Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 1 (2025) 151-160 https://doi.org/10.56947/gjom.v20i.2892

SOME ANALYTICAL PROPERTIES OF THE NIELSEN'S BETA FUNCTION

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Abstract. In this paper, we establish some analytical properties such as limits, monotonicity, convexity, concavity, and inequalities involving derivatives of the Nielsen's beta function. The results obtained include a solution to an open problem and generalisations of some known results.

1. Introduction and Preliminaries

The classical Nielsen's beta function, otherwise known as the incomplete beta function, is defined for $z > 0$ by either of the following representations.

$$\beta(z) = \int_0^1 \frac{t^{z-1}}{1+t}dt, \tag{1.1}$$

$$= \int_0^\infty \frac{e^{-zt}}{1+e^{-t}}dt, \tag{1.2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+z}, \tag{1.3}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right\}, \tag{1.4}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is the digamma function and $\Gamma(z)$ is the gamma function. See [2], [4], [18, p.16] and the related references therein. It satisfies the properties

$$\beta(z+1) = \frac{1}{z} - \beta(z), \tag{1.5}$$

$$\beta(z) + \beta(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.6}$$

Date: Received: Apr 21, 2025; Accepted: May 12, 2025.
2020 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48, 26D07.
Key words and phrases. Nielsen's beta function; polygamma function; Dirichlet's eta function; Riemann zeta function; inequality.

Derivatives of the function are respectively given as

$$\beta^{(s)}(z) = \int_0^1 \frac{(\ln t)^s t^{z-1}}{1+t} dt, \quad s \in \mathbb{N}_0, \tag{1.7}$$

$$= (-1)^s \int_0^\infty \frac{t^s e^{-zt}}{1+e^{-t}} dt, \quad s \in \mathbb{N}_0, \tag{1.8}$$

$$= (-1)^s s! \sum_{k=0}^\infty \frac{(-1)^k}{(k+z)^{s+1}}, \quad s \in \mathbb{N}_0, \tag{1.9}$$

$$= \frac{1}{2^{s+1}} \left\{ \psi^{(s)}\left(\frac{z+1}{2}\right) - \psi^{(s)}\left(\frac{z}{2}\right) \right\}, \quad s \in \mathbb{N}_0, \tag{1.10}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and $\psi^{(0)}(z) \equiv \psi(z)$. Clearly, $\beta^{(0)}(z) = \beta(z)$. The functional equation (1.5) implies that

$$\beta^{(s)}(z+1) = \frac{(-1)^s s!}{z^{s+1}} - \beta^{(s)}(z). \tag{1.11}$$

In particular,

$$\beta^{(s)}(1) = (-1)^s s! \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^{s+1}} = (-1)^s s! \eta(s+1), \quad s \in \mathbb{N}_0, \tag{1.12}$$

$$= (-1)^s s! \left(1 - \frac{1}{2^s}\right) \zeta(s+1), \quad s \in \mathbb{N}, \tag{1.13}$$

where $\eta(z)$ and $\zeta(z)$ are respectively the Dirichlet's eta function and the Riemann zeta function which are given by

$$\eta(z) = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^z},\ z > 0 \quad \text{and} \quad \zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z},\ z > 1.$$

The Nielsen's beta function comes in handy when evaluating certain integrals, and it is closely associated with several prominent mathematical constants. See [2] and [4, p. 906]. Its relation with other special functions makes it very useful in mathematical analysis. As a result of this, it has been investigated along different directions in recent times. For example, see [1], [5], [7], [8], [9], [10], [12], [13], [14], [15], [22] and [23]. In [16], the author posed four open problems concerning the Nielsen's beta function. In the paper [21], among other things, the author provided solutions to two of the problems. That is, [16, Problem 2.3] and [16, Problem 2.4]. It is mentioned at the conclusion part of [16] that, solutions to those problems will provide a good foundation for further studies on the function. Indeed, in this paper, the solutions provided by [21] shall pave the way for more interesting discoveries. Riding on the solution to [16, Problem 2.3], we also provide a solution to [16, Problem 2.2] and further establish some properties (such as monotonicity, convexity, concavity and inequalities) involving the Nielsen's beta function. In some cases, the established results provide generalisations for some known results in the literature.

2. Main Results

We now present our findings in this section. We begin with the following limit properties.

Theorem 2.1. *The following limits are valid for all $s \in \mathbb{N}$.*

$$\lim_{z\to 0} z^{s+1}\beta^{(s)}(z) = (-1)^s s!, \tag{2.1}$$

$$\lim_{z\to \infty} z^{s}\beta^{(s)}(z) = 0, \tag{2.2}$$

$$\lim_{z\to 0} z\frac{\beta^{(s+1)}(z)}{\beta^{(s)}(z)} = -(s+1), \tag{2.3}$$

$$\lim_{z\to 0} \frac{\left[\beta^{(s+1)}(z)\right]^2}{\beta^{(s)}(z)\beta^{(s+2)}(z)} = \frac{s+1}{s+2}, \tag{2.4}$$

$$\lim_{z\to \infty} \frac{\left[\beta^{(s+1)}(z)\right]^2}{\beta^{(s)}(z)\beta^{(s+2)}(z)} = \frac{s+1}{s+2}. \tag{2.5}$$

Proof. From the functional equation (1.11), we obtain

$$\lim_{z\to 0} z^{s+1}\beta^{(s)}(z) = \lim_{z\to 0}\big((-1)^s s! - z^{s+1}\beta^{(s)}(z+1)\big) = (-1)^s s!.$$

Next, it is known (see [3]) that

$$\psi^{(s)}(z) = (-1)^{s-1}\left[\frac{(s-1)!}{z^s} + \frac{s!}{2z^{s+1}} + \int_0^\infty K(t)t^{s-1}e^{-zt}dt\right]$$

where $K(t) = \frac{t}{e^t-1} - 1 + \frac{t}{2}$. Then

$$\psi^{(s)}(\frac{z+1}{2}) = (-1)^{s-1}\left[\frac{2^s(s-1)!}{(z+1)^s} + \frac{2^s s!}{(z+1)^{s+1}} + \int_0^\infty K(t)t^{s-1}e^{-(\frac{z+1}{2})t}dt\right]$$

which implies that

$$z^s\psi^{(s)}(\frac{z+1}{2})$$

$$= (-1)^{s-1}\left[2^s\left(\frac{z}{z+1}\right)^s(s-1)! + 2^s\left(\frac{z}{z+1}\right)^s\frac{s!}{z+1} + \int_0^\infty K(t)t^{s-1}z^s e^{-(\frac{z+1}{2})t}dt\right].$$

Hence

$$\lim_{z\to\infty}\psi^{(s)}(\frac{z+1}{2}) = (-1)^{s-1}2^s(s-1)!.$$

Similarly,

$$z^s\psi^{(s)}(\frac{z}{2}) = (-1)^{s-1}\left[2^s(s-1)! + 2^s\frac{s!}{z} + \int_0^\infty K(t)t^{s-1}z^s e^{-(\frac{z}{2})t}dt\right]$$

which implies that

$$\lim_{z\to\infty}\psi^{(s)}(\frac{z}{2}) = (-1)^{s-1}2^s(s-1)!.$$

Therefore by (1.10), we obtain

$$\lim_{z\to\infty} z^s\beta^{(s)}(z) = \frac{1}{2^{s+1}}\left\{\lim_{z\to\infty} z^s\psi^{(s)}\left(\frac{z+1}{2}\right) - \lim_{z\to\infty} z^s\psi^{(s)}\left(\frac{z}{2}\right)\right\} = 0.$$

Next, by applying (2.1), we obtain

$$\lim_{z\to 0} z\frac{\beta^{(s+1)}(z)}{\beta^{(s)}(z)} = \lim_{z\to 0}\frac{z^{s+2}\beta^{(s+1)}(z)}{z^{s+1}\beta^{(s)}(z)} = \frac{(-1)^{s+1}(s+1)!}{(-1)^s s!} = -(s+1). \tag{2.6}$$

Likewise, by applying (2.1), we obtain

$$\begin{aligned}\lim_{z\to 0}\frac{\left[\beta^{(s+1)}(z)\right]^2}{\beta^{(s)}(z)\beta^{(s+2)}(z)} &= \lim_{z\to 0}\frac{z^{s+2}\beta^{(s+1)}(z)z^{s+2}\beta^{(s+1)}(z)}{z^{s+1}\beta^{(s)}(z)z^{s+3}\beta^{(s+2)}(z)}\\ &= \frac{\lim_{z\to 0}(z^{s+2}\beta^{(s+1)}(z))\lim_{z\to 0}(z^{s+2}\beta^{(s+1)}(z))}{\lim_{z\to 0}(z^{s+1}\beta^{(s)}(z))\lim_{z\to 0}(z^{s+3}\beta^{(s+2)}(z))}\\ &= \frac{(-1)^{s+1}(s+1)!(-1)^{s+1}(s+1)!}{(-1)^s s!(-1)^{s+2}(s+2)!}\\ &= \frac{s+1}{s+2}.\end{aligned}$$

By using the asymptotic formulas

$$\psi\left(z+\frac{1}{2}\right)\sim \ln z+\frac{1}{24z^2} \quad \text{and} \quad \psi(z)\sim \ln z-\frac{1}{2z} \quad \text{as} \quad z\to\infty,$$

we have

$$\psi\left(\frac{z}{2}+\frac{1}{2}\right)-\psi\left(\frac{z}{2}\right)\sim\frac{1}{z} \quad \text{as} \quad z\to\infty,$$

which implies that

$$\frac{1}{2^s}\left\{\psi^{(s)}\left(\frac{z}{2}+\frac{1}{2}\right)-\psi^{(s)}\left(\frac{z}{2}\right)\right\}\sim\frac{(-1)^s s!}{z^{s+1}} \quad \text{as} \quad z\to\infty.$$

Thus

$$\beta^{(s)}(z)=\frac{1}{2^{s+1}}\left\{\psi^{(s)}\left(\frac{z}{2}+\frac{1}{2}\right)-\psi^{(s)}\left(\frac{z}{2}\right)\right\}\sim\frac{(-1)^s s!}{2z^{s+1}} \quad \text{as} \quad z\to\infty.$$

Hence

$$\begin{aligned}\lim_{z\to\infty}\frac{\left[\beta^{(s+1)}(z)\right]^2}{\beta^{(s)}(z)\beta^{(s+2)}(z)} &= \lim_{z\to\infty}\frac{\left(\frac{(-1)^{s+1}(s+1)!}{2z^{s+2}}\right)^2}{\left(\frac{(-1)^s s!}{2z^{s+1}}\right)\left(\frac{(-1)^{s+2}(s+2)!}{2z^{s+3}}\right)}\\ &= \frac{(s+1)!(s+1)!}{s!(s+2)!}\\ &= \frac{s+1}{s+2}.\end{aligned}$$

□

Remark 2.2. The limits (2.4) and (2.5) imply that the function

$$\mathcal{H}(z)=\frac{\left[\beta^{(s+1)}(z)\right]^2}{\beta^{(s)}(z)\beta^{(s+2)}(z)}$$

is not monotonic. This conclusion has been given in [21].

Lemma 2.3 ([21])**.** *Let $s \in \mathbb{N}_0$ and $z > 0$. Then the inequality*

$$\beta^{(s)}(z)\beta^{(s+2)}(z) - 2\left(\beta^{(s+1)}(z)\right)^2 < 0 \tag{2.7}$$

holds.

Proposition 2.4. *Let $s \in \mathbb{N}_0$ and $z > 0$. Then the function*

$$\mathcal{V}(z) = \frac{1}{\beta^{(s)}(z)}$$

is strictly convex if s is even and strictly concave if s is odd.

Proof. Differentiating and applying (2.7) yields

$$\mathcal{V}''(z) = \frac{2\left(\beta^{(s+1)}(z)\right)^2 - \beta^{(s)}(z)\beta^{(s+2)}(z)}{\left(\beta^{(s)}(z)\right)^3}$$

which implies that $\mathcal{V}''(z) > 0$ if s is even and $\mathcal{V}''(z) < 0$ if s is odd. □

Lemma 2.5. *Let $s \in \mathbb{N}_0$ and $z > 0$. Then the function*

$$\mathcal{K}(z) = \frac{\beta^{(s)}(z)}{\beta^{(s+1)}(z)} + z \tag{2.8}$$

is strictly increasing and positive.

Proof. By differentiating and applying (2.7), we have

$$\mathcal{K}'(z) = \frac{2\left(\beta^{(s+1)}(z)\right)^2 - \beta^{(s)}(z)\beta^{(s+2)}(z)}{\left(\beta^{(s)}(z)\right)^2} > 0$$

which shows that $\mathcal{K}(z)$ is strictly increasing. Next, the increasing nature of $\mathcal{K}(z)$ and the limit (2.1) imply that

$$\begin{aligned}
\mathcal{K}(z) > \lim_{z\to 0}\mathcal{K}(z) &= \lim_{z\to 0}\frac{\beta^{(s)}(z) + z\beta^{(s+1)}(z)}{\beta^{(s+1)}(z)} \\
&= \lim_{z\to 0}\frac{zz^{s+1}\beta^{(s)}(z) + zz^{s+2}\beta^{(s+1)}(z)}{z^{s+2}\beta^{(s+1)}(z)} \\
&= \frac{0.(-1)^s s! + 0.(-1)^{s+1}(s+1)!}{(-1)^{s+1}(s+1)!} \\
&= 0
\end{aligned}$$

completing the proof. □

Lemma 2.6. *Let $z > 0$ and $k \in \mathbb{N}_0$. Then*

$$\beta^{(k)}(z) + z\beta^{(k+1)}(z) < 0 \tag{2.9}$$

holds if k is even and

$$\beta^{(k)}(z) + z\beta^{(k+1)}(z) > 0 \tag{2.10}$$

holds if k is odd.

Proof. These follow directly from the positivity of the function $\mathcal{K}(z)$ in Lemma 2.5. Recall that $\beta^{(c)}(z)$ is positive if c is even and $\beta^{(c)}(z)$ is negative if c is odd. □

Lemma 2.7. *Let $z > 0$ and $s \in \mathbb{N}_0$. Then the function*

$$\mathcal{G}(z) = z\beta^{(s+1)}(z)$$

is strictly increasing if s is even and strictly decreasing if s is odd.

Proof. Direct computation yields

$$\mathcal{G}'(z) = \beta^{(s+1)}(z) + z\beta^{(s+2)}(z)$$

and by virtue of Lemma 2.6, we arrive at the conclusion that $\mathcal{G}'(z) > 0$ if s is even and $\mathcal{G}'(z) < 0$ if s is odd. □

Theorem 2.8. *Let $z > 0$ and $s \in \mathbb{N}_0$. Then*

$$\beta^{(s)}(z) + \beta^{(s)}(1/z) \geq 2s!\eta(s+1) \tag{2.11}$$

holds when s is even and

$$\beta^{(s)}(z) + \beta^{(s)}(1/z) \leq -2s!\eta(s+1) \tag{2.12}$$

holds when s is odd. In each case, equality is arrived at when $z = 1$.

Proof. It is easy to establish the cases for equality. For this reason, let $\mathcal{U}(z) = \beta^{(s)}(z) + \beta^{(s)}(1/z)$ for $s \in \mathbb{N}_0$ and $z \in (0,1) \cup (1,\infty)$. Then upon differentiating, we get

$$\begin{aligned} z\mathcal{U}'(z) &= z\beta^{(s+1)}(z) - \frac{1}{z}\beta^{(s+1)}(1/z) \\ &\triangleq \mathcal{D}(z). \end{aligned}$$

At this point, assume that s is even. Since $\mathcal{G}(z)$ is increasing for even s, the conclusion is that $\mathcal{D}(z) < 0$ if $z \in (0,1)$ and $\mathcal{D}(z) > 0$ if $z \in (1,\infty)$. Consequently, $\mathcal{U}(z)$ is decreasing on the interval $(0,1)$ and increasing on the interval $(1,\infty)$. Therefore, for $z \in (0,1) \cup (1,\infty)$, we have

$$\mathcal{U}(z) > \lim_{z\to 1}\mathcal{U}(z) = 2\beta^{(s)}(1) = 2s!\eta(r+1)$$

which gives (2.11).

In a similar fashion, assume that s is odd. Since $\mathcal{G}(z)$ is decreasing for odd s, the conclusion is that $\mathcal{D}(z) > 0$ if $z \in (0,1)$ and $\mathcal{D}(z) < 0$ if $z \in (1,\infty)$. Consequently, $\mathcal{U}(z)$ is increasing on the interval $(0,1)$ and decreasing on the interval $(1,\infty)$. Therefore, for $z \in (0,1) \cup (1,\infty)$, we have

$$\mathcal{U}(z) < \lim_{z\to 1}\mathcal{U}(z) = 2\beta^{(s)}(1) = -2s!\eta(r+1)$$

which gives (2.12). □

Remark 2.9. In particular, if $s = 0$ in Theorem 2.8, then we obtain the inequality

$$\beta(z) + \beta(1/z) \geq 2\ln 2$$

as established in [12]. If $s = 1$ in Theorem 2.8, then we obtain the inequality

$$\beta'(z) + \beta'(1/z) \leq -\frac{\pi^2}{6}.$$

Lemma 2.10. *Let the function $\mathcal{Q}(z)$ be defined by*

$$\mathcal{Q}(z) = \frac{z\beta^{(s+1)}(z)}{\left(\beta^{(s)}(z)\right)^2} \tag{2.13}$$

for $z > 0$ and $s \in \mathbb{N}_0$. Then $\mathcal{Q}(z)$ decreasing when s is even and increasing when s is odd.

Proof. By differentiating $\mathcal{Q}(z)$ and employing inequality (2.7), we have

$$\begin{aligned}
[\beta^{(s)}(z)]^3\mathcal{Q}'(z) &= \beta^{(s)}(z)\beta^{(s+1)}(z) + z\beta^{(s)}(z)\beta^{(s+2)}(z) - 2z\left[\beta^{(s+1)}(z)\right]^2 \\
&= \beta^{(s)}(z)\beta^{(s+1)}(z) + z\left[\beta^{(s)}(z)\beta^{(s+2)}(z) - 2\left(\beta^{(s+1)}(z)\right)^2\right] \\
&< 0.
\end{aligned}$$

Thus, we arrive at the conclusion that $\mathcal{Q}'(z) < 0$ if s is even, and $\mathcal{Q}'(z) > 0$ if s is odd. By this, we complete the proof. □

Remark 2.11. Lemma 2.10 provides an answer to [16, Problem 2.2].

Theorem 2.12. *Let $z > 0$ and $s \in \mathbb{N}_0$. Then the inequality*

$$\frac{2\beta^{(s)}(z)\beta^{(s)}(1/z)}{\beta^{(s)}(z) + \beta^{(s)}(1/z)} \leq s!\eta(s+1) \tag{2.14}$$

holds when s is even and the inequality

$$\frac{2\beta^{(s)}(z)\beta^{(s)}(1/z)}{\beta^{(s)}(z) + \beta^{(s)}(1/z)} \geq -s!\eta(s+1) \tag{2.15}$$

holds when s is odd. In each case, equality is arrived at when $z = 1$.

Proof. The condition for equality in (2.14) and (2.15) is straightforward. In view of this, we shall only prove the results for $z \in (0,1) \cup (1,\infty)$. Now we define

$$\mathcal{A}(z) = \frac{2\beta^{(s)}(z)\beta^{(s)}(1/z)}{\beta^{(s)}(z) + \beta^{(s)}(1/z)}$$

for $s \in \mathbb{N}_0$ and $z \in (0,1) \cup (1,\infty)$. Then, direct differentiation results to

$$\frac{\mathcal{A}'(z)}{\mathcal{A}(z)} = \frac{\beta^{(s+1)}(z)}{\beta^{(s)}(z)} - \frac{1}{z^2}\frac{\beta^{(s+1)}(1/z)}{\beta^{(s)}(1/z)} - \frac{\beta^{(s+1)}(z) - \frac{1}{z^2}\beta^{(s+1)}(1/z)}{\beta^{(s)}(z) + \beta^{(s)}(1/z)}$$

which can be written as

$$z\left[\beta^{(s)}(z) + \beta^{(s)}(1/z)\right]\frac{\mathcal{A}'(z)}{\mathcal{A}(z)} = z\frac{\beta^{(s+1)}(z)}{\beta^{(s)}(z)}\beta^{(s)}(1/z) - \frac{1}{z}\frac{\beta^{(s+1)}(1/z)}{\beta^{(s)}(1/z)}\beta^{(s)}(z).$$

This can further be written as

$$\begin{aligned}
z\left[\frac{1}{\beta^{(s)}(z)} + \frac{1}{\beta^{(s)}(1/z)}\right]\frac{\mathcal{A}'(z)}{\mathcal{A}(z)} &= z\frac{\beta^{(s+1)}(z)}{\left(\beta^{(s)}(z)\right)^2} - \frac{1}{z}\frac{\beta^{(s+1)}(1/z)}{\left(\beta^{(s)}(1/z)\right)^2} \\
&\triangleq \mathcal{D}(z).
\end{aligned}$$

It is worth noting that if $z \in (0,1)$, then $z < 1/z$ and if $z \in (1,\infty)$, then $z > 1/z$. Additionally, if s is even, then $\beta^{(s)}(z) > 0$ and if s is odd, then $\beta^{(s)}(z) < 0$.

Now let s be even. The decreasing property of $\mathcal{Q}(z)$ for even s implies that, $\mathcal{D}(z) > 0$ if $z \in (0,1)$ and $\mathcal{D}(z) < 0$ if $z \in (1,\infty)$. These imply that $\mathcal{A}'(z) > 0$ if $z \in (0,1)$ and $\mathcal{A}'(z) < 0$ if $z \in (1,\infty)$. Thus, $\mathcal{A}(z)$ is increasing on the interval $(0,1)$ and decreasing on the interval $(1,\infty)$. Hence, on both intervals, we obtain

$$\mathcal{A}(z) < \lim_{z\to 1} \mathcal{A}(z) = \beta^{(s)}(1) = s!\eta(s+1)$$

which leads to (2.14).

In like manner, let s be odd. The increasing property of $\mathcal{Q}(z)$ for odd s implies that, $\mathcal{D}(z) < 0$ if $z \in (0,1)$ and $\mathcal{D}(z) > 0$ if $z \in (1,\infty)$. These imply that $\mathcal{A}'(z) < 0$ if $z \in (0,1)$ and $\mathcal{A}'(z) > 0$ if $z \in (1,\infty)$. Thus, $\mathcal{A}(z)$ is decreasing on the interval $(0,1)$ and increasing on the interval $(1,\infty)$. Hence, on both intervals, we obtain

$$\mathcal{A}(z) > \lim_{z\to 1} \mathcal{A}(z) = \beta^{(s)}(1) = -s!\eta(s+1)$$

which leads to (2.15). □

Remark 2.13. Particularly, if $s = 0$ in Theorem 2.12, then we obtain

$$\frac{2\beta(z)\beta(1/z)}{\beta(z)+\beta(1/z)} \leq \ln 2$$

which is the main result of [6]. See also Conjecture 3.7 of [12].

If $s = 1$ in Theorem 2.12, then we obtain

$$\frac{2\beta'(z)\beta'(1/z)}{\beta'(z)+\beta'(1/z)} \geq -\frac{\pi^2}{12}.$$

Theorem 2.12 therefore provides a generalization of the results of [6].

Theorem 2.14. *Let $r \in [0,1)$ and $s \in \mathbb{N}_0$. Then the inequality*

$$\beta^{(s)}(1+r)\beta^{(s)}(1-r) \geq \left(s!\eta(s+1)\right)^2 \tag{2.16}$$

holds if s is even.

Proof. It has been established in [11] that, $\beta^{(s)}(z)$ is logarithmically convex if s is even. This translates to

$$\beta^{(s)}\left((1-\lambda)x+\lambda y\right) \leq \left(\beta^{(s)}(x)\right)^{1-\lambda}\left(\beta^{(s)}(y)\right)^{\lambda} \tag{2.17}$$

where $\lambda \in [0,1]$, $x > 0$ and $y > 0$. Now, by letting $r \in [0,1)$, $\lambda = \frac{1}{2}$, $x = 1+r$, $y = 1-r$ in (2.17), we obtain the desired result (2.16). □

Finally, we pose the following conjecture.

Conjecture 2.15. Let $z > 0$ and $s \in \mathbb{N}$. Then inequality

$$\beta^{(s)}(z)\beta^{(s)}(1/z) \leq \left(s!\eta(s+1)\right)^2 \tag{2.18}$$

holds.

Remark 2.16. Results analogous to Theorem 2.8, Theorem 2.12 and Conjecture 2.15 involving the polygamma functions can be found in [17]. Also, some analytical properties concerning beta-logarithmic function and Dirichlet's eta function can be found in [19] and [20] respectively.

3. Conclusion

We have established analytical properties such as limits, monotonicity, convexity, concavity, and inequalities regarding derivatives of the Nielsen's beta function. The results obtained include a solution to an open problem and generalisations of some known results. Some of the results obtained provide bounds (in respect of Dirichlet's eta function) for arithmetic, harmonic and geometric means associated with the Nielsen's beta function. We have the firm belief that the present results will lay a good foundation for further research on the subject.

Acknowledgement. We would like to thank the anonymous reviewers and the editor for their useful comments and suggesstions.

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