sigma\leq T}\|\mathscr{N}(\sigma,h^1(\sigma))-\mathscr{N}(\sigma,h^2(\sigma))\|\\ &\quad+T\sup_{0\leq\sigma\leq T}\|\mathscr{N}(\sigma,h^1(\sigma))-\mathscr{N}(\sigma,h^2(\sigma))\|\\ &\leq \left(1+\frac{1}{1-e^{(|\mathscr{B}|+\nu)T}}\right)T\ K_{R,\mathscr{N}}\sup_{0\leq\sigma\leq T}\|h^1(\sigma)-h^2(\sigma)\|.\end{aligned}$$

Hence

$$\|\mathcal{Q}(h^1)-\mathcal{Q}(h^2)\|_\infty \leq \Upsilon K_{R,\mathscr{N}}\|h^1-h^2\|_\infty.$$

Since $K_{R,\mathscr{N}} < \dfrac{1}{\Upsilon}$, from contraction mapping theorem we deduce that $\mathcal{Q}$ admits a unique fixed point $v \in \mathscr{H}_R$. Thus, Eq. (1.1) verifies the property ($\mathcal{UP}$). □

5. Application

We now turn to the practical application of our findings. We explore the dynamics of a mathematical model that incorporate periodic interactions. Consider the following diffusive logistic equation:

$$\begin{cases}\dfrac{\partial}{\partial t}W(t,\zeta)=\Delta W(t,\zeta)-\alpha W(t,\zeta)\left(1-\dfrac{W(t,\zeta)}{b(t)}\right), \quad t\geq 0, \quad \zeta\in[0,1],\\ W(t,0)=W(t,1)=0, \quad t\geq 0,\\ W(0,\zeta)=W_0(\zeta), \quad \zeta\in[0,1],\end{cases} \tag{5.1}$$

where $\alpha > 0$, $b : \mathbb{R}^+ \to \mathbb{R}_+^*$ is continuous and T-periodic, and there exists $b_0 > 0$ such that for all $t \geq 0$, $b(t) \geq b_0$. Let $\mathscr{E} = \mathbf{C}([0,1],\mathbb{R})$ equipped with the uniform norm topology and let Δ be the linear operator defined from $\mathscr{D}(\Delta) \subset \mathscr{E}$ to $\mathscr{E}$ as:

$$\mathscr{D}(\Delta) = \{v \in \mathcal{C}^2([0,1],\mathbb{R}) : v(0) = v(1) = 0\} \text{ and } \Delta v = v''.$$

According to [8], the operator Δ verifies:

$$\rho(\Delta) \supset (0, +\infty) \text{ and } |(\varsigma I - \Delta)^{-1}| \leq \frac{1}{\varsigma} \text{ for } \varsigma > 0.$$

In addition, one has

$$\overline{\mathscr{D}(\Delta)} = \{v \in \mathscr{E} : v(0) = v(1) = 0\} \neq \mathscr{E}.$$

Furthermore, we introduce Δ_0 the part of Δ as follows:

$$\mathscr{D}(\Delta_0) = \{v \in \mathcal{C}^2([0,1], \mathbb{R}) : v(0) = v(1) = v^{"}(0) = v^{"}(1) = 0\} \text{ and } \Delta_0 v = v^{"}.$$

So, Δ_0 generates a C_0-semigroup denoted by $\{\mathscr{O}(t), t \geq 0\}$ on $\overline{\mathscr{D}(\Delta)}$ such that

$$|\mathscr{O}(t)| \leq e^{-t}, \quad t \geq 0.$$

On the other hand, let $\mathscr{B}$ be a bounded linear operator defined from $\overline{\mathscr{D}(\Delta)}$ to $\mathscr{E}$ such that $\mathscr{B}v = -\alpha v$. Moreover, let $\mathscr{N} : \mathbb{R}^+ \times \overline{\mathscr{D}(\Delta)} \to \mathscr{E}$ be defined as: $\mathscr{N}(t, v) = \alpha \dfrac{v^2}{b(t)}$.

So, Eq. (5.1) takes the abstract form Eq. (1.1).

Theorem 5.1. *If*

$$\alpha < \min\left\{1; \frac{1}{4} \ln\left(\frac{1 - e^{-T}}{2} + 1\right)\right\}. \tag{5.2}$$

Moreover, if there exists $R > 0$ satisfying

$$\frac{2R\alpha}{b_0} < \frac{1}{4\left(\dfrac{1}{1 - e^{(\alpha-1)T}} + 1\right)}, \tag{5.3}$$

then, Eq. (5.1) verifies the property $(\mathcal{UP})$.

Proof. We need to only examine the hypotheses given in Theorem (4.1). Let $x, y \in \overline{\mathscr{D}(\Delta)}$ such that $\|x\| \leq R$ and $\|y\| \leq R$, then, we have

$$\|\mathscr{N}(t, x) - \mathscr{N}(t, y)\| \leq \frac{\alpha}{b_0}\|x^2 - y^2\| \leq \frac{2R\alpha}{b_0}\|x - y\|.$$

This implies that $\mathscr{N}(t, \cdot)$ is locally Lipschitz with $K_{R,\mathscr{N}} = \dfrac{2R\alpha}{b_0}$. Furthermore, for $0 \leq t \leq T$ and $x \in \overline{\mathscr{D}(\Delta_0)}$ with $\|x\| \leq R$, one has that

$$\|\mathscr{N}(t, x)\| \leq \frac{R^2\alpha}{b_0} = N_{R,\mathscr{N}}.$$

Therefore, it is enough to compute the constants in Theorem (4.1) to ensure that the bounds (5.2) and (5.3) validate its conditions. Consequently, Eq. (5.1) verifies the property $(\mathscr{U}\mathscr{P})$. $\square$

We now verify the theoretical results through numerical simulations. The model was discretized using an explicit Euler scheme for temporal integration and second order finite differences for spatial approximation, implemented in

MATLAB. The computational grid employed $n_\zeta = 20$ nodes ($\Delta\zeta = 1/19$) with $n_t = 60{,}000$ time steps ($\Delta t \approx 2.7 \times 10^{-4}$), satisfying the CFL stability condition:

$$\Delta t < \frac{(\Delta\zeta)^2}{2},$$

which ensures numerical stability. For the numerical verification of Eq. (5.1), we select parameters fulfilling all requirements of Theorem 5.1. Specifically, we set:

- $T = 4$
- $\alpha = 0.095$
- $b(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$
- Initial condition: $S_0(\theta, \zeta) = 0.1\sin(\pi\zeta)\cos\left(\frac{\pi}{2}\theta\right)$.

As a result, for $R = 5$, the bounds (5.2) and (5.3) hold. This result is illustrated by Figure 1 in $3D$ and figure 2 in $2D$ for 0.5.

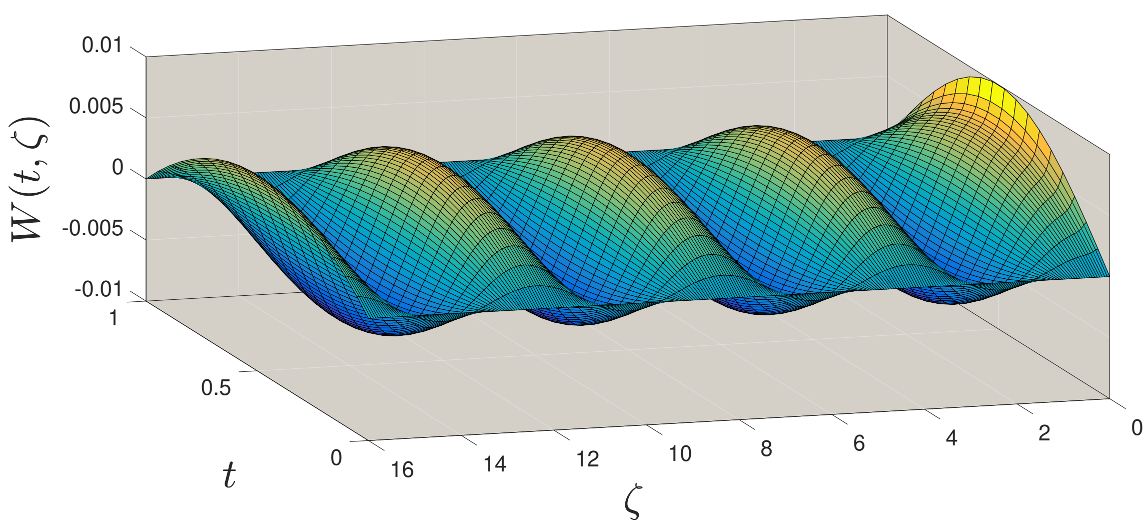

FIGURE 1. The evolution of the solution to the system (5.1) in 3D.

6. Conclusion

We have developed a framework for proving the existence and uniqueness of periodic solutions in reaction-diffusion systems, utilizing semi-Fredholm operators and the contraction mapping principle. Our results ensure periodicity under small perturbations and controlled nonlinearities. As an application, we established periodic oscillations in a diffusive logistic model, providing insights into biologically meaningful dynamics. This approach opens avenues for further exploration in mathematical biology and related fields.

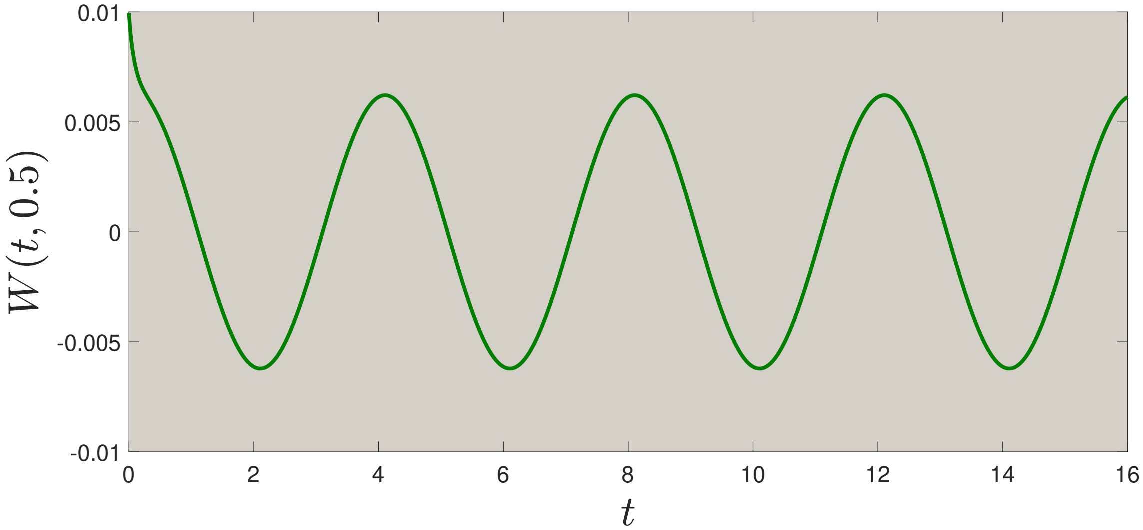

FIGURE 2. The evolution of the solution to the system (5.1) in 2D such that $\zeta = 0.5$.

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