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ON NIL-SEMICOMMUTATIVE MODULES

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Abstract. In this paper, we introduce the concept of nil-semicommutative modules and present it as an extension of nil-semicommutative rings to modules. We prove that the class of nil-semicommutative modules is contained in the class of weakly semicommutative modules, while that of the converse may not be true. We also show that in case of semicommutative modules and nil-semicommutative modules, one does not imply the other. Moreover, for a given nil-semicommutative ring, we provide the conditions under which the same can be extended to a nil-semicommutative module and show that the converse may not be true in general. We also find the conditions under which the quotient module M/N is nil-semicommutative if and only if M is nil-semicommutative. Further, we also prove that for a left R-module M, ${}_RM$ is nil-semicommutative iff its localization $S^{-1}M$ over the ring $S^{-1}R$ is also nil-semicommutative. Lastly, we explore whether the nil-semicommutative property is preserved under direct product of modules.

1. Introduction and preliminaries

Throughout the study, the ring R is associative with identity, and the module M is a unitary left R-module. $M_n(R)$, $T_n(R)$, and $S_n(R)$ denote respectively the ring of $n \times n$ matrices, upper triangular matrices, and special upper triangular matrices with the same diagonal entries over R. By a domain D, we mean a ring without zero divisors i.e. $\forall\ ab = 0$, we have either $a = 0$ or $b = 0$. And by a commutative domain, we mean a commutative ring without zero divisors. An element m is torsion if $\exists\ 0 \neq r \in R$ such that $rm = 0$ and the collection of such elements is denoted by $Tor(M)$. A torsion-free module M is a module satisfying the condition $Tor(M) = 0$. For S_0 denoting the set of non-zero zero divisors of a ring R, $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in S_0\}$. $Nil(R)$ denotes the set of all nilpotent elements in the ring R and $Nil_R(M)$ denotes the set of all nilpotent elements in the left R-module M.

The concept of $Nil(R)$ has been there for long, and the utilization of this concept by many algebraists can be seen in numerous concepts in the form of reduced rings, semicommutative rings, weakly semicommutative rings, nil-semicommutative rings, etc. In fact, it has been studied and proved in [7] that

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semicommutative ring $\Rightarrow$ nil-semicommutative ring $\Rightarrow$ weakly semicommutative ring with the converses not holding in each of the cases. Several other results based on the concept of nilpotency in rings are also being studied and discussed presently.

However, the concept of nilpotent elements in a module was defined and studied by Ssevviiri and Groenewald [12] in 2013, and with that, a breakthrough in extending existing results on nilpotency in rings to their module counterparts was achieved . An element $m \in M$ is said to be a nilpotent element if either $m = 0$ or if for $m \neq 0$, $\exists\ r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0$ but $rm \neq 0$.

Moreover, using this definition, Ansari and Herachandra in [2] came up with a result that further simplified the finding of nilpotent elements in modules. In their paper, they proved the following proposition.

"Let m be an element of the left R-module M. Then, the following conditions are equivalent.

(i) $\exists\ r \in R$ and $n \geq 2$ such that $r^n m = 0; r^{n-1} m \neq 0$.
(ii) $\exists\ t \in R$ such that $t^2 m = 0; tm \neq 0$" (Nazeer and Herachandra, 2020, p. 323).

So, if one wants to check whether an element $m \in Nil_R(M)$, it suffices to check whether either of the above conditions holds.

While reviewing the literature, it is learned that after inspiration from a result on reduced rings (rings with no non-zero nilpotent elements) in [1] by E. Armendariz, Rege, and Chhawchharia introduced the concept of Armendariz rings in [13]. A ring R is Armendariz if given any polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ with coefficients in R satisfying $f(x).g(x) = 0$, we have $a_i b_j = 0\ \forall\ i, j$. Further, many new studies were made on this for some time, and in [5], Antoine introduced the concept of nil-Armendariz rings as a generalization of Armendariz rings. It is defined that "a ring R is nil-Armendariz if for polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ with co-efficients in R satisfying the condition $f(x).g(x) \in Nil(R)[x]$, we have $a_i b_j \in Nil(R)\ \forall i, j$" (Antoine, 2008, p.3130). Then, in [11], Liu and Zhao introduced weak Armendariz rings and proved it as a further generalization of nil-Armendariz rings. In their words, "a ring R is weak Armendariz ring if for polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ with coefficients in R satisfying the condition $f(x).g(x) = 0$, we have $a_i b_j \in Nil(R)$" (Liu and Zhao, 2006, p.2608). Thus, in short, one can observe that Armendariz rings $\Rightarrow$ nil-Armendariz rings $\Rightarrow$ weak Armendariz rings.

Similarly, in [9], Lee and Zhou introduced the concept of reduced modules and provided numerous generalizations in the module counterparts of the previously mentioned rings. In [9], a left R-module M is reduced if it satisfies any of the following conditions.

(i) whenever $a \in R, m \in M$ satisfy $a^2 m = 0$, we have $aRm = 0$
(ii) whenever $a \in R, m \in M$ satisfy $am = 0$, we have $aM \bigcap Rm = 0$

Moreover, the concept of Armendariz modules was also introduced in [13] and numerous studies on the relationship of this class of modules with the class of reduced modules and semicommutative modules were done by Rege and Buhphang in [6]. In [13], a left R-module M is said to be Armendariz if whenever

$f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{m} m_j x^j \in M[x]$ satisfy $f(x).m(x) = 0$, then $a_i m_j = 0 \ \forall \ i, j$.

Then, Ansari and Herachandra, in [3], made further studies on the concept of nilpotent elements in modules and their role and impact on various types of modules. In fact, it was proved in [3] that for a reduced module, there are no non-zero nilpotent elements. Further, the extension of the concept of weak Armendariz rings to weak Armendariz modules was also done in [3], thereby generalizing the said ring theoretical concept. A left R-module M is said to be weak Armendariz if whenever $f(x) = \sum_{i=0}^{n} a_i x^j \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$, we have $a_i m_j \in Nil_R(M) \ \forall \ i, j$. Moreover, in [4] Ansari et.al introduced the concept of nil-Armendariz modules as a smaller class of weak Armendariz modules. However, it has also been shown in [4] that between Armendariz modules and nil-Armendariz modules, one may not imply the other. A left R-module M is said to be nil-Armendariz if whenever $f(x) = \sum_{i=0}^{n} a_i x^j \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) \in Nil_R(M)[x]$, we have $a_i m_j \in Nil_R(M) \ \forall \ i, j$.

On the other hand, semicommutative rings were studied in [8] by Huh et al. in 2002, and various other studies on it were also done under the names of IFP rings and zero-insertive(ZI) rings in the literature. A generalization on the same - semicommutative modules were then introduced and studied by Buhphang and Rege [6] in the same year. In [2], it is defined that a ring is said to be semicommutative if whenever $a, b \in R$ satisfy $ab = 0$, we have $aRb = 0$. And in [6], it is defined that a left R-module M is said to be semicommutative module if whenever $a \in R$ and $m \in M$ satisfy $am = 0$, we have $aRm = 0$.

Weakly semicommutative rings were introduced in [10], and numerous studies were done on it by Liang et al. A ring is said to be weakly semicommutative if whenever $a, b \in R$ satisfy $ab = 0$, we have $aRb \subseteq Nil(R)$. Inspired by this, in [2] Ansari and Herachandra introduced weakly semicommutative modules as an extension of weakly semicommutative rings. A left R module M is said to be weakly semicommutative if whenever $a \in R$ and $m \in M$ satisfy $am = 0$, then $aRm \subseteq Nil_R(M)$.

Lastly, nil-semicommutative rings were defined and studied by Chen [7] in 2011, with further contributions made by Moussavi et al. in 2012. In [7], a ring is said to be nil-semicommutative if whenever $a, b \in R$ satisfy $ab \in Nil(R)$, we have $aRb \subseteq Nil(R)$.

2. Main results

In this section, we introduce nil-semicommutative module and present it as an extension of the concept of nil-semicommutative rings to modules. We also investigate numerous preliminary results on the new class of modules and show that many properties of semicommutative, weakly semicommutative and Armendariz modules can be extended to that of nil-semicommutative modules under the general setting.

Definition 2.1. A left R- module M is said to be nil-semicommutative if whenever $a \in R$ and $m \in M$ satisfy $am \in Nil_R(M)$, we have $aRm \subseteq Nil_R(M)$.

Recall that in the case of modules, reduced $\Rightarrow$ semicommutative $\Rightarrow$ weakly semicommutative. Similarly, by definition of nil-semicommutative modules, it is clear to say that reduced $\Rightarrow$ nil-semicommutative $\Rightarrow$ weakly semicommutative. However, as we will show in example 2.7, the converse may not be true in general. Moreover, in the case of rings, we know that every semicommutative ring is nil-semicommutative, and it looks like the same follows in the case of modules. But, the Example 2.7 which we will discuss below will eliminate such possibilities, i.e., there exist modules that are nil-semicommutative but not semicommutative, and there are modules that are semicommutative but not nil-semicommutative.

Remark 2.2. A weakly semicommutative module may not necessarily be a nil-semicommutative module. The following Example 2.7 is one such example.

Remark 2.3. A semicommutative module may not necessarily be a nil-semicommutative module. The following Example 2.7 is one such example.

Lemma 2.4. *For the left $\mathbb{Z}_n$- module $\mathbb{Z}_n$, $\bar{1} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n) \iff n$ is not square-free.*

Proof. ($\Rightarrow$) Suppose, n is square-free. Then, $n = p_1p_2p_3...p_k$ for primes p_i and $k \in \mathbb{N}$. Now, as $\bar{1} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$ $\exists$ $\bar{r} \in \mathbb{Z}_n$ and $k_0 \in \mathbb{N}$ such that $\bar{r}^{k_0}.\bar{1} = \bar{0}$ but $\bar{r}.\bar{1} \neq \bar{0}$. This means that $r^{k_0} = n.l$ for some $l \in \mathbb{Z}$ and so, $n|r^k$ *i.e.* $(p_1p_2...p_k)|r^k$. This implies that $p_i \mid r^{k_0}$ $\forall$ $i = 1$ to k. Consequently, $p_i \mid r$ $\forall$ i ($\because$ all $p_i's$ are primes) and we have $r = (p_1p_2p_3...p_k).t$ for some $t \in \mathbb{Z}$. However, this means that $\bar{r}.\bar{1} = \bar{0}$ which is a contradiction. Hence, n cannot be square-free.

For the converse, since n is not square-free, $n = p_1^k.s$ for $k \geq 2$ and some $s \in \mathbb{Z}$. Now, $\exists$ $(\bar{p_1.s}) \in \mathbb{Z}_n$ such that $(\bar{p_1.s})^k.\bar{1} = \bar{0}$ but $(\bar{p_1.s}).\bar{1} \neq \bar{0}$. Hence, $\bar{1} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$. ∎

Lemma 2.5. *For the left $\mathbb{Z}_n$- module $\mathbb{Z}_n$, $Nil_{\mathbb{Z}_n}(\mathbb{Z}_n) = \{0\} \iff n$ is square-free.*

Proof. ($\Rightarrow$) Suppose n is not square-free, then $\bar{1} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$ by the above lemma. But, this is a contradiction as $Nil_{\mathbb{Z}_n}(\mathbb{Z}_n) = \{0\}$. So, we have n is square-free.

For the converse, assume $\exists$ $\bar{0} \neq \bar{x} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$. Then, $\exists$ $\bar{r} \in \mathbb{Z}_n, k \in \mathbb{N}$ such that $(\bar{r})^k.\bar{x} = \bar{0}$ but $(\bar{r}).\bar{x} \neq \bar{0}$. But, as n is square free, $n = p_1.p_2.p_3...p_k$ for some primes p_i and $k \in \mathbb{N}$. So, we have $p_i \mid r^kx$ $\forall$ i. But, all $p_i's$ are primes. So, $p_i \mid r^k$ or $p_i \mid x$ $\forall$ i. In either case, $r.x = (p_1.p_2...p_k).l = \bar{0}$ which is a contradiction. Hence, $Nil_{\mathbb{Z}_n}(\mathbb{Z}_n) = \{\bar{0}\}$. ∎

Lemma 2.6. *For the left $\mathbb{Z}_n$- module $\mathbb{Z}_n$, ${}_{\mathbb{Z}_n}\mathbb{Z}_n$ is nil-semicommutative $\iff n$ is square-free.*

Proof. ($\Rightarrow$) Suppose n is not square-free. Then, $n = p^k.t$ for some $k \geq 2$ and $t \in \mathbb{Z}$. Also, by Lemma 2.4 $\bar{1}.\bar{1} \in Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$. We claim that $\bar{p}^{k-1} \notin Nil_{\mathbb{Z}_n}(\mathbb{Z}_n)$. Suppose this is not the case. Then, $\exists$ $\bar{r} \in {}_{\mathbb{Z}_n}\mathbb{Z}_n$ such that $\bar{r}^k.\bar{p}^{k-1} = \bar{0}$ but

$\bar{r}.\bar{p}^{k-1} \neq \bar{0}$. This implies that $r^k.p^{k-1} = p^k.l$ and hence, $p \mid r$. So, $\bar{r}.\bar{p}^{k-1} = \bar{0}$ which is a contradiction. Hence, n should be square-free. Converse follows easily from Lemma 2.5. □

Example 2.7. Consider the $\mathbb{Z}_{p^n}$-module $\mathbb{Z}_{p^n}$. Then, $\mathbb{Z}_{p^n}$ being a module over a commutative ring is semicommutative, and also weakly semicommutative. However, for $n \geq 2$, $\mathbb{Z}_{p^n}$ is not nil-semicommutative.

Proof. The proof follows easily from the fact that p^n is not square-free for $n \geq 2$. □

We now quote a lemma from [4] which shows that the matrix module ${}_{M_n(R)}M_n(M)$ is a nil module for any left R-module M.

Lemma 2.8. *([4], Lemma 1) For a left R-module M, $Nil_{M_n(R)}(M_n(M)) = {}_{M_n(R)}M_n(M)$ $\forall\, n \geq 2$.*

Proof. Let $L = [l_{ij}]_{n\times n}$ be a non-zero matrix. Then, $l_{ij} \neq 0$ for some $1 \leq i,j \leq n$. So, we have :

(i) When $i \neq j$, take $r = E_{ji}$ (the elementary matrix with ji^{th} entry $= 1$). Then, $r^2L = (E_{ji})^2L = 0$; $rL = E_{ji}L \neq 0$.
(ii) When $i = j$, take $r = E_{ki}$ such that $k \neq i$ and $1 \leq k,i \leq n$. Then, $r^2L = (E_{ki})^2L = 0$; $rL = E_{ki}L \neq 0$.

Therefore, $L \in Nil_{M_n(R)}(M_n(M))$ and hence, $Nil_{M_n(R)}(M_n(M)) = M_n(M)$ ([4], Pg-3) □

Remark 2.9. Nil modules are always nil-semicommutative and also weakly semicommutative.

Remark 2.10. Submodules of nil-semicommutative modules are nil-semicommutative.

Remark 2.11. A nil-semicommutative module need not necessarily be a semicommutative module. The following is one such example.

Example 2.12. Consider a left R-module M. Then, by Lemma 2.8 $M_n(M)$ is nil-semicommutative over $M_n(R)$ for $n \geq 2$. However, as shown below, it is not semicommutative for $n \geq 4$.

Proof. Let $A = \begin{pmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ and $K = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in$ $M_n(M)$ where $m \neq 0$.

Then, we can see that

$$AK = \begin{pmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} = 0$$

But, for $L = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(R)$, we have

$$ALK = \begin{pmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

Hence, ${}_{M_n(R)}M_n(M)$ is not semicommutative for $n \geq 4$.

□

Proposition 2.13. *Let R be a reduced ring and M be a torsion-free left R-module. Then, $Nil_R(M) = \{0\} = Tor(M)$.*

Proof. Let R be a reduced ring and $m \in Nil_R(M)$. Then, $\exists\ r_0 \in R$ and $k_0 \in \mathbb{N}$ such that $r_0^{k_0}.m = 0$; $r_0.m \neq 0$. But, $r^{k_0} \neq 0$ and M is torsion-free, so $m = 0$. So, $Nil_R(M) = \{0\}$.

□

Proposition 2.14. *Let M be left R-module. If M is torsion-free, then the following statements hold and are also equivalent.*
(i) ${}_RM$ is semicommutative
(ii) ${}_RM$ is nil-semicommutative
(iii) ${}_RM$ is weakly semicommutative

Proof. The proof follows easily from the fact that if M is torsion-free, then $Nil_R(M) = \{0\}$.

$(i) \Rightarrow (ii)$ Let $a \in R, m \in M$ be such that $am \in Nil_R(M)$. But, M is torsion-free, so $Nil_R(M) = \{0\}$ and so, $am = 0$. Thus, $aRm = \{0\} \subseteq Nil_R(M)$. Hence, M is nil-semicommutative module.

$(ii) \Rightarrow (iii)$ This follows from definition.

$(iii) \Rightarrow (i)$ Let $a \in R$ and $m \in M$ be such that $am = 0$. Then, as ${}_RM$ is weakly semicommutative module, $aRm \subseteq Nil_R(M) = \{0\}$. So, $aRm = \{0\}$. Hence, ${}_RM$ is semicommutative module. □

Proposition 2.15. *For a ring R and a left R - module M, the following conditions are equivalent.*
(i) M is nil-semicommutative module.
(ii) Every submodule of ${}_RM$ is nil-semicommutative module.
(iii) Every finitely generated submodule of ${}_RM$ is nil-semicommutative.
(iv) Every cyclic submodule of ${}_RM$ is nil-semicommutative module.

Proof. $(i) \Rightarrow (ii)$ Let $N \subseteq M$ be a submodule and let $a \in R$, $n \in N$ be such that $an \in Nil_R(N)$.Then, $an \in Nil_R(M)$. So, $aRn \subseteq Nil_R(M)$. Thus, $\exists\ r_0 \in R$ and $k_0 \in \mathbb{N}$ such that $r_0^{k_0}(atn) = 0, r_0(atn) \neq 0\ \forall\ t \in R$. But, $atn \in N$, so we have $atn \in Nil_R(N)$. Hence, $aRn \subseteq Nil_R(N)$. Thus, N is nil-semicommutative module.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ follow clearly.

$(iv) \Rightarrow (i)$ Let $a \in R$ and $m \in M$ be such that $am \in Nil_R(M)$. This means that $\exists\ r_0 \in R$ and $k_0 \in \mathbb{N}$ such that $r_0^{k_0}(am) = 0$; $r_0(am) \neq 0$. Now, $m = 1.m \in Rm \leq M$. This implies that $am \in Rm$ and consequently, $am \in Nil_R(Rm)$. But, as Rm is cyclic submodule of M, we have $aRm \subseteq Nil_R(Rm) \subseteq Nil_R(M)$. So, we have $aRm \subseteq Nil_R(M)$. Hence, M is nil-semicommutative.

□

Although we have seen in Lemma 2.8 that for any given ${}_RM$ module and $n \geq 2$, ${}_{M_n(R)}M_n(M)$ is always a nilmodule and hence, a nil-semicommutative module. However, this may not be so in case of ${}_{T_n(R)}T_n(M)$. The following is one such example.

Example 2.16. Consider the $\mathbb{Z}_{p^n}$ - module $\mathbb{Z}_{p^n}$. Then, for $n \geq 2$, ${}_{T_n(\mathbb{Z}_{p^n})}T_n(\mathbb{Z}_{p^n})$ is not nil-semicommutative.

Proof. Let $A = \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix}$ and $K = \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} \in {}_{T_n(\mathbb{Z}_{p^n})}T_n(\mathbb{Z}_{p^n})$

Then, $AK = \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix} \in Nil_{T_n(\mathbb{Z}_{p^n})}(T_n(\mathbb{Z}_{p^n}))$.

as $\exists\ P = \begin{pmatrix} \bar{p} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix} \in T_n(\mathbb{Z}_{p^n})$ such that $P^n(AK) = 0$ but $P(AK) \neq 0$

But, for $L = \begin{pmatrix} \bar{p}^{n-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix}$, we claim that $ALK = \begin{pmatrix} \bar{p}^{n-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix}$ $\notin Nil_{T_n(\mathbb{Z}_{p^n})}(T_n(\mathbb{Z}_{p^n}))$.

For this, suppose $\exists\ S = \begin{pmatrix} \bar{s_1} & * & \dots & * \\ \bar{0} & \bar{s_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{s_n} \end{pmatrix} \in T_n(\mathbb{Z}_{p^n})$

such that $S^k(ALK) = 0$ but $S(ALK) \neq 0$.

Now, $S^k(ALK) = 0$

$$\implies \begin{pmatrix} \bar{s_1} & * & \dots & * \\ \bar{0} & \bar{s_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{s_n} \end{pmatrix}^k \begin{pmatrix} \bar{p}^{n-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} \bar{s_1}^k \bar{p}^{n-1} & * & \dots & * \\ \bar{0} & \bar{0} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \bar{0} \end{pmatrix} = 0$$

$\implies {s_1}^k p^{n-1} = p^n s$ for some $s \in \mathbb{Z}$.

$\implies p \mid s_1^k$

Consequently, $p \mid s_1$ and thus, $s_1 = pl$ for some $l \in \mathbb{Z}$.

So, $S(ALK) = \begin{pmatrix} \bar{pl} & * & \dots & * \\ \bar{0} & \bar{a_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{a_n} \end{pmatrix} \begin{pmatrix} \bar{p}^{n-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} = 0$, which is a contradiction.

Therefore, there does not exist such S and so, $ALK \notin Nil_{T_n(\mathbb{Z}_{p^n})}(T_n(\mathbb{Z}_{p^n}))$. Hence, ${}_{T_n(\mathbb{Z}_{p^n})}T_n(\mathbb{Z}_{p^n})$ is not nil-semicommutative. $\square$

Moreover, unlike in weakly semicommutative case as in [2], ${}_RM$ being a nil-semicommutative module may not necessarily imply that ${}_{T_n(R)}T_n(M)$ is also nil-semicommutative. The following is one such example.

Example 2.17. Consider the $\mathbb{Z}_p$-module $\mathbb{Z}_p$. Then, $\mathbb{Z}_p$ is nil-semicommutative as it is torsion-free. However, for $n \geq 2$, ${}_{T_n(\mathbb{Z}_p)}T_n(\mathbb{Z}_p)$ is not nil-semicommutative.

Proof. Let $A = \begin{pmatrix} \overline{p-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \overline{p-1} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \overline{p-1} \end{pmatrix}$ and $K = \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{1} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{1} \end{pmatrix} \in {}_{T_n(\mathbb{Z}_p)}T_n(\mathbb{Z}_p)$

Then, $AK = \begin{pmatrix} \overline{p-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \overline{p-1} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \overline{p-1} \end{pmatrix} \in Nil_{T_n(\mathbb{Z}_p)}(T_n(\mathbb{Z}_p))$

as $\exists\ P = \begin{pmatrix} \bar{0} & \bar{0} & \dots & \bar{1} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} \in {}_{T_n(\mathbb{Z}_p)}T_n(\mathbb{Z}_p)$
such that $P^2(AK) = 0$ but $P(AK) \neq 0$.

But, for $L = \begin{pmatrix} \bar{0} & \bar{0} & \dots & \overline{p-1} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} \in T_n(\mathbb{Z}_p)$, as in Example 2.16, we can easily prove that

$$ALK = \begin{pmatrix} \overline{p-1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \overline{p-1} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \overline{p-1} \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{0} & \dots & \overline{p-1} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} \\ \bar{0} & \bar{1} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{1} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{0} & \bar{0} & \dots & (\overline{p-1})^2 \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{0} & \bar{0} & \dots & \bar{1} \\ \bar{0} & \bar{0} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{0} \end{pmatrix} \notin Nil_{T_n(\mathbb{Z}_p)}(T_n(\mathbb{Z}_p))$$

Hence, ${}_{T_n(\mathbb{Z}_p)}T_n(\mathbb{Z}_p)$ is not nil-semicommutative.

□

Further, recall from [9] that for a given left R-module M and $A = (a_{ij}) \in M_n(R)$, the ring of $n \times n$ matrices over R, $AM = \{(a_{ij}m) : m \in M\}$. Moreover, for $V = \sum_{i=1}^{n-1} E_{i,i+1}, n \geq 2$ where $E_{i,j}$ are matrix units, $V_n(R) = RI_n + RV + RV^2 + \ldots + RV^{n-1}$ forms a ring and $V_n(M) = MI_n + MV + MV^2 + \ldots + MV^{n-1}$ forms a left module over the ring $V_n(R)$. In addition to this, there exists a ring isomorphism $\theta : V_n(R) \to \frac{R[x]}{(x^n)}$ defined by $\theta(r_0 I_n + r_1 V + r_2 V^2 + \ldots + r_{n-1}V^{n-1}) = r_0 + r_1 x + \ldots r_{n-1}x^{n-1} + (x^n)$ and an abelian group isomorphism $\phi : V_n(M) \to \frac{M[x]}{(M[x](x^n))}$ defined by $\phi(m_0 + m_1 V + m_2 V^2 + \ldots + m_1 V^{n-1}) = m_0 + m_1 x + m_2 x^2 + \ldots + m_{n-1}x^{n-1} + M[x](x^n)$ such that $\phi(AW) = \theta(A)\phi(W) \ \forall \ A \in V_n(R), W \in V_n(M)$.

Remark 2.18. For a left R - module M, ${}_RM$ is nil-semicommutative may not imply that ${}_{V_n(R)}V_n(M)$ is nil-semicommutative. We can consider the following Example 2.19 in this regard.

Example 2.19. Consider the $\mathbb{Z}_p$- module $\mathbb{Z}_p$. Then, ${}_{\mathbb{Z}_p}\mathbb{Z}_p$ is nil-semicommutative. However, for $n \geq 2$, ${}_{V_n(\mathbb{Z}_p)}V_n(\mathbb{Z}_p)$ is not nil-semicommutative.

Proof. The proof is similar to that of Example 2.17.

□

In Proposition 3.15 of [2], it was proved that if ${}_RM$ is weakly semicommutative, then for $n \geq 2$ $\frac{M[x]}{(M[x](x^n))}$ is also a weakly semicommutative module over $\frac{R[x]}{(x^n)}$. However, as stated in the following remark, this may not be true in the nil-semicommutative case.

Remark 2.20. For a left R- module M, ${}_RM$ is nil-semicommutative may not imply that $\frac{M[x]}{(M[x](x^n))}$ is nil-semicommutative over $\frac{R[x]}{(x^n)}$. For this, as we have seen in Example 2.19, it does not hold in the ${}_{V_n(R)}V_n(M)$ case. Hence, it will not hold in this case.

We recall from [12] that for S_0 denoting the set of nonzero divisors of R, $T(M) = [m \in M : rm = 0$ for some $r \in S_0]$. Also, for a module M over a general ring R, $T(M) \subseteq Tor(M)$ and for R-domain (ring without zero divisors), $T(M) = Tor(M)$. If R is not domain, then the equality may not hold. For instance, in the $\mathbb{Z}_6$ - module $\mathbb{Z}_6$, $\bar{2}.\bar{3} = \bar{0} \Rightarrow \bar{3} \in Tor(M)$ but $\bar{3} \notin T(M)$ as $\bar{3}$ is a zero divisor in $\mathbb{Z}_6$. Moreover, if R is a commutaive domain then, $Tor(M)$ is a submodule of M. Further, Rege and Bhuphang in [6] proved that for a left R-module M, ${}_RM$ is semicommutative module $\Rightarrow T(M)$ is a submodule of M. In addition to this, Ansari and Herachandra in [12] generalised this result when they proved that for a weakly semicommutative module M over a domain D, $T(M)$ is a submodule of M. Similarly, we have continued the trend and proved that the result holds true in case of nil-semicommutative modules as well.

Proposition 2.21. *For a nil-semicommutative module M over a domain D, $T(M)$ is a submodule of M.*

Proof. The proof follows easily from the fact that every nil-semicommutative module is weakly semicommutative.

□

Proposition 2.22. *Let D be a domain and M be a left module over D. Then, M is nil-semicommutative $\Rightarrow M/T(M)$ is semicommutative module.*

Proof. Let $a(m + T(M)) = T(M)$ for $a \in D$ and $m + T(M) \in M/T(M)$. This implies that $am \in T(M)$ and so, $\exists$ some $s \in D$ such that $s(am) = 0$. Also, since M is nil-semicommutative, we have $(sa)r(m) \in Nil_R(M)$ for any $r \in D$. Consequently, we have $t \in D$ and $k \in \mathbb{N}$ such that $t^k(sarm) = 0$ and $t(sarm) \neq 0$. Moreover, D being domain implies that $t^k s \neq 0$ and so, $arm \in T(M)$. This imples that $ar(m + T(M)) = T(M)$ and hence, $M/T(M)$ is semicommutative.

□

Proposition 2.23. *Let R be a domain and ${}_RM$ be nil-semicommutative. Then, ${}_RM$ is Armendariz $\iff$ $T(M)$ is Armendariz.*

Proof. ($\Rightarrow$) Since, $T(M) = Tor(M) \leq {}_RM$ and every submodule of an Armendariz module is Armendariz. Hence, $T(M)$ is Armendariz.

For the converse, let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. Then, for r = smallest non-negative number for which

$a_r \neq 0, r \in \{0, 1, ..., n\}$, we have

$$\begin{aligned} a_r m_0 &= 0 \\ a_{r+1} m_0 + a_r m_1 &= 0 \\ a_{r+2} m_0 + a_{r+1} m_1 + a_r m_2 &= 0 \\ &\vdots \\ a_n m_k &= 0 \end{aligned}$$

So, $m_0 \in T(M)$ and as ${}_R M$ is nil-semicommutative, we have $a_r a_{r+1} m_0 \in Nil_R(M)$. This implies that $\exists\ s \in R$ and $k \in \mathbb{N}$ such that $s^k a_r a_{r+1} m_0 = 0$ and $s a_r a_{r+1} m_0 \neq 0$. Moreover, mutltiplying second equation by a_r, we get $a_r a_{r+1} m_0 + a_r^2 m_1 = 0$. Consequently, we get $s^k a_r^2 m_1 = 0$ and as a result $m_1 \in T(M)$. Proceeding this way, we get $m_i \in T(M)\ \forall\ i = 1, ..., k$ and eventually, $m(x) \in T(M)[x]$. This implies that $f(x)m(x) = 0$ for $f(x) \in R[x]$ and $m(x) \in T(M)[x]$. But, $T(M)$ is Armendariz module, so we have $a_i m_j = 0\ \forall\ i, j$. Hence, ${}_R M$ is Armendariz module.

□

Remark 2.24. For a torsion-free module M, the following are equivalent
(i) M is nil-semicommutative module
(ii) $M/Tor(M)$ is nil-semicommutative module.

Proposition 2.25. *Let R be a commutative domain. Then, ${}_R M$ is nil-semicommutative $\iff$ Tor(M) is nil-semicommutative.*

Proof. ($\Rightarrow$) The proof follows easily from the fact that if R is commutative domain, then $Tor_R(M) \leq {}_R M$ and every submodule of a nil-semicommutative module is nil-semicommutative.

For the converse, let $a \in R$ and $m \in M$ such that $am \in Nil_R(M)$. Then, $\exists\ r \in R$ and $k \in \mathbb{N}$ such that $r^k(am) = 0, r(am) \neq 0$. This implies that $am \in Tor(M)$. So, $am \in Nil_R(Tor(M))$ and hence, $aRm \subseteq Nil_R(M)$. Therefore, M is nil-semicommutative. □

Proposition 2.26. *Let R be a commutative domain. Then, ${}_R M$ is nil-semicommutative $\Rightarrow M/Tor(M)$ is nil-semicommutative.*

Proof. Let $a \in R, \bar{m} \in M/Tor(M)$ such that $a\bar{m} \in Nil_R(\bar{M})$ where $\bar{M} = M/Tor(M)$. Then, $\exists\, k \in R$ and $n \in \mathbb{N}$ such that $k^n am + Tor(M) = Tor(M); kam + Tor(M) \neq Tor(M)$. But, $Tor(M) \leq {}_R M$. So, we have $rk^n am = k^n(arm) \in Tor(M)$ for some $r \in R$. Moreover, with a simple proof by contradiction, we can show that $rkam = k(arm) \notin Tor(M)$. Hence, $arm \in Nil_R(\bar{M})$. Thus, $\bar{M}$ or $M/Tor(M)$ is nil-semicommutative.

□

Proposition 2.27. *Let R be a commutative domain and $N \leq {}_R M$ such that $N \subseteq Nil_R(M)$. Then, $Nil_R(M/N) = Nil_R(M)/N$.*

Proof. Let $m+N \in Nil_R(M/N)$. Then, $\exists\ s_0 \in R, k_0 \in \mathbb{N}$ such that $s_0^{k_0}m+N = N; s_0m+N \neq N$. This implies that $s_0^{k_0}m \in N; s_0m \notin N$. So, $\exists\ r_0 \in R, n_0 \in \mathbb{N}$ such that $r_0^{n_0}(s_0{}^{k_0}m) = 0; r_0(s_0{}^{k_0}m) \neq 0$. Now, for $l_0 = r_0s_0{}^{k_0}$, $l_0{}^{n_0}m = 0; l_0m \neq 0$. Thus, $m \in Nil_R(M)$ implying $m+N \in Nil_R(M)/N$.
For the converse, let $m+N \in Nil_R(M)/N$. Then, $\exists\ s_0 \in R, n_0$ such that $s_0{}^{n_0}m = 0; s_0m \neq 0$. This implies that $s_0{}^{n_0}(m+N) = N; s_0(m+N) \neq N$. Hence, $m+N \in Nil_R(M/N)$.

$\square$

Proposition 2.28. *Let R be a commutative domain and $N \leq {}_RM$ such that $N \subseteq Nil_R(M)$. Then, ${}_RM$ is nil-semicommutative $\iff$ M/N is nil-semicommutative over R.*

Proof. The proof follows easily from Proposition 2.27.

$\square$

Proposition 2.29. *Let R be a commutative ring in which the degree of nilpotency for any $r \in Nil(R)$ is > 2. Then, R is nil-semicommutative ring $\Rightarrow$ ${}_RR$ is nil-semicommuatative module.*

Proof. Let a $\in R$ and $m \in R$ satisfy $am \in Nil_R(R)$. Then, two cases arise. For $am = 0$, the proof is easy. For $am \neq 0$, $\exists\ r \in R$ and $k \in \mathbb{N}$ s.t. $r^k(am) = 0$ but $r(am) \neq 0$. So, for $p = r^ka$ we have, $pm = 0$. This implies that $pm \in Nil(R)$ and hence, $pRm \subseteq Nil(R)$. Consequently, we have $(r^ka)(t)m \in Nil(R)$. Now, if $r^k(atm) = 0$, then we have $atm \in Nil_R(R)$ and so, $aRm \subseteq Nil_R(R)$. And if $r^k(atm) \neq 0$, then $\exists\ k_2 \in \mathbb{N}$ such that $\left(r^k(atm)\right)^{k_2} = 0$. Consequently, we have $r^{k.k_2}(atm)^{k_2}(atm)^{k_2(k-1)+1} = 0$ and thus, $(ratm)^{kk_2}(atm) = 0$. Now, for $r_* = ratm, q = kk_2 \in \mathbb{N}$, we have $(r_*)^q(atm) = 0$ .We then claim that $r_*(atm) \neq 0$. For if $r_*(atm) = 0$ then, $(ratm)(atm) = r(atm)^2 = 0$ which implies that $r^{2k-1}.r(atm)^2 = 0$ and so, $(r^k(atm))^2 = 0$, which is a contradiction.Therefore, in both cases, $atm \in Nil_R(R) \Rightarrow aRm \subseteq Nil_R(R)$. Hence, ${}_RR$ is nil-semicommutative module. $\square$

Remark 2.30. The converse of Proposition 2.29 may not be true in general. The following is one such example.

Example 2.31. Consider the ring $M_3(\mathbb{Z})$. Then, as per Lemma 2.8, ${}_{M_3(\mathbb{Z})}M_3(\mathbb{Z})$ is a nil module and hence nil-semicommutative module. But $M_3(\mathbb{Z})$ is not nil-semicommutative ring.

Proof. For $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = I_3$, we have $AB \in Nil(M_3(\mathbb{Z}))$.
But, for $L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$, $ALB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin Nil(M_3(\mathbb{Z}))$
$(\because \notin k \in \mathbb{N}\ s.t.\ (ALB)^k = 0)$.
Hence, $M_3(\mathbb{Z})$ is not nil-semicommutative ring.

$\square$

Proposition 2.32. *Let $\phi : R \to R'$ be a ring homomorphism. Then for a left R'-module M, M is also a left R - module via $rm = \phi(r)m$. Moreover, if ϕ is onto then, the following are equivalent.*
(i) ${}_{R'}M$ is nil-semicommutative.
(ii) ${}_RM$ is nil-semicommutative.

Proof. (i) $\Rightarrow$ (ii) Let $a \in R$, $m \in M$ such that $am \in Nil_R(M)$. Then, $\exists\ r \in R, n \in \mathbb{N}$ such that $r^k(am) = 0; r(am) \neq 0$. So, $(r^k a)m = 0$; $(ra)m \neq 0$ and consequently, $\phi(r^k a)m = 0$; $\phi(ra)m \neq 0$.This eventually implies that $\phi(r)^k\phi(a)m = 0$; $\phi(r)\phi(a)m \neq 0$ and hence, $\phi(a)m \in Nil_{R'}(M)$. But, M is nil-semicommutative R' - module. So, $\phi(a)R'm \subseteq Nil_{R'}(M)$ and $\phi(a)r'_1m \in Nil_{R'}(M)\ \forall r'_1 \in R'$. But ϕ is onto so $\exists\ r_1 \in R$ such that $\phi(r_1) = r'_1$. So, $\phi(a)\phi(r_1)m \in Nil_{R'}(M)$ and consequently, $\exists\ \phi(r_0) \in R', k \in \mathbb{N}$ s.t $\phi(r_0)^{k_0}(\phi(a)\phi(r_1)m) = 0$; $\phi(r_0)(\phi(a)\phi(r_1)m) \neq 0$ Eventually, we then have $\phi(r_0^{k_0}ar_1)m = 0$; $\phi(r_0ar_1)m \neq 0$ and hence, ${r_0}^{k_0}(ar_1m) = 0$; $r_0(ar_1m) \neq 0$. Thus, we have $ar_1m \in Nil_R(M)$ and consequently, $aRm \subseteq Nil_R(M)$. Hence, M is nil-semicommutative R- module.

(ii) $\Rightarrow$ (i) Let $b \in R'$, $m \in M$ such that $bm \in Nil_{R'}(M)$.Then, $\exists\ r' \in R', k \in \mathbb{N}$ such that $(r')^k bm = 0$; $r'(bm) \neq 0$. Now, ϕ is onto so $\exists\ a \in R$ and $r \in R$ such that $b = \phi(a)$ and $r' = \phi(r)$. So, we have $\phi(r)^k\phi(a)m = 0$; $\phi(r)(\phi(a)m) \neq 0$ and eventually, $\phi(r^k a)m = 0$; $\phi(ra)m) \neq 0$. This implies that $(r^k a)m = 0; (ra)m) \neq 0$ and consequently, $am \in Nil_R(M)$ and $aRm \subseteq Nil_R(M)$. Further, we also have $(r_0^{k_0}ar)m = 0$; $(r_0ar)m \neq 0$ for some $r_0 \in R, k_0 \in \mathbb{N}$. This implies that $\phi(r_0^{k_0}ar)m = 0$; $\phi(r_0ar)m \neq 0$ and eventually, $\phi(r_0)^{k_0}(\phi(a)\phi(r)m) = 0$; $\phi(r_0)(\phi(a)\phi(r))m) \neq 0$. Thus, we have $br'm \in Nil'_R(M)$ and hence, M is nil-semicommutative R'- module.

□

Proposition 2.33. *For a ring R and a multiplicatively closed subset S of $C(R)\backslash\{0\}$ containing 1, ${}_RM$ is nil-semicommutative module $\iff$ $S^{-1}M$ is nil-semicommutative $S^{-1}R$ module.*

Proof. ($\Rightarrow$) Let $\frac{r}{s} \in S^{-1}R, \frac{m}{q} \in S^{-1}M$ be such that $\frac{r}{s}.\frac{m}{q} \in Nil_{S^{-1}R}(S^{-1}M)$. Then, we have $(\frac{r_0}{s_0})^{k_0}(\frac{r}{s}.\frac{m}{q}) = 0$; $(\frac{r_0}{s_0})(\frac{r}{s}.\frac{m}{q}) \neq 0$ for some $\frac{r_0}{s_0} \in S^{-1}R,\ k_0 \in \mathbb{N}$. This implies that $u_1.r_0^{k_0}rm = 0$ for some $u_1 \in S$; $u.r_0rm \neq 0\ \forall\ u \in S$. Now, since $u_1 \in S \subseteq C(R)$, we have $u_1rm \in Nil_R(M)$. Consequently, we have $u_1rlm \in Nil_R(M)\ \forall\ l \in R$. Therefore, $\exists\ t_1 \in R, k_1 \in \mathbb{N}$ such that $t_1^{k_1}(u_1rlm) = 0$; $t_1(u_1rlm) \neq 0$. This implies that for s_1, s, p and $q \in S$, we have $u_1(t_1^{k_1}rlm.1 - 0.s_1spq) = 0$; $u_1(t_1rlm.1 - 0.s_1spq) \neq 0$. Thus, we have $\frac{t_1^{k_1}rlm}{s_1^{k1}spq} = 0$ and with an easy 'proof by contradiction' we also see that $\frac{t_1rlm}{s_1spq} \neq 0$.

Hence, we have $\frac{r}{s}\frac{l}{p}\frac{m}{q} \in Nil_{S^{-1}R}(S^{-1}M)$ for arbitrary $\frac{l}{p} \in S^{-1}R$. And consequently, we have ${}_{S^{-1}R}S^{-1}(M)$ is nil-semicommutative module.

For the converse, let $a \in R, m \in M$ such that $am \in Nil_R(M)$. Then $\exists\ r_0 \in R, k_0 \in \mathbb{N}$ such that $r_0^{k_0}am = 0$; $ram \neq 0$. This implies that for s_0, s and $q \in S$, we have $u_1(r_0^{k_0}am.1 - 0.s_0^{k_0}sq) = 0$ for some $u_1 \in S$; $u(ram.1 - 0.s_0sq) \neq 0\ \forall\ u \in S$. Consequently, we have $(\frac{r_0}{s_0})^{k_0}(\frac{a}{s}.\frac{m}{q}) = 0$; $(\frac{r_0}{s_0})(\frac{a}{s}.\frac{m}{q}) \neq 0$. Therefore, $\frac{a}{s}.\frac{m}{q} \in Nil_{S^{-1}R}(S^{-1}M)$ and consequently, $\frac{a}{s}(S^{-1}R)\frac{m}{q} \subseteq Nil_{S^{-1}R}(S^{-1}M)$. This implies that $\frac{a}{s}\frac{l}{p}\frac{m}{q} \in Nil_{S^{-1}R}(S^{-1}M)\ \forall\ \frac{l}{p} \in S^{-1}R$ and hence, we have $\frac{t_1^{k_1}alm}{s_1^{k_1}spq} = 0$ and $\frac{t_1alm}{s_1spq} \neq 0$. Eventually, we then arrive at $u_1t_1alm = 0$ for some $u_1 \in S$ and $ut_1alm \neq 0\ \forall\ u \in S$. Then, for $u = 1$ and with the help of an easy proof, we finally have $t_1^{k_1}alm = 0$; $t_1alm \neq 0$. Therefore, $aRm \subseteq Nil_R(M)$ and thus, ${}_RM$ is nil-semicommutative module. □

We recall from [9] that for a left R-module M, $R[x, x^{-1}]$ is the ring of Laurent polynomials in x with coefficients in R and $M[x, x^{-1}]$ forms a left module over the ring $R[x, x^{-1}]$. Now, as a corollary to the above proposition, we deduce a relation between the ${}_{R[x]}M[x]$ module and the ${}_{R[x,x^{-1}]}M[x, x^{-1}]$ module.

Corollary 2.34. *For a left R-module M, ${}_{R[x]}M[x]$ is nil-semicommutative $\iff$ ${}_{R[x,x^{-1}]}M[x, x^{-1}]$ is nil-semicommutative.*

Proof. Let $S = \{1, x, x^2, x^3, ...\}$. Then S is a multiplicatively closed subset of $C(R)\backslash\{0\}$ containing 1. Now, for $a_{-m}x^{-m} + a_{-(m-1)}x^{-(m-1)} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots a_{-n}x^n \in R[x, x^{-1}]$, it is easy to see that $\frac{1}{x^m}[a_{-m} + a_{-(m-1)}x + \cdots a_{-1}x^{m-1} + a_0x^m + a_1x^{m+1} + \cdots a_{-n}x^{m+n}] \in S^{-1}R[x]$. So, $R[x, x^{-1}] \subseteq S^{-1}R[x]$. Similarly, $S^{-1}R[x] \subseteq R[x, x^{-1}]$. Therefore, $S^{-1}R[x] = R[x, x^{-1}]$ and in the same way, $S^{-1}M[x] = M[x, x^{-1}]$. Hence, by Proposition 2.33 the result follows clearly. □

3. Direct product of nil-semicommutative modules

In this section we examine whether the nil-semicommutative property is preserved under direct products and find some conditions under which it does so.

Remark 3.1. For given left R-modules M_1 and M_2, the equality $Nil_R(M_1 \times M_2) = Nil_R(M_1) \times Nil_R(M_2)$ may not hold for some M_1 and M_2. The following is one such example.

Example 3.2. Consider $M_1 = M_2 = {}_{\mathbb{Z}_4}\mathbb{Z}_4$. Then, $Nil_{\mathbb{Z}_4}(\mathbb{Z}_4) = \{\bar{0}, \bar{1}, \bar{3}\}$. So, $Nil_{\mathbb{Z}_4}(\mathbb{Z}_4) \times Nil_{\mathbb{Z}_4}(\mathbb{Z}_4) = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{0}, \bar{3}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1}), (\bar{1}, \bar{3}), (\bar{3}, \bar{0}), (\bar{3}, \bar{1}), (\bar{3}, \bar{3})\}$

But, for $(\bar{1},\bar{2}) \in M_1 \times M_2$, $\exists\ \bar{2} \in \mathbb{Z}_4$ such that $\bar{2}^2(\bar{1},\bar{2}) = (\bar{0},\bar{0})$ but $\bar{2}(\bar{1},\bar{2}) = (\bar{2},\bar{0}) \neq (\bar{0},\bar{0})$. So, $(\bar{1},\bar{2}) \in Nil_{\mathbb{Z}_4}(\mathbb{Z}_4 \times \mathbb{Z}_4)$ but $\notin Nil_{\mathbb{Z}_4}(\mathbb{Z}_4) \times Nil_{\mathbb{Z}_4}(\mathbb{Z}_4)$.

However, we recall that for R-nonreduced and M torsion-free, M becomes a nilmodule, i.e. $Nil_R(M) = M$. Similarly, we prove that the same holds in the $M_1 \times M_2$ module if both M_1 and M_2 are torsion-free modules over a nonreduced commutative ring R.

Proposition 3.3. *If M_1 and M_2 are two torsion-free modules over a nonreduced commutative ring R, then $M_1 \times M_2$ is a nilmodule, i.e. $Nil_R(M_1 \times M_2) = M_1 \times M_2$.*

Proof. Let $(m_1, m_2) \in M_1 \times M_2$. Then, $m_i \in Nil_R(M_i)$ for $i = 1, 2$. This implies that $\exists\ r_i \in R$ and $k_i \in \mathbb{N}$ such that $r_i^{k_i} m_i = 0, r_i m_i \neq 0$. Now, as R is nonreduced commutative ring and M_i are torsion-free, for $r_0 = r_1 r_2 \in R$ and $k_0 = max\{k_1, k_2\}$, we have $r_0^{k_0}(m_1, m_2) = (0, 0)$ but $r_0(m_1, m_2) \neq (0, 0)$. So, $(m_1, m_2) \in Nil_R(M_1 \times M_2)$. Hence, $Nil_R(M_1 \times M_2) = M_1 \times M_2$ □

We now utilize this proposition to find the condition under which the direct product of two nil-semicommutative modules is also nil-semicommutative.

Proposition 3.4. *If M_1 and M_2 are two torsion-free modules over a nonreduced commutative ring R, then M_1 and M_2 are nil-semicommutative $\iff$ $M_1 \times M_2$ is also nil-semicommutative.*

Proof. The proof follows from the fact that nilmodules are nil-semicommutative. □

Proposition 3.5. *For a commutative ring R, $Tor(M_1 \times M_2) = Tor(M_1) \times Tor(M_2)$.*

Corollary 3.6. *If M_1 and M_2 are two torsion-free modules over a commutative ring R, then $M_1 \times M_2$ is also torsion-free.*

Proposition 3.7. *If M_1 and M_2 are two torsion-free modules over a reduced commutative ring R, then M_1 and M_2 are nil-semicommutative $\iff$ $M_1 \times M_2$ is also nil-semicommutative.*

Proof. The proof follows easily from the fact that $Nil_R(M_1 \times M_2) = \{(0, 0)\} = Tor(M_1 \times M_2)$ □

4. Conclusion and future scope

In this study, we have introduced and examined the structure and properties of nil-semicommutative modules with respect to its submodules, quotient modules, matrix modules and its localizations. Also, we have developed several relationships which the class of nil-semicommutative modules hold with the likes of semicommutative, weakly semicommutative and Armendariz modules. Moreover, we have also explored whether the direct product of this new class of modules preserves its nil-semicommutative nature and found results, and examples which assures the preservation. The preservation of these properties under various algebraic settings further strengthens the significance of this new class of

modules in module theory as a whole.
For the way forward, we believe there are many more exciting ventures waiting to be explored with respect to nil-semicomutative modules. In fact, a deeper dive into the realm of tensor products of the new class is one immediate target. Moreover, examining the relation of nil-semicommutativity with other existing modules like projective and injective modules is another potential venture worth exploring. Further, there is also the scope of generalizing the new concept by utilizing the concept of generalized nilpotent elements of a module and introducing a more generalized concept - 'Generalized Nil-semicommutative modules'. Lastly, the study of tensor products of nil-semicommutative modules along with the application of Tor - functors can also propel us towards greater findings in homological algebra as well.

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