Gulf Journal of Mathematics
Vol 20, Issue 1 (2025) 289-302

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i.2907

A STUDY ON FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS USING TEMPERED ψ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. This study explores the existence and uniqueness of solutions for fractional integrodifferential equations involving the tempered ψ-Caputo fractional derivative. These results are derived by employing the Banach fixed point theorem and Schaefer's fixed point theorem. We also examine the stability of the solutions. Finally, an example is provided to highlight the key findings.

1. Introduction

Fractional differential equations have proven to be excellent tools for modelling a wide range of phenomena in domains such as control theory, signal processing, rheology, fractals, chaotic dynamics, modelling, bioengineering, biomedical applications and more [1, 4, 8, 10, 14]. Fractional differential and integrodifferential equations have received a lot of attention in recent years, which leads to a number of intriguing findings about their existence and uniqueness. For more in-depth details, look to [1, 19, 14, 15, 20, 27] and the associated references.

Tempered fractional derivatives and integrals, which multiply fractional derivatives and integrals by an exponential term, are useful in modelling turbulence in geophysical flows and Levy processes like Brownian motion [11]. Tempered fractional calculus has been rediscovered under different names, such as generalized proportional fractional calculus [12] and substantial fractional calculus [13, 23].

Li et al. [14] studied the features of tempered fractional derivatives, including the existence, uniqueness, and stability of specific tempered fractional ordinary differential equations [7]. Zhao et al. [27] investigated function space and composition features before focusing on variational calculus and spectral analysis for Riemann-Liouville-type tempered fractional equations. Morgado and Rebelo [18] discussed nonlinear Caputo-type tempered fractional differential equations with terminal conditions, analyzing solution existence and uniqueness, and proposing three numerical approximation strategies [5].

Date: Received: April 27, 2025; Accepted: May 16, 2025.

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2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 47H10.

Key words and phrases. tempered fractional calculus, fractional differential equation, fixed point theorem, stability.

Kilbas et al. [14] proposed ψ-fractional operators as an extension of Riemann-Liouville operators. Unlike conventional fractional operators, these fractional operators include a kernel stated in terms of another function. Almeida [4, 5, 6] proposed the ψ-Caputo fractional derivative, which extends the traditional fractional derivative by taking into account a different function. He also examined a variety of properties, including Taylor's theorem and the semigroup law. Several studies [2, 3, 9, 16, 21, 22] investigated the existence and uniqueness results for ψ-Caputo fractional differential equations.

Medved and Bresto [17] introduced a variation of the ψ-Caputo derivative, termed the tempered ψ-Caputo derivative, and conducted an analysis of the Cauchy problem concerning fractional differential equations employing this particular fractional derivative.

The study of stability results in fractional differential equations improves our understanding of complex systems, including biological and physical systems [2, 3, 7, 24, 25, 26]. It also improves our ability to predict their behavior. Several authors have investigated various Ulam-Hyers Stability problems concerning different types of fractional integral and fractional differential equations utilizing a range of approaches [2, 3, 24].

Inspired by the preceding discussion, we delve into the analysis of the following tempered ψ-Caputo fractional integrodifferential equation

$$
\begin{aligned}
{}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}u(t) =& \mathfrak{f}\left(t, u(t), \int_0^t \mathfrak{g}(t,\tau,u(\tau))d\tau\right),\ t\in \mathsf{J}=[0,b],\ 0<\alpha<1, \qquad (1.1)\\
u(0) =& u_0 \in \mathbb{R}^n, \qquad (1.2)
\end{aligned}
$$

where $(\mathbb{R}^n, |\cdot|)$ symbolizes a Banach space and ${}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}$ symbolizes the tempered ψ-Caputo fractional derivative with order α. Let $\mathfrak{f}: \mathsf{J}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $\mathfrak{g}: \mathsf{J}\times\mathsf{J}\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous functions, ψ be a continuously differentiable, increasing function with $\psi(0)=0, \psi'(t)>0$ on $\mathbb{R}^+=[0,\infty)$ and $\lim\limits_{t\to\infty}\psi(t)=\infty$. We take $\mathsf{K}u(t) = \int_0^t \mathfrak{g}(t,\tau,u(\tau))d\tau$. Assume that the collection $\mathcal{E}=\mathcal{C}(\mathsf{J},\mathbb{R}^n)$ of all continuous functions from J into $\mathbb{R}^n$ is the Banach space having the norm

$$\|u\|_{\mathcal{E}} = \sup\{|u(\tau)| : \tau\in\mathsf{J}\},\ \text{for}\ \ u\in\mathcal{E}.$$

Structure of the paper: The notations, fundamental concepts and preliminary facts that will be used throughout the work are introduced in Section 2. In Section 3, classical fixed point theorems are employed to investigate the existence and uniqueness results for the problems (1.1)−(1.2). Section 4 is dedicated to the study of stability of the proposed problem. Finally, in Section 5, an illustration is provided to validate the theory presented in the paper.

2. Preliminaries

Definition 2.1. [17] Let $\alpha\in\mathbb{R}^+\backslash\{0\}$, $u(t)$ be the continuous real function on J, and $\psi\in\mathcal{C}^1(\mathsf{J})$ is an increasing differentiable function with $\psi'(t)\neq 0$ on J. Then, the ψ-Riemann-Liouville fractional integral with order α is defined by:

$$I_0^{\alpha,\psi}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}\psi'(\tau)u(\tau)d\tau.$$

Definition 2.2. [17] Suppose the assumptions of Definition 2.1 are satisfied and $\beta \in \mathbb{R}^+$. Then, the tempered ψ-fractional integral with order α is defined as:

$$\begin{aligned} I_0^{\alpha,\beta,\psi}u(t) =& e^{-\beta\psi(t)} I_\psi^\alpha (e^{\beta\psi(t)}u(t)) \\ =& \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)u(\tau)d\tau. \end{aligned}$$

Definition 2.3. [17] Suppose $\psi \in \mathcal{C}^n(\mathsf{J})$ is an increasing and differentiable function with $\psi'(t) \neq 0$ on J. For $\alpha \in (n-1,n)$, the tempered ψ-Caputo fractional derivative with order α is defined as:

$$^C\mathcal{D}_0^{\alpha,\beta,\psi}u(t) = \frac{e^{-\beta\psi(t)}}{\Gamma(n-\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{n-\alpha-1} u_{\beta,\psi}^{[n]}(\tau)d\tau,$$

where $n \in \mathbb{N}, \beta \in \mathbb{R}^+$ and

$$u_{\beta,\psi}^{[n]}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n (e^{\beta\psi(t)}u(t)).$$

Definition 2.4. [25] The equation (1.1)-(1.2) is Ulam-Hyers Stable (UHS) if there is $\mathcal{C}_f \in \mathbb{R}^+\backslash\{0\}$ with for any $\epsilon > 0$ and for any solution $\widehat{u} \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ satisfying the inequality

$$\left|{}^C\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t) - \mathfrak{f}(t,\widehat{u}(t),\mathsf{K}\widehat{u}(t))\right| \leq \epsilon, \forall\, t \in \mathsf{J}, \tag{2.1}$$

then there is a solution $u \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ of (1.1)-(1.2) with

$$|\widehat{u}(t) - u(t)| \leq \mathcal{C}_f\epsilon, \forall\, t \in \mathsf{J}.$$

Definition 2.5. [25] The equation (1.1)-(1.2) is Generalized Ulam-Hyers Stable (GUHS) if there is $\psi_f \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\psi_f(0) = 0$ such that for any solution $\widehat{u} \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ satisfying the inequality

$$\left|{}^C\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t) - \mathfrak{f}(t,\widehat{u}(t),\mathsf{K}\widehat{u}(t))\right| \leq \epsilon, \forall\, t \in \mathsf{J},$$

then there is a solution $u \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ of (1.1)-(1.2) with

$$|\widehat{u}(t) - u(t)| \leq \psi_f(\epsilon), \forall\, t \in \mathsf{J}.$$

Definition 2.6. [25] The equation (1.1)-(1.2) is Ulam-Hyers Rassias Stable (UHRS) with respect to $\phi \in \mathcal{C}(\mathsf{J},\mathbb{R}^+)$ if there is $C_f \in \mathbb{R}^+\backslash\{0\}$ such that for any $\epsilon > 0$ and for any solution $\widehat{u} \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ satisfying the condition

$$\left|{}^C\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t) - \mathfrak{f}(t,\widehat{u}(t),\mathsf{K}\widehat{u}(t))\right| \leq \epsilon\phi(t), \forall\, t \in \mathsf{J},$$

then there is a solution $u \in \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ of (1.1)-(1.2) with

$$|\widehat{u}(t) - u(t)| \leq C_f\epsilon\phi(t), \forall\, t \in \mathsf{J}.$$

Definition 2.7. [25] The equation (1.1)-(1.2) is Generalized Ulam-Hyers Rassias Stable (GUHRS) with respect to $\phi \in \mathcal{C}(\mathsf{J}, \mathbb{R}^+)$ if there is $C_{f,\phi} \in \mathbb{R}^+ \backslash \{0\}$ such that for any solution $\widehat{u} \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$ satisfying the condition

$$\left| {}^C\mathcal{D}_0^{\alpha,\beta,\psi} \widehat{u}(t) - \mathfrak{f}(t, \widehat{u}(t), \mathsf{K}\widehat{u}(t)) \right| \leq \phi(t), \forall\ t \in \mathsf{J},$$

then there is a solution $u \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$ of (1.1)-(1.2) with

$$|\widehat{u}(t) - u(t)| \leq C_{f,\phi}\ \ \phi(t), \forall\ t \in \mathsf{J}.$$

Theorem 2.8. *(Schaefer's Theorem) Let* H *be a completely continuous operator from a Banach space* X *to itself. If*

$$\mathsf{H}(X) = \{x \in X : x = \mu(\mathsf{H}x) \text{ and } 0 < \mu < 1\}$$

is bounded, then H *has at least one fixed point.*

3. Existence Results

We present the following 3 hypotheses to illustrate key findings.

(H1) The function $\mathfrak{f}$ is continuous and $\forall\ t \in \mathsf{J}$ and $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^n$

$$|\mathfrak{f}(t, x, y) - \mathfrak{f}(t, \tilde{x}, \tilde{y})| \leq \mathfrak{L}_1(|x - \tilde{x}| + |y - \tilde{y}|),$$

where $\mathfrak{L}_1$ is a positive constant.

(H2) The function $\mathfrak{g}$ is continuous and $\forall\ t, s \in \mathsf{J}$ and $x, \tilde{x} \in \mathbb{R}^n$

$$\left| \int_0^t \Big(\mathfrak{g}(t, s, x) - \mathfrak{g}(t, s, \tilde{x}) \Big) ds \right| \leq \mathfrak{L}_2(|x - \tilde{x}|),$$

where $\mathfrak{L}_2$ is a positive constant.

(H3) The function $\mathfrak{f}$ is bounded and

$$\mathsf{M}_{\mathfrak{f}} = \sup\{|\mathfrak{f}(t, x, \tilde{x}| :\ t \in \mathsf{J} \text{ and } x, \tilde{x} \in \mathbb{R}^n\}.$$

Lemma 3.1. *Suppose the mappings* $\mathfrak{f} : \mathsf{J} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\mathfrak{g} : \mathsf{J} \times \mathsf{J} \times \mathbb{R}^n \to \mathbb{R}^n$ *are bounded. Then* $u(t)$ *is a solution of the problem* (1.1)-(1.2) *on the interval* J *if and only if it satisfies:*

$$\begin{aligned} u(t) =& e^{-\beta\psi(t)} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) \\ & \mathfrak{f}\left(\tau, u(\tau), \mathsf{K}u(\tau)\right) d\tau, \forall t \in \mathsf{J}. \end{aligned} \tag{3.1}$$

Proof. Necessity. By applying $I_0^{\beta,\alpha,\psi}$ on both sides of (1.1) and utilizing the composition properties, we derive equation (3.1).
Sufficiency. Since ${}^C\mathcal{D}_0^{\alpha,\psi}(u_0) = 0$, we obtain

$${}^C\mathcal{D}_0^{\alpha,\beta,\psi}(e^{-\beta\psi(t)} u_0) = e^{-\beta\psi(t)}\ {}^C\mathcal{D}_0^{\alpha,\psi} \left(e^{\beta\psi(t)} e^{-\beta\psi(t)}(u_0) \right) = 0. \tag{3.2}$$

Applying operator ${}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}$ in equation (3.1), we obtain

$$\begin{aligned}{}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}u(t) =& e^{-\beta\psi(t)C}\mathcal{D}_0^{\alpha,\beta,\psi}u_0 + {}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}\frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1} \\ & e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\mathfrak{f}\left(\tau, u(\tau), \mathsf{K}u(\tau)\right) d\tau \\ =& \mathfrak{f}\left(t, u(t), \int_0^t \mathfrak{g}(t,\tau,u(\tau))d\tau\right).\end{aligned}$$

Substituting $t=0$ in (3.1), yields $u(0)=u_0$. □

Theorem 3.2. *Let $\psi \in \mathcal{C}^1(\mathbb{R}^+)$ be a non-negative increasing function with $\psi'(t) \neq 0$ on $\mathbb{R}^+$ and $\lim_{t\to\infty}\psi(t)=\infty$. Assume that (H1)-(H3) are satisfied. If*

$$\frac{\mathfrak{L}_1(1+\mathfrak{L}_2)}{\beta^\alpha} < 1, \tag{3.3}$$

then there is a unique solution to (1.1) − (1.2).

Proof. Let $\delta = \frac{\mathsf{M}_\mathfrak{f}}{\beta^\alpha}$. Define the closed subset B_δ of $\mathcal{C}(\mathsf{J},\mathbb{R}^n)$ by

$$\mathsf{B}_\delta = \{u \in \mathcal{C}(\mathsf{J},\mathbb{R}^n) : \sup_{t\in\mathsf{J}} |u(t) - u_0 e^{-\beta\psi(t)}| \leq \delta\}$$

and $\mathsf{H} : \mathcal{C}(\mathsf{J},\mathbb{R}^n) \to \mathcal{C}(\mathsf{J},\mathbb{R}^n)$ by

$$\begin{aligned}(\mathsf{H}u)(t) =& e^{-\beta\psi(t)}u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau) \\ & \mathfrak{f}\left(\tau, u(\tau), \mathsf{K}u(\tau)\right) d\tau.\end{aligned}$$

First, we prove that $\mathsf{H}(\mathsf{B}_\delta) \subset \mathsf{B}_\delta$. Let $t \in \mathsf{J}$ and $u \in \mathsf{B}_\delta$. We obtain

$$\begin{aligned}|(\mathsf{H}u)(t) - e^{-\beta\psi(t)}u_0| \leq& \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau) \\ & |\mathfrak{f}\left(\tau, u(\tau), \mathsf{K}u(\tau)\right)| d\tau \\ \leq& \frac{\mathsf{M}_\mathfrak{f}}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)d\tau \\ \leq& \frac{\mathsf{M}_\mathfrak{f}}{\Gamma(\alpha)}\int_0^{\psi(t)} \eta^{\alpha-1}e^{-\beta\eta}d\eta \\ \leq& \frac{\mathsf{M}_\mathfrak{f}}{\beta^\alpha} = \delta.\end{aligned}$$

Hence, $\mathsf{H}(\mathsf{B}_\delta) \subset \mathsf{B}_\delta$. Now, we will demonstrate that $\mathsf{H} : \mathsf{B}_\delta \to \mathsf{B}_\delta$ is a contraction. Let $t \in \mathsf{J}$ and $u, \tilde{u} \in \mathsf{B}_\delta$. We get

$$\begin{aligned}|(\mathsf{H}u)(t) - (\mathsf{H}\tilde{u})(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) \\ & \left[\left| \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau)) - \mathfrak{f}(\tau, \tilde{u}(\tau), \mathsf{K}\tilde{u}(\tau)) \right| \right] d\tau \\ \leq & \frac{1}{\beta^\alpha} \mathfrak{L}_1(1 + \mathfrak{L}_2) \|u - \tilde{u}\|_\varepsilon \\ \leq & \frac{\mathfrak{L}_1(1 + \mathfrak{L}_2)}{\beta^\alpha} \|u - \tilde{u}\|_\varepsilon .\end{aligned}$$

Given the assumption $\frac{\mathfrak{L}1(1+\mathfrak{L}2)}{\beta^\alpha} < 1$, $\mathsf{H} : \mathsf{B}_\delta \to \mathsf{B}_\delta$ becomes a contraction mapping. Therefore, by the Banach fixed point theorem, H possesses a unique fixed point. Consequently, there exists a unique solution $u(t)$ to the problem (1.1)-(1.2) on J. □

Theorem 3.3. *Let $\psi \in \mathcal{C}^1(\mathsf{J})$ be a non-negative increasing function such that $\psi'(t) > 0$ on J, $\lim_{t\to\infty} \psi(t) = \infty$ and*

(H4) *$\mathfrak{f} : \mathsf{J} \times \mathbb{R}^n \times \mathbb{R}^n \to \times \mathbb{R}^n$ be a continuous function satisfying the condition*

$$|\mathfrak{f}(t, x, \tilde{x})| \leq c_0 + c_1|x| + c_2|\tilde{x}|, \forall\ x, \tilde{x} \in \mathbb{R}^n \text{ and } t \in \mathsf{J}, \tag{3.4}$$

where c_0, c_1 and c_2 are positive constants,

(H5) *$\mathfrak{g} : \mathsf{J} \times \mathsf{J} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and there exists a positive constant c_3 such that*

$$\left| \int_0^t \mathfrak{g}(t, \tau, u(\tau)) d\tau \right| \leq c_3[1 + |u|], \forall\ u \in \mathbb{R}^n \text{ and } t, \tau \in \mathsf{J},$$

then there exists at least one solution to the problem (1.1)-(1.2) over J.

Proof. Let $\mathcal{Q} := \{u \in \mathcal{C}^1(\mathsf{J}, \mathbb{R}^n) : {}^C\mathcal{D}_0^{\alpha,\beta,\psi} u \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)\}$ and let $\mathsf{H} : \mathcal{Q} \to \mathcal{Q}$ be the operator defined by the following expression

$$\begin{aligned}(\mathsf{H}u)(t) = & e^{-\beta\psi(t)} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) \\ & \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\, d\tau.\end{aligned}$$

The mapping $t \to (\mathsf{H}u)(t)$ is continuous for all $u \in \mathcal{C}^1(\mathsf{J}, \mathbb{R}^n)$ and ${}^C\mathcal{D}_0^{\alpha,\beta,\psi}(\mathsf{H}u)(t) = \mathfrak{f}(t, u(t), \mathsf{K}u(t))$ is also continuous on J. This indicates that the operator H is well-defined, i.e., $\mathsf{H}(\mathcal{Q}) \subset \mathcal{Q}$.

Claim I: The operator H is continuous. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $\mathcal{Q}$ such that $u_n \to u$ in $\mathcal{Q}$. Then

$$\begin{aligned}|\mathsf{H}(u_n(t)) - \mathsf{H}(u(t))|_{\mathcal{E}} =& \Big|\frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\\ &[\mathfrak{f}(\tau, u_n(\tau), \mathsf{K}u_n(\tau)) - \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))]d\tau\Big|\\ \leq& \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\\ &\Big|[\mathfrak{f}(\tau, u_n(\tau), \mathsf{K}u_n(\tau)) - \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))]\Big|d\tau\\ &\to 0 \ \ as\ n \to \infty,\end{aligned}$$

because $\lim_{n\to\infty} \mathfrak{f}(\tau, u_n(\tau), \mathsf{K}u_n(\tau)) = \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))$. Therefore, the operator H is continuous.

Claim II: H maps every bounded set into a bounded set in $\mathcal{Q}$. Let $u \in B_r = \{u \in \mathcal{Q} : \|u\| \leq r\}$. By (3.4), we get

$$\begin{aligned}|\mathsf{H}(u)|_{\mathcal{E}} \leq& |e^{-\beta\psi(t)}u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}\\ &e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau\\ \leq& |e^{-\beta\psi(t)}u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}\\ &e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\ (c_0 + c_1|u(\tau)| + c_2|\mathsf{K}u(\tau)|)d\tau\\ \leq& \overline{r},\end{aligned}$$

where $\overline{r} = |u_0| + \frac{1}{\beta^\alpha}\left[c_0 + c_1 r + c_2 c_3[1+r]\right]$. Since the number $\overline{r}$ is independent of t and u, this inequality implies that the mapping H is uniformly bounded.

Claim-III: H maps every bounded set into a equicontinuous set in $\mathcal{Q}$. Let $t, \tilde{t} \in \mathsf{J}$ such that $t < \tilde{t}$ and

$$\mathsf{M}_\psi = \beta \max_{\tau\in\mathsf{J}}\left(\psi'(\tau)e^{-\beta\psi(\tau)}\right).$$

Therefore, by the mean value theorem we have $|e^{-\beta\psi(\tilde{t})} - e^{-\beta\psi(t)}| \leq \mathsf{M}_\psi|\tilde{t} - t|$ and

$$\begin{aligned}|(\mathsf{H}u)(\tilde{t}) - \mathsf{H}(u)(t)| \leq& \mathsf{M}_\psi|\tilde{t}-t|u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t [\psi(\tilde{t})-\psi(\tau)]^{\alpha-1}\\ &e^{-\beta[\psi(\tilde{t})-\psi(\tau)]}\psi'(\tau)\left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau\\ &-\int_0^t [\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\psi'(\tau)\\ &\left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau + \int_t^{\tilde{t}} [\psi(\tilde{t})-\psi(\tau)]^{\alpha-1}\\ &e^{-\beta[\psi(\tilde{t})-\psi(\tau)]}\psi'(\tau)\left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau\end{aligned}$$

$$\begin{aligned}
\le & \mathsf{M}_\psi |\tilde{t} - t||u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t \Big[[\psi(\tilde{t}) - \psi(\tau)]^{\alpha-1} \\
& e^{-\beta[\psi(\tilde{t})-\psi(\tau)]} - [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \Big] \\
& \psi'(\tau) \left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau \\
& + \int_t^{\tilde{t}} [\psi(\tilde{t}) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(\tilde{t})-\psi(\tau)]} \psi'(\tau) \\
& \left|\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))\right| d\tau. \qquad (3.5)
\end{aligned}$$

The right-hand side of (3.5) is independent of H and evidently tends to 0 when $\tilde{t} \to t$. Therefore, the set $\mathsf{H}(B_r)$ is equicontinuous. By applying Claim I-III and the Arzela-Ascoli theorem, we infer that H is completely continuous.

Claim IV: The set

$$\mathcal{T} := \{u \in \mathcal{Q} : u = \mu \mathsf{H}(u) \text{ and } 0 < \mu < 1\}$$

is bounded. By utilizing condition (3.4), we get for $t \in \mathsf{J}$ and $u \in \mathcal{T}$

$$\begin{aligned}
|u(t)| =& |(\mathsf{H}u)(t)||\mu| \\
\le & |e^{-\beta\psi(t)} u_0| + \frac{c_0}{\Gamma(\alpha)} \left[\int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) d\tau \right] \\
& + \frac{c_1}{\Gamma(\alpha)} \left[\int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) |u(\tau)| d\tau \right] \\
& + \frac{c_2}{\Gamma(\alpha)} \left[\int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) |\mathsf{K}u(\tau)| d\tau \right] \\
\le & |e^{-\beta\psi(t)}||u_0| + \frac{c_0}{\beta^\alpha} + \\
& \frac{c_1}{\Gamma(\alpha)} \left[\int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) |u(\tau)| d\tau \right] \\
& + \frac{c_2}{\Gamma(\alpha)} \left[\int_0^t [\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \psi'(\tau) |\mathsf{K}u(\tau)| d\tau \right] \\
\|u\|_{\mathcal{E}} \le & |u_0| + \frac{c_0}{\beta^\alpha} + \frac{1}{\beta^\alpha} \Big[[c_1 \|u\|_{\mathcal{E}} + c_2 c_3 [1 + \|u\|_{\mathcal{E}}] \Big] \\
\|u\|_{\mathcal{E}} \le & \frac{|u_0| + \frac{c_0}{\beta^\alpha} + \left(\frac{c_2 c_3}{\beta^\alpha} \right)}{\left(1 - \left(\frac{c_2 c_3}{\beta^\alpha} + \frac{c_1}{\beta^\alpha} \right) \right)},
\end{aligned}$$

provided that $\left(\frac{c_2 c_3}{\beta^\alpha} + \frac{c_1}{\beta^\alpha} \right) < 1$, which demonstrates that $\mathcal{T}$ is bounded. According to Schaefer's fixed point theorem, the operator H possesses at least one fixed point. □

4. Stability Results

Remark 4.1. The function $\widehat{u} \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$ satisfies the inequality

$$\left| {}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t) - \mathfrak{f}(t, \widehat{u}(t), \mathsf{K}\widehat{u}(t))\right| \leq \epsilon, \forall\, t \in \mathsf{J},$$

if and only if there exists $h \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$

(i) $\|h(t)\| \leq \epsilon, \quad \forall\, t \in \mathsf{J}$,

(ii) ${}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t) = \mathfrak{f}(t, \widehat{u}(t), \mathsf{K}\widehat{u}(t)) + h(t), \quad \forall\, t \in \mathsf{J}$.

(H6) There is an increasing function $\phi \in \mathcal{C}(\mathsf{J}, \mathbb{R}^+)$ and $\lambda_\phi \in \mathbb{R}^+\backslash\{0\}$ such that

$$I_0^{\alpha,\beta,\psi}\phi(t) \leq \lambda_\phi\, \phi(t), \forall\, t \in \mathsf{J}.$$

Theorem 4.2. *Assume that the inequality (3.3) and (H1)-(H2) hold. Then the problem (1.1)-(1.2) is UHS.*

Proof. Let $\epsilon > 0$ and $\widehat{u} \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$ be a function which satisfies the inequality (2.1) and $u \in \mathcal{C}(\mathsf{J}, \mathbb{R}^n)$ be a unique solution of the problem

$$\begin{aligned} {}^{c}\mathcal{D}_0^{\alpha,\beta,\psi}u(t) =& \mathfrak{f}(t, u(t), \mathsf{K}u(t)) \\ u(0) =& \widehat{u}(0) = u_0. \end{aligned} \tag{4.1}$$

Hence, we obtain

$$u(t) = A_u + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))d\tau,$$

where $A_u = e^{-\beta\psi(t)}u_0$. If $u(0) = \widehat{u}(0) = u_0$, then we can determine

$$\begin{aligned} |A_{\widehat{u}} - A_u| \leq& |e^{-\beta\psi(t)}u_0 - e^{-\beta\psi(t)}u_0| = 0 \\ A_u =& A_{\widehat{u}}. \end{aligned}$$

Then

$$u(t) = A_{\widehat{u}} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))d\tau.$$

It follows from Remark 4.1, we derive

$$\begin{aligned} &\left|\widehat{u}(t) - A_{\widehat{u}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau, \widehat{u}(\tau), \mathsf{K}\widehat{u}(\tau))d\tau\right| \\ &\quad\leq \int_0^t \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]}|h(\tau)|d\tau \leq \frac{\epsilon}{\beta^\alpha} \end{aligned}$$

and

$$\begin{aligned} \left|\widehat{u}(t) - u(t)\right| =& \Big|\widehat{u}(t) - A_{\widehat{u}} - \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t) - \psi(\tau)]^{\alpha-1} e^{-\beta[\psi(t)-\psi(\tau)]} \\ & \mathfrak{f}(\tau, u(\tau), \mathsf{K}u(\tau))d\tau\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq |\widehat{u}(t)-A_{\widehat{u}}-\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\\
&\quad \mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))d\tau| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}\\
&\quad e^{-\beta[\psi(t)-\psi(\tau)]}\Big|\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))-\mathfrak{f}(\tau,u(\tau),\mathsf{K}u(\tau))\Big|d\tau\\
&\leq \frac{\epsilon}{\beta^\alpha}+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\\
&\quad \left(\mathfrak{L}_1\left(\Big|\widehat{u}(\tau)-u(\tau)\Big|+\mathfrak{L}_2\Big|\widehat{u}(\tau)-u(\tau)\Big|\right)\right)d\tau\\
\|\widehat{u}-u\|_{\mathcal{E}} &\leq \frac{\epsilon}{\beta^\alpha-(\mathfrak{L}_1(1+\mathfrak{L}_2))}\\
&\leq \epsilon\lambda_{\mathfrak{f}},
\end{aligned}
$$

provided that $0<\mathfrak{L}_1(1+\mathfrak{L}_2)<\beta^\alpha$, where $\lambda_{\mathfrak{f}}=\dfrac{1}{\beta^\alpha-(\mathfrak{L}_1(1+\mathfrak{L}_2))}$. We conclude that (1.1) – (1.2) exhibits UHS. □

Theorem 4.3. *If the hypotheses (H1)-(H2) hold and there exists $\phi_{\mathfrak{f}}\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ such that $\phi_{\mathfrak{f}}(0)=0$, then (1.1)-(1.2) is GUHS.*

Proof. Following the same method as in Theorem 4.2 by letting $\phi_{\mathfrak{f}}(\epsilon)=\lambda_{\mathfrak{f}}(\epsilon)$ and $\phi_{\mathfrak{f}}(0)=0$. We have

$$\Big|\widehat{u}(t)-u(t)\Big|\leq\phi_{\mathfrak{f}}(\epsilon).$$

□

Lemma 4.4. *Let $\widehat{u}\in\mathcal{C}(\mathsf{J},\mathbb{R}^n)$ be a solution of the inequality*

$$\Big|{}^{c}\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t)-\mathfrak{f}(t,\widehat{u}(t),\mathsf{K}\widehat{u}(t)\Big|\leq\ \epsilon\phi(t). \tag{4.2}$$

Then $\widehat{u}$ is a solution of the inequality

$$
\begin{aligned}
&\Big|\widehat{u}(t)-A_{\widehat{u}}-\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))d\tau\Big|\\
&\quad\leq\epsilon\lambda_\phi\phi(t),
\end{aligned}
$$

where $A_{\widehat{u}}=e^{-\beta\psi(t)}u_0$.

Proof. Indeed by Remark 4.1, we have

$${}^{C}\mathcal{D}_0^{\alpha,\beta,\psi}\widehat{u}(t)=\mathfrak{f}(t,\widehat{u}(t),\mathsf{K}\widehat{u}(t))+h(t),\forall\ t\in\mathsf{J}. \tag{4.3}$$

Applying $I_0^{\alpha,\beta,\psi}$ to both sides of (4.3), we obtain:

$$
\begin{aligned}
\widehat{u}(t)-e^{-\beta\psi(t)}u_0 &= \int_0^t \psi'(\tau)\frac{[\psi(t)-\psi(\tau)]^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}h(\tau)d\tau
\end{aligned}
$$

It follows from $(H6)$, that

$$\begin{aligned}\Big|\widehat{u}(t)-e^{-\beta\psi(t)}u_0-&\int_0^t\psi'(\tau)\frac{(\psi(t)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta[\psi(t)-\psi(\tau)]}\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))d\tau\Big|\\ \leq&\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\Big|h(\tau)\Big|d\tau\\ \leq&\frac{\epsilon}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\phi(\tau)d\tau\\ \leq&\epsilon\lambda_\phi\ \phi(t).\end{aligned}$$

□

Theorem 4.5. *If the inequality (4.2) and the hypotheses $(H1)-(H2)$ hold, then (1.1)-(1.2) is UHRS and GUHRS.*

Proof. Let $\epsilon>0$ and $\widehat{u}\in\mathcal{C}(\mathsf{J},\mathbb{R}^n)$ satisfies (4.2) and $u\in\mathcal{C}(\mathsf{J},\mathbb{R}^n)$ be a unique solution of (1.1)-(1.2). By Theorem 3.2 and $A_u=A_{\widehat{u}}$, we obtain

$$u(t)=A_{\widehat{u}}+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{-\beta(\psi(t)-\psi(\tau))}\mathfrak{f}(\tau,u(\tau),\mathsf{K}u(\tau))d\tau.$$

By $(H1)-(H2)$ and Lemma 3.9, we have

$$\begin{aligned}|\widehat{u}(t)-u(t)|=&\Big|\widehat{u}(t)-A_{\widehat{u}}-\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\\ &\mathfrak{f}(\tau,u(\tau),\mathsf{K}u(\tau))d\tau\Big|\\ \leq&\Big|\widehat{u}(t)-A_{\widehat{u}}-\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\\ &\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))d\tau\Big|+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}\\ &e^{-\beta[\psi(t)-\psi(\tau)]}\Big|\mathfrak{f}(\tau,\widehat{u}(\tau),\mathsf{K}\widehat{u}(\tau))-\mathfrak{f}(\tau,u(\tau),\mathsf{K}u(\tau))\Big|d\tau\\ \leq&\epsilon\lambda_\phi\phi(t)+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}e^{-\beta[\psi(t)-\psi(\tau)]}\\ &\Big(\mathfrak{L}_1\Big(\Big|\widehat{u}(\tau)-u(\tau)\Big|+\mathfrak{L}_2\Big|\widehat{u}(\tau)-u(\tau)\Big|\Big)\Big)d\tau\\ \leq&\epsilon\lambda_\phi\phi(t)+\frac{\mathfrak{L}_1(1+\mathfrak{L}_2)}{\beta^\alpha}\|\widehat{u}-u\|_{\mathcal{E}}\\ \|\widehat{u}-u\|_{\mathcal{E}}\leq&\epsilon C_{f,\phi}\ \phi(t),\end{aligned}$$

provided that $\frac{\mathfrak{L}_1(1+\mathfrak{L}_2)}{\beta^\alpha}<1$, where $C_{f,\phi}=\frac{\lambda_\phi}{1-\left(\frac{\mathfrak{L}_1(1+\mathfrak{L}_2)}{\beta^\alpha}\right)}$. This means that problem (1.1)-(1.2) UHRS.

By the same way, with letting $\epsilon=1$, we can determine that

$$\|\widehat{u}-u\|_{\mathcal{E}}\leq C_{f,\phi}\ \phi(t).$$

This yields that problem (1.1)-(1.2) is GUHRS. □

5. Example

Example 5.1. Consider the fractional differential equation given below:

$$\begin{aligned} {}^{C}\mathcal{D}_0^{1/2,\beta,\psi}u(t) =& \frac{1}{10} + \frac{e^{-\beta\, t}}{2}u(t) + \int_0^t \frac{t}{2}u(\tau)d\tau, \quad t \in [0,1] = \mathsf{J}, \qquad (5.1)\\ u(0) =& u_0, \qquad (5.2) \end{aligned}$$

here, we take

$$\alpha = 0.5, \quad \beta = 2, \quad \psi(t) = 2^t \text{ and}$$

$$\begin{aligned} \mathfrak{f}(t, u(t), \mathsf{K}u(t)) =& \frac{1}{10} + \frac{e^{-\beta\, t}}{2}u(t) + \int_0^t \frac{t}{2}u(\tau)d\tau,\\ \mathsf{K}u(t) =& \int_0^t \frac{t}{2}u(\tau)d\tau. \end{aligned}$$

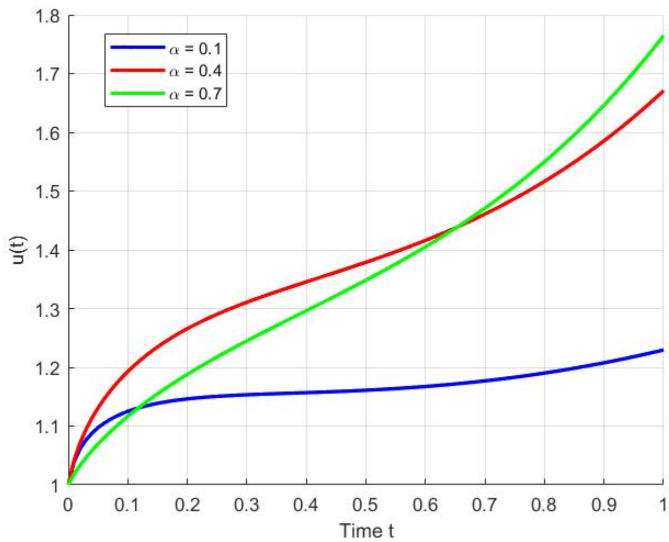

Figure 1. Graph of u for different values of α

Let $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}^n$ and $t \in \mathsf{J}$, then we have

$$\begin{aligned} |\mathfrak{f}(t, u(t), \mathsf{K}v(t)) - \mathfrak{f}(t, \tilde{u}(t), \mathsf{K}\tilde{v}(t))| \leq& \frac{1}{2}(|u - \tilde{u}| + |v - \tilde{v}|)\\ |\mathfrak{g}(t, \tau, u) - \mathfrak{g}(t, \tau, \tilde{u})| \leq& \frac{1}{2}(|u - \tilde{u}|). \end{aligned}$$

Thus, the hypotheses (H1)-(H2) are satisfied with $\mathfrak{L}_1 = \frac{1}{2}$ and $\mathfrak{L}_2 = \frac{1}{2}$. By simple calculation, we obtain

$$\frac{\mathfrak{L}_1(1 + \mathfrak{L}_2)}{\beta^\alpha} = \frac{1/2(1 + 1/2)}{2^{0.5}} < 0.53 < 1.$$

Therefore, based on Theorem 3.2, there is a unique solution to the problem (5.1)-(5.2) on J. Additionally, according to Theorem 4.2, (5.1) – (5.2) exhibits UHS.

6. Conclusion

In this work, we investigated the existence and uniqueness of solutions for fractional integrodifferential equations involving the tempered ψ-Caputo fractional derivative using fixed point theorems. Additionally, we analyzed the stability of the solutions under specific assumptions. An example is given to demonstrate the theoretical results.

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