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EXTENSION OF KOSAMBI-KARHUNEN-LOÈVE EXPANSION IN L^p; $1 < p < 2$

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Abstract. This study presents an extension of the Kosambi-Karhunen-Loève (KKL) decomposition to accommodate stochastic processes with potentially infinite variance. We propose a nonlinear homeomorphic transformation upon which the classical KKL expansion may be applied. Subsequently, we establish the existence of specific subsets wherein this transformation preserves isometric properties. The theoretical framework is illustrated through an application to symmetric α-stable processes, demonstrating the practical utility of the proposed methodology.

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1. Introduction

Let

$$X : (\Omega \times \mathbb{T},\, \mathcal{A} \otimes \mathcal{T}) \longmapsto (\mathbb{R},\, \mathcal{B}_{\mathbb{R}}) \tag{1.1}$$

be a stochastic process where $\mathbb{T} \subset \mathbb{R}_+$. We assume that $X(.,t)$, denoted X_t, is in $L^2(\Omega)$. Kosambi-Karhunen-Loève (KKL) expansion(Kosambi[11], Karhunen [8], Loève [12]), usually named Karhunen-Loève (KL) expansion, consists of expanding a continuous stochastic process in $L^2(\Omega)$ to some infinite linear combination of orthogonal functions. This technique is well established for processes X, where $X_t \in L^2(\Omega)$, leveraging the Hilbert space structure to derive spectral representations. Dauxois et al.[3], in their seminal work (1982), developed the asymptotic framework for Principal Component Analysis (PCA) applied to random functions. Their results can be applied easily on KKL expansion of processes defined in $L^2(\Omega) \times L^2(\mathbb{T})$. For periodic processes, the KKL expansion reduces to a Fourier-like series: the orthogonal basis becomes restricted to sinusoidal functions with period T and the coefficients must be sorted decreasingly (Panaretos and Tavakoli [15]). However, significant limitations arise when $X_t \in L^p(\Omega)$, $1 < p < 2$ or $X(\omega,.) \in L^p(\mathbb{T})$, $1 < p < 2$. In these settings, neither $L^p(\Omega)$ nor $L^p(\mathbb{T})$ admits an isometric embedding into a Hilbert space, contrary to the construction of Kim

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et al. ([9]). In their approach, functional principal component analysis (FPCA) is extended to handle data taking values in non-Euclidean spaces, such data in Aitchison geometry on the simplex for example, and to embed them isometrically into some Euclidean or Hilbert space. Thereafter, standard FPCA techniques can be applied. When data are represented in a space L^p, $1 < p < 2$, we can't use this type of transformation because there isn't a canonical isometric isomorphism between L^p, $1 < p < 2$ and L^2. Moreover, the covariance operator doesn't exist rendering classical KKL expansion (or FPCA) inoperative. To overcome this problem, Kokoszka, P. and Kulik, R. (2023) make significant advances in the analysis of regularly varying functional data [10]. Their exploration of Principal Component Analysis (PCA) for infinite-variance processes demonstrates that the sample covariance operator and its eigenfunctions remain well-defined even in such settings. However, the asymptotic behavior of these critical statistics under infinite variance has long remained an open question, necessitating the assumption of regularly varying tails in the observations.

In this work, we propose a novel extension of the Kosambi-Karhunen-Loève (KKL) expansion tailored to stochastic processes $X \in L^p(\Omega) \times L^2(\mathbb{T})$ where $1 < p < 2$ addressing the limitations of classical Hilbert-space-based methods. Our approach is centered on a principal stage, the application of a Nonlinear Homeomorphic Transformation : We introduce a nonlinear homeomorphism that transforms the process into a Hilbert space where classical KKL techniques become directly applicable, circumventing the limitations of $L^p(\Omega)$.

Finally, we identify a restricted subspace of $L^p(\Omega)$ where this nonlinear transformation acts as an isometry, enabling the transfer of asymptotic results from Hilbertian settings. Before introducing our approach, we first present, in the next section, the classical KKL expansion using our notations, ensuring clarity in the stage of our methodology.

2. KKL Expansion of a Second-Order Process

Let $(X_t)_{t\in\mathbb{T}}$ be a zero-mean stochastic process with $X_t \in L^2(\Omega)$ for all $t \in \mathbb{T}$, where $L^2(\Omega)$ denotes a separable Hilbert space endowed with the standard inner product

$$\langle X_t, X_s\rangle_{L^2(\Omega)} := E(X_t X_s) = \int_\Omega X(\omega,t)X(\omega,s)\,\mathbb{P}(d\omega).$$

On the other hand, we consider that $X(\omega,\cdot)$, denoted by X^ω, belongs to some separable Hilbert space $L^2(\mathbb{T})$, endowed with the inner product

$$\langle X^{\omega_1}, X^{\omega_2}\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} X^{\omega_1}(t)X^{\omega_2}(t)\,\mu(dt),$$

where μ is a σ-finite measure.

We consider the linear continuous operator $U : L^2(\Omega) \longrightarrow L^2(\mathbb{T})$ defined by

$$\forall h \in L^2(\Omega) \quad U(h)(t) = E(X_t h) =< X(.,t), h >_{L^2(\Omega)}=< X_t, h >_{L^2(\Omega)} \quad (\in L^2(\mathbb{T}))$$

The adjoint operator of U, denoted U^*, is given by

$$\forall u \in L^2(\mathbb{T}) \quad U^*(u)(\omega) =< X(\omega,.), u >_{L^2(\mathbb{T})}=< X^\omega, u >_{L^2(\mathbb{T})} \qquad (\in L^2(\Omega))$$

Identifying $L^2(\mathbb{T})$ and $L^2(\Omega)$ with their respective duals, the covariance operator of the process, denoted by $\mathbf{\Gamma} = U \circ U^*$, is defined as:

$$\begin{array}{llllll} \mathbf{\Gamma}: & L^2(T) & \longrightarrow & L^2(T) & & \\ & f & \longmapsto & \mathbf{\Gamma} f = g: & \mathbb{T} & \rightarrow \mathbb{R} \\ & & & & t & \mapsto g(t) = \int_{\mathbb{T}} E(X_t X_s) f(s) \mu(ds) \end{array}$$

$\mathbf{\Gamma}$ is defined as the integral operator with kernel :

$$\gamma(t,s) = E(X_t X_s).$$

It is a self-adjoint, bounded, and nonnegative operator. Therefore, its spectrum, denoted by $\sigma(\mathbf{\Gamma})$, is a discrete subset of $\mathbb{R}_+$. On the other hand, the operator $U^*U : L^2(\Omega) \to L^2(\Omega)$ is also bounded, self-adjoint, and non-negative, and admits the following spectral decomposition:

$$U^*U(h) = \sum_k \lambda_k \, \psi_k \otimes \psi_k \, h = \sum_k \lambda_k \langle h, \psi_k \rangle_{L^2(\Omega)} \, \psi_k,$$

where the spectrum of U^*U is the same as that of $\mathbf{\Gamma}$, and ψ_k is a normalized eigenfunction associated with the eigenvalue λ_k, satisfying

$$\psi_k = \frac{U^* f_k}{\|U^* f_k\|_{L^2(\Omega)}},$$

with $(f_k)_k$ the orthonormal system of eigenfunctions of $\mathbf{\Gamma}$.

We can easily decompose the process X as follows:

$$X(\omega, t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \, \psi_k(\omega) \, f_k(t), \tag{2.1}$$

which is the well-known Kosambi–Karhunen–Loève (KKL) expansion in the Hilbert spaces $L^2(\Omega)$ and $L^2(\mathbb{T})$.

According to spectral theory, for all $f \in L^2(\mathbb{T})$, we have

$$\mathbf{\Gamma}(f) = \sum_k \lambda_k \, (f_k \otimes f_k)(f) = \sum_k \lambda_k \langle f, f_k \rangle_{L^2(\mathbb{T})} \, f_k,$$

where $f_k \in L^2(\mathbb{T})$ is a normed eigenfunction associated to λ_k and the spectrum of $\mathbf{\Gamma}$ is taken such that each eigenvalue is repeated with the same times as its multiplicity order and the eigenvalues are sorted decreasingly.

In the sequel, we will particularly consider the case where $\mathbf{\Gamma}$ is a nuclear operator. This means that

$$\sum_k \lambda_k = \mathrm{Tr}(\mathbf{\Gamma}) < \infty,$$

which implies that U (or equivalently U^*) is a Hilbert–Schmidt operator.

Linear regression. Let's denote $\mathbf{B}(L^2(\mathbb{T}))$ the algebra of bounded operators on $L^2(\mathbb{T})$. Since $\mathbf{\Gamma}$ is self-adjoint operator, considered bounded, there exists a spectral measure ν on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ vanishing on the complement of the spectrum of $\mathbf{\Gamma}$ (which is in $\mathbb{R}$) and a function $\Lambda : \sigma(\mathbf{\Gamma}) \longrightarrow \mathbf{B}(L^2(\mathbb{T}))$ such that Vakhania et al. (1987) :

- for all $\lambda \in \sigma(\mathbf{\Gamma})$, $\Lambda(\lambda)$ is an orthogonal projector (self-adjoint and idempotent),
- if $\lambda_1 \neq \lambda_2$, $\Lambda(\lambda_1) \circ \Lambda(\lambda_2) = 0$,
- The integral $\displaystyle\int_{\mathbb{C}} \Lambda(\lambda) d\nu(\lambda)$ converges strongly to the identity operator,
- $\mathbf{\Gamma} = \displaystyle\int_{\mathbb{C}} \lambda\ \Lambda(\lambda)\ d\nu(\lambda) = \int_{\sigma(\mathbf{\Gamma})} \lambda\ \Lambda(\lambda)\ d\nu(\lambda)$ is called the spectral decomposition of $\mathbf{\Gamma}$.

Assuming that the spectrum $\sigma(\mathbf{\Gamma})$ consists of a decreasing sequence of eigenvalues λ_k, the operator $\mathbf{\Gamma}$ can be represented as

$$\mathbf{\Gamma} = \sum_k \lambda_k \, \Lambda(\lambda_k),$$

where $\Lambda(\lambda_k)$ denotes the orthogonal projection onto the eigenspace corresponding to the eigenvalue $\lambda_k \in \sigma(\mathbf{\Gamma})$. .

Proposition 2.1. *If $Y(.,t) = P(X(.,t))$ is an orthogonal projection of $X(.,t)$ with covariance operator $\mathbf{\Gamma}$, then covariance operator of Y is an orthogonal projection of $\mathbf{\Gamma}$ in $\mathbf{B}(L^2(\mathbb{T}))$, expressed $P\mathbf{\Gamma}P$ which is self-adjoint.*

More generally, if Y is a linear transformation $Y = AX$ and $A \in \mathbf{B}(L^2(\mathbb{T}))$, the space of bounded linear operators, one has

Proposition 2.2. *Let $X \in L^2(\mathbb{T})$ be a square-integrable, mean-zero stochastic process with covariance operator $\mathbf{\Gamma}$, and let $A \in \mathbf{B}(L^2(\mathbb{T}))$ be a bounded linear operator. Define $Y = AX$. Then the covariance operator of Y is expressed $\mathbf{\Gamma}_Y = A\mathbf{\Gamma}A^*$, where A^* is the adjoint of A.*

Using these notations, we can generally express all PCA-based methods in a unified framework.

Empirical Estimation. In practical case, we estimate the covariance operator $\mathbf{\Gamma}$ by an empirical covariance operator $\hat{\mathbf{\Gamma}}_n$. If $\hat{\mathbf{\Gamma}}_\mathbf{n}$ is an estimate of $\mathbf{\Gamma}$, then for all sample $X^{(1)}, \ldots, X^{(n)}$ i.i.d. as X, one has

$$\hat{\mathbf{\Gamma}}_\mathbf{n} = \frac{1}{n} \sum_{j=1}^{n} X^{(j)} \otimes X^{(j)} \tag{2.2}$$

As in practice it's too hard to obtain the $X^{(j)}$, one uses cylindrical projections of X denoted $\pi_{t_1,\ldots,\ldots t_k}$ such that $X \circ \pi_{t_1,\ldots,\ldots t_k}$ is represented as a vector in $\mathbb{R}^n$.

So for all $j \in \{1, \ldots, n\}$ one has

$$X^{(j)} \circ \pi_{t_1,\ldots,\ldots t_k} := \begin{pmatrix} X_{t_1}^{(j)} \\ \vdots \\ X_{t_k}^{(j)} \end{pmatrix}$$

We estimate $\mathbf{\Gamma}$ under the assumption that $E(X) = 0$ by

$$\begin{aligned}\hat{\Gamma}_{n,k} &= \frac{1}{n}\sum_{j=1}^{n}(X^{(j)} \circ \pi_{t_1,\ldots,\ldots t_k}) \otimes (X^{(j)} \circ \pi_{t_1,\ldots,\ldots t_k}) \\ &= \frac{1}{n}\sum_{j=1}^{n}\begin{pmatrix} X_{t_1}^{(j)} \\ \vdots \\ X_{t_k}^{(j)}\end{pmatrix}\left(X_{t_1}^{(j)}, \ldots, X_{t_k}^{(j)}\right).\end{aligned}$$

Denoting

$$X_{n,k} = \begin{pmatrix} X_{t_1}^{(1)} & \ldots & X_{t_k}^{(1)} \\ \vdots & \vdots & \vdots \\ X_{t_1}^{(n)} & \ldots & X_{t_k}^{(n)}\end{pmatrix}$$

The KKL expansion can be approximated in practice by applying PCA to the empirical matrix $X_{n,k}$.

In this framework, the operator U is approximated by

$$\frac{1}{\sqrt{n}} X_{n,k}^{\top}, \quad \text{and} \quad U^* \text{ by } \frac{1}{\sqrt{n}} X_{n,k},$$

where the empirical duality between $L^2(\Omega)$ and $L^2(\mathbb{T})$ is captured by this matrix form.

The operator $\hat{\Gamma}_{n,k}$ is an unbiased symmetric estimator of $\mathbf{\Gamma}$. The following theorem establishes the almost sure convergence of $\hat{\Gamma}_{n,k}$ to $\mathbf{\Gamma}$, along with a convergence rate [2].

Theorem 2.3. *Assume that*

$$\mathbb{E}\left[\int_{\mathbb{T}} \|X(t)\|^4 dt\right] < \infty.$$

holds. If $n \to \infty$ *and* $\sup |t_{i+1} - t_i| \to 0$ *we have:*

$$\frac{n^{1/4}}{(\log n)^{\beta}} \left\|\hat{\Gamma}_{n,k} - \mathbf{\Gamma}\right\|_{HS} \longrightarrow 0 \quad \textit{almost surely,} \tag{2.3}$$

for all $\beta > \frac{1}{2}$.

Moreover, the convergence rate satisfies:

$$\left\|\hat{\Gamma}_{n,k} - \mathbf{\Gamma}\right\|_{HS}^2 = \mathcal{O}\left(\left(\frac{\log n}{n}\right)^{1/2}\right). \tag{2.4}$$

Spectral Estimators. Let $\mathbf{\Gamma}$ be a nuclear , self-adjoint, and positive operator on a Hilbert space $L^2(\mathbb{T})$. It admits the spectral decomposition:

$$\mathbf{\Gamma} = \sum \lambda_k \langle \cdot, f_k \rangle f_k, \tag{2.5}$$

where (f_k) is a complete orthonormal system in $L^2(\mathbb{T})$, and (λ_K) is a decreasing sequence of positive real numbers such that:

$$\sum_{k=1}^{\infty} \lambda_k < \infty. \tag{2.6}$$

A natural estimator of these spectral elements is given by the empirical eigen elements obtained from the empirical covariance operator $\hat{\Gamma}_{n,k}$, which satisfy:

$$\hat{\Gamma}_{n,k}\hat{f}_{k_n} = \hat{\lambda}_{k_n}\hat{f}_{k_n}, \quad k \geq 1, \tag{2.7}$$

where $\hat{\lambda}_{1_n} \geq \hat{\lambda}_{2_n} \geq \cdots \geq \hat{\lambda}_{n_n} \geq 0 = \hat{\lambda}_{(n+1)_n} = \hat{\lambda}_{(n+2)_n} = \cdots$, and $(\hat{f}_{k_n})$ forms a complete orthonormal system in $L^2(\mathbb{T})$ verifying :

$$\hat{\Gamma}_{n,k} = \sum_{k_n=1}^{n} \hat{\lambda}_{k_n}\hat{f}_{k_n} \otimes \hat{f}_{k_n},$$

We take $\hat{\lambda}_{k_n}$ as the estimators of λ_k and $\hat{f}_{k_n}$ as the estimators of the f_k and we have

$$\frac{\langle X_i, f_k\rangle_{L^2(\mathbb{T})}}{\sqrt{\lambda_k}} = \psi_{ik}$$

$$\frac{\langle X_{i,k}, \hat{f}_{k_n}\rangle}{\sqrt{\lambda_k}} = \hat{\psi}_{ik_n}$$

Convergence of Spectral Estimators. The asymptotic behavior of the empirical eigenvalues $\hat{\lambda}_{k_n}$ and eigenfunctions $\hat{f}_{k_n}(t)$ associated with the empirical covariance operator $\hat{\Gamma}_{n,k}$ has been thoroughly studied by Bosq (2000)[2], Hsing, Eubank (2015)[6], and Horváth and Kokoszka (2012)[7]. Under the following assumptions:

H1: Finite fourth moment:

$$\mathbb{E}\left[\int_{\mathbb{T}} \|X(t)\|^4\, dt\right] < \infty.$$

H2: Spectral gap condition:

$$\min_{1\leq l\leq N}(\lambda_l - \lambda_{l+1}) > 0, \quad \text{for some fixed } N \geq 1.$$

For all $k = 1, \ldots, N$, the following convergence results hold at a parametric rate $\mathcal{O}_P(n^{-1})$:

- Eigenvalues:

$$\left|\hat{\lambda}_{k_n} - \lambda_k\right| = \mathcal{O}_P(n^{-1}),$$

- Eigenfunctions:

$$\int_{\mathbb{T}} \left|\hat{f}_{k_n}(t) - f_k(t)\right|^2 dt = \mathcal{O}_P(n^{-1}).$$

Furthermore, under the additional assumption:

H3: Higher moment condition:

$$\mathbb{E}\left[\int_{\mathbb{T}} |X(t)|^{2\beta}\, dt\right] < \infty, \quad \text{for some } \beta \geq 2,$$

the estimated functional scores satisfy the uniform convergence rate:

$$\max_{1\leq i\leq n}\left\|\hat{\psi}_{ik_n}-\psi_{ik}\right\|_{L^2(\Omega)}=\mathcal{O}_P\left(n^{-\frac{\beta-1}{2\beta}}\right).$$

From Theorem 2.4 in Horváth and Kokoszka (2012)[7], if $\mathbb{E}\|X\|^4<\infty$ implies $\mathbb{E}\|\hat{\Gamma}_n\|_{HS}^2<\infty$, Specifically:

Theorem 2.4. *Let $X_1, X_2, \ldots, X_n \in L^2(\mathbb{T})$ be observations with the same distribution as X, assumed to be square integrable with $\mathbb{E}(X) = 0$ and $\mathbb{E}\|X\|^4 < \infty$. Then:*

$$\mathbb{E}\|\hat{\Gamma}_n\|_{HS}^2 \leq \mathbb{E}\|X\|^4.$$

Proof. By the triangle inequality, we have

$$\mathbb{E}\left\|\hat{\Gamma}_n\right\|_{\mathcal{HS}}=\mathbb{E}\left\|\frac{1}{N}\sum_{n=1}^{N}\langle X_n,\cdot\rangle X_n\right\|_{\mathcal{HS}}\leq\mathbb{E}\left\|\langle X,\cdot\rangle X\right\|_{\mathcal{HS}}$$

for any orthonormal basis $\{f_j\}_{j\geq 1}$, we compute:

$$\|\langle X,\cdot\rangle X\|_{\mathcal{HS}}^2=\sum_{j=1}^{\infty}\|\langle X,f_j\rangle X\|^2=\|X\|^2\sum_{j=1}^{\infty}|\langle X,f_j\rangle|^2=\|X\|^4$$

Taking expectations on both sides yields:

$$\mathbb{E}\left\|\langle X,\cdot\rangle X\right\|_{\mathcal{HS}}^2\leq\mathbb{E}\|X\|^4$$

□

When the covariance operator $\mathbf{\Gamma}$ is nuclear, the convergence rate theorems of Bosq et al. provide explicit rates for $\hat{\Gamma}_{n,k}$, $\hat{\lambda}_{k_n}$, and $\hat{f}_{k_n}$. If $\mathbf{\Gamma}$ is of Hilbert-Schmidt type, these rates are improved due to the condition $\sum_{k=1}^{\infty}\lambda_k^2 < \infty$, enabling more accurate spectral approximations and faster convergence of the empirical estimators.

By Mercer's theorem (1909)[14], if $\mathbf{\Gamma}$ is Hilbert-Schmidt, then $\hat{\Gamma}_{n,k}$ converges strongly to $\mathbf{\Gamma}$ in the operator norm. The best n-term approximation (in the mean square error sense) of the process X is given by:

$$X_n(\omega,t)=\sum_{k_n=1}^{n}\sqrt{\hat{\lambda}_{k_n}}\hat{\psi}_{k_n}(\omega)\hat{f}_{k_n}(t), \tag{2.8}$$

which converges uniformly to $X(\omega, t)$ as $n \to \infty$.

The function $X_n(\cdot, t)$ (respectively, $X_n(\omega, \cdot)$) provides the best approximation (in the mean squared sense) of $X(\cdot, t)$ (respectively, $X(\omega, \cdot)$) in an n-dimensional subspace of $L^2(\Omega)$ (respectively, $L^2(\mathbb{T})$), as established by Mercer's theorem. The quality of this approximation is quantified by the ratio:

$$\frac{\sum_{k_n=1}^{n}\hat{\lambda}_{k_n}}{\operatorname{tr}(\hat{\Gamma}_{n,k})}\longrightarrow\frac{\sum_{k=1}^{n}\lambda_k}{\operatorname{tr}\mathbf{\Gamma}}\qquad a.s.$$

representing the proportion of total variance explained by the first n components. Generally, in practice, we choose n such that the cumulative proportion, between 0 and 1, is sufficiently large.

In the next section, we develop our methodology and explore it application.

3. Kosambi-Karhunen-Loève expansion for process in $L^{p,2}(\Omega \times \mathbb{T})$ (an extension)

Let $(X_t)_{t\in\mathbb{T}}$ a stochastic process such that, for all $t \in \mathbb{T}$, $E(X_t) = 0$ and $X_t \in L^p(\Omega)$, denoted E, $1 < p < 2$, which is a separable reflexive Banach space. It can be also represented as $X = (X^\omega)_{\omega\in\Omega}$ defined in $(\mathbb{T}, \mathcal{T}, \mu)$ such that $\forall\omega \in \Omega$, $X^\omega \in L^2(\mathbb{T})$ (X^ω represents a trajectory).

3.1. **KKL using a non-linear transformation approach.** Let $(X_t)_{t\in\mathbb{T}}$ a stochastic process such that, for all $t \in \mathbb{T}$, $E(X_t) = 0$ and $X_t \in L^p(\Omega)$, $1 < p < 2$.

We seek to construct a homeomorphism from $L^p(\Omega)$ to $L^2(\Omega)$ to ensure a KKL expansion on the transformed process. To do this, we carry out the following transformation :

$$\begin{array}{rccl} R: & L^p(\Omega) & \longrightarrow & L^2(\Omega) \\ & x & \longmapsto & Rx = x_+^{\frac{p}{2}} - x_-^{\frac{p}{2}} \end{array}$$

Each random element $x \in L^p$ can be uniquely decomposed (a.s.) in a difference of two positive random elements x_+ and x_- such that $\forall\, w \in \Omega$, $x_+(\omega) = \max(0, x(\omega))$ and $x_-(\omega) = -\min(0, x(\omega)) = \max(0, -x(\omega))$.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence of random elements converging a.s. to x then $(x_{+,n})_{n\in\mathbb{N}} \longrightarrow x_+$ and $(x_{-,n})_{n\in\mathbb{N}} \longrightarrow x_-$ a.s..
Let S another transformation from $L^2(\Omega)$ to $L^p(\Omega)$ which verifies

$$\begin{array}{rccl} S: & L^2(\Omega) & \longrightarrow & L^p(\Omega) \\ & x & \longmapsto & S(x) = x_+^{\frac{2}{p}} - x_-^{\frac{2}{p}} \end{array}$$

It is obvious that R and S are continuous.

We verify also that $R \circ S = I_{L^2(\Omega)}$ and $R' \circ R = I_{L^p(\Omega)}$ where $I_{L^2(\Omega)}$ (resp. $I_{L^p(\Omega)}$) is the identity in $L^2(\Omega)$ (resp. $L^p(\Omega)$).

Furthermore, we confirm that R (resp. S) is a transformation which conserves the sign.
Based on the previous transformation, we can build easily KKL expansion of RX in $L^2(\Omega)$.

In order to express linear operator $U : L^2(\Omega) \to L^2(\mathbb{T})$ of our analysis, we proceed as follows :

$$\begin{array}{rccl} U: & L^2(\Omega) & \longrightarrow & L^2(\mathbb{T}) \\ & \xi & \longmapsto & \displaystyle\int_\Omega RX(\omega, \bullet)\, \xi(\omega)\ P(d\omega) \end{array}$$

One verifies easily that U is continuous.

U^* verifies :

$$\begin{array}{rccl} U^*: & L^2(\mathbb{T}) & \longrightarrow & L^2(\Omega) \\ & \nu & \longmapsto & \displaystyle\int_\Omega RX(\bullet, t)\ \nu(t)\ \mu(dt) \end{array}$$

We verify that U^* is continuous also.

With the help of following diagram

$$\begin{tikzcd}
 & L^2(\Omega) \arrow[d, "I"'] & L^2(\mathbb{T}) \arrow[l, "U^*"'] \arrow[d, shift left, "\mathbf{\Gamma}"] \\
L^p(\Omega) \arrow[r, shift left, "R"] & L^2(\Omega) \arrow[l, shift left, "S"] \arrow[r, "U"] & L^2(\mathbb{T}) \arrow[u, shift left, "I"]
\end{tikzcd}$$

where $\mathbf{\Gamma} = U \circ U^*$ is covariance operator of RX, supposed nuclear, we construct KKL expansion of RX as follows :

$$RX(\omega, t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\psi_k(\omega) f_k(t) \tag{3.1}$$

The sequence $(\lambda_k)_k$ forms the spectra of $\mathbf{\Gamma}$ arranged in decreasing order such that each eigenvalue is repeated according to its multiplicity,

$(f_k)_k$ constitutes an orthonormal system in $L^2(\mathbb{T})$ consisting of eigenfunctions of $\mathbf{\Gamma}$ corresponding to eigenvalues $(\lambda_k)_k$ (the expansion is unique up to sign if all eigenvalues are simple),

$(\psi_k)_k$ constitutes an orthonormal system in $L^2(\Omega)$ as eigenfunctions of $U^* \circ U$ associated to the same eigenvalues $(\lambda_k)_k$.

The n^{th} best approximation of RX is written

$$(RX)_n(\omega, t) = \sum_{k=1}^{n} \sqrt{\lambda_k}\psi_k(\omega) f_k(t). \tag{3.2}$$

In the approximation (3.2), when $n = 1$, the inverse transformation gives

$$S(RX)_1(\omega, t) = S(\sqrt{\lambda_1}\psi_1(\omega) f_1(t)) \tag{3.3}$$

so if $X \equiv X_1$

$$X(\omega, t) = \sqrt{\tilde{\lambda_1}}\tilde{\psi_1}(\omega)\tilde{f_1}(t) \tag{3.4}$$

where $\tilde{\lambda_1} = \lambda_1^{\frac{2}{p}}$, $\tilde{\psi_1} = \psi_1^{\frac{2}{p}}$ and $\tilde{f_1} = f_1^{\frac{2}{p}}$.

But when $n > 1$, as S is non-linear

$$S(\sum_{k=1}^{n} \sqrt{\lambda_k}\psi_k(\omega) f_k(t)) \neq \sum_{k=1}^{n} S(\sqrt{\lambda_k}\psi_k(\omega) f_k(t))$$

If X_1 is a good approximation of X, assuming S is differentiable in a neighborhood of X_1, a first-order Taylor expansion yields:

$$S(\sum_{k=1}^{n} \sqrt{\lambda_k}\psi_k f_k) = S(X_1 + h_X),$$

where h_X is small in L^2, implying

$$S(RX) = S(X_1 + h_X).$$

with

$$X_1 = \sqrt{\lambda_1}\,\psi_1 f_1, \quad h_X = \sum_{k=2}^{n} \sqrt{\lambda_k}\,\psi_k f_k.$$

(Note: the ψ_k are eigenfunctions in $L^2(\Omega)$ of the operator RX.) A Taylor expansion is then applied:

$$S(X_1 + h_X) = S(X_1) + S'(X_1)h_X + o(h_X).$$

Assuming $S(x) = x_+^{2/p} - x_-^{2/p}$ in the scalar case, one has

$$S'(x) = \frac{2}{p} \cdot \mathrm{sign}(x) \cdot |x|^{\frac{2-p}{p}}.$$

Under the assumption that $X_1 > 0$, the paper then writes the following :

$$S(X_1 + h_X) = S(X_1) + \frac{2}{p} X_1^{2/p-1} h + o(h_X),$$

$$X(\omega,t) = \lambda_1^{\frac{1}{p}} \psi_1(\omega)^{\frac{2}{p}} f_1^{\frac{2}{p}}(t) + \sum_{k=2}^{n} \frac{2}{p} \sqrt{\lambda_k} \lambda_1^{\frac{2-p}{p}} \psi_k(\omega) \psi_1^{\frac{2-p}{p}} f_k(t) f_1^{\frac{2-p}{p}}(t) + o(h_X)$$

However, the final expansion is

$$X(\omega,t) \approx \sum_k \sqrt{\tilde{\lambda}_k}\,\tilde{\psi}_k(\omega) \tilde{f}_k(t), \tag{3.5}$$

where

$$\tilde{\lambda}_1 = \lambda_1^{2/p}, \quad \tilde{\psi}_1 = \psi_1^{2/p}, \quad \tilde{f}_1 = f_1^{2/p},$$

and for $k > 1$,

$$\tilde{\lambda}_k = \frac{4}{p^2} \lambda_k^{2/p} \lambda_1^{(2-p)/p}, \quad \tilde{\psi}_k = \psi_k^{2/p} \psi_1^{(2-p)/p}, \quad \tilde{f}_k = f_k^{2/p} f_1^{(2-p)/p}.$$

(p here refers to the integrability parameter from the L^p space.)
Remark. If $p = 2$, one recovers the exact Kosambi–Karhunen–Loève (KKL) expansion, since the nonlinear transformation S reduces to the identity map.

However, this technique becomes more difficult if we develop Taylor expansion beyond the first order.

3.2. **Partial isometry.** It is obvious that R (resp. S) is not an isometry. Furthermore, to obtain some partial isometry between $L^2(\Omega)$ (containing the range of R) and $L^p(\Omega)$ (space containing variables with infinite variance), we must find subset $\mathcal{I} \subset L^p(\Omega) \times L^2(\Omega)$ such that

$$\mathcal{I} = \{(x, Rx) \in L^p(\Omega) \times L^2(\Omega);\ \|x\|_{L^p(\Omega)} = \|Rx\|_{L^2(\Omega)}.\}$$

Sets which verify this condition are only unit spheres of both the spaces $L^p(\Omega)$ and $L^2(\Omega)$. The norm is also preserved at 0, but this is not interesting.

Omitting $\{0\}$, vectors on spheres of both these spaces are constructed by random variables $\dfrac{X(.,\,t)}{\|X(.,\,t)\|_{L^p(\Omega)}}$ and $\dfrac{RX(.,\,t)}{\|RX(.,\,t)\|_{L^2(\Omega)}}$ which are reduced variables of both the spaces.

So we can affirm that KKL expansion on X and its transformation RX are equivalent, up the isometry, if X is reduced.

4. Application

The Lévy α-stable process is a stochastic process characterized by heavy-tailed distributions and self-similarity. It's characterized by four parameters $(\alpha, \mu, \beta, \sigma)$ where :

$\alpha \in (0,\,2]$ is the stability index governing tail heaviness (smaller α implies heavier tails). $\mu \in \mathbb{R}$ is the location parameter; if $\alpha > 1$ μ indicates the mean.

$\beta \in [-1,\,1]$ is the skewness parameter ($\beta = 0$ ensures that the process is symmetric);

σ is the scale parameter controlling some dispersion.

If $X \sim S(\alpha, \mu, \beta, \sigma)$ and $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables following the same law as that X, then

$$S_n = \sum_{i=1}^{n} X_i \quad \sim \quad S(\alpha, n\mu, \beta, n^{\frac{1}{\alpha}}\sigma) \qquad \text{(Summation Stability)}$$

$$\forall \lambda > 0, \quad \lambda X \sim S(\alpha, \lambda\mu, \beta, \lambda^{\frac{1}{\alpha}}\sigma) \qquad \text{(Scaling)}$$

For all $t \in \mathbb{T}$ and for all constant $c > 0$,

$$X(ct) \stackrel{d}{=} c^{\frac{1}{\alpha}} X(t) \quad \text{where } \stackrel{d}{=} \text{ denotes equality in distribution} \quad \text{(Self-similarity)}$$

Referring to Taqqu's works, particularly in Samorodnitsky and Taqqu 1994 , we have the following property

$$\begin{cases} E|X|^m < \infty, & \text{if } m < \alpha, \\ E|X|^m = \infty, & \text{if } m > \alpha. \end{cases}$$

In the sequel, we choose $1 < \alpha < 2$. To illustrate our methodology, we generated a sample of symmetric self-similar Lévy α-stable process trajectories. Specifically, we simulated 700 trajectories, each comprising 1001 discrete time points ($t \in [0,\,1]$ with uniform spacing). Each trajectory represents the cumulative sum of increments drawn from a symmetric α-stable distribution.

Taking $\epsilon > 0$ and $p = \alpha - \epsilon \geq 1$, $X(t) \in L^p(\Omega)$ Samorodnitsky and Taqqu (1994). and has infinite variance.

The KKL expansion is applied on symmetric Lévy processes, we use non-linear transformation approach transformation of Lévy α-stable process $X \in L^{p\times 2}(\Omega \times \mathbb{T})$ to $RX \in L^{2\times 2}(\Omega \times \mathbb{T})$;with $\alpha = 1.51$ The KKL expansion of the α stable process $1 < \alpha < 2$, which infinite variance for all $t \in \mathbb{T}$ ($X \in L^{p\times 2}(\Omega \times \mathbb{T})$ $1 < p < \alpha < 2$) verifies

For all $p \in [1,\,\alpha)$, $RX \in L^{2\times 2}(\Omega \times \mathbb{T})$. Therefore, the stochastic process RX admits a classical KKL expansion for all $p \in [1,\,\alpha)$ Samorodnitsky and Taqqu

(1994). By way of example, we display graphical illustrations relative to the first two principal axes for three specific values of p, p within the range $[1, \alpha)$.

As $\alpha = 1.51$ in this simulation study, we consider $RX \in L^{2\times 2}(\Omega \times \mathbb{T})$ with $RX = X_+^{\frac{p}{2}} - X_-^{\frac{p}{2}}$ for $p = \alpha - 0.01$; $p = \alpha - 0.21$ and $p = \alpha - 0.41$.

On the first bi-dimensional subspace of the KKL expansion for simulated realizations, we obtain these graphical illustrations.

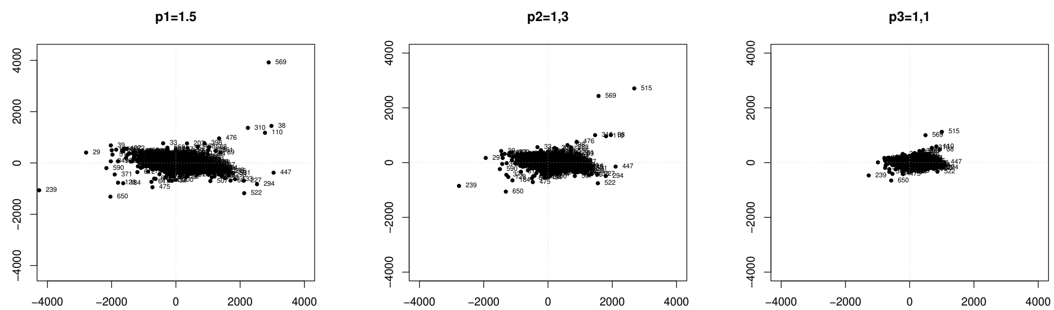

FIGURE 1. The first bi-dimensional subspaces of the KKL expansion of RX for $p = 1.5$; 1.3 and 1.1

The point cloud of initial trajectories represented in different L^p spaces ($p < \alpha$) has different transformations in L^2 with variances decreasing with p; however, the positions maintain the same form up to a homothetic transformation as illustrated in Figure 1.

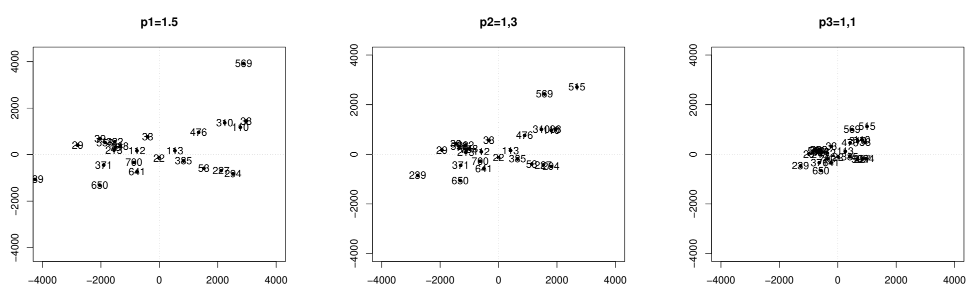

FIGURE 2. Some trajectories projected

For illustration purposes, by taking a few transformed trajectories (22, 29, 33, 53, 113, 239, 476, ..., 569, ..., 650,700), we observe that the global phenomenon can be treated on a case-by-case basis. Thus, by adjusting with respect to a homothetic factor linked to the variance, we notice that we obtain an equivalence of results in the first two principal components. This equivalence has indeed been demonstrated in the isometric approach between unit spheres of the different representations. As isometry is verified on both reduced transformed trajectories in L^2 and all their representations in L^p ($p \in [1, \alpha)$).

When we have a mixture of α-stable processes, one uses $p \in [1,\ min(\alpha))$.

5. Conclusion

For Gaussian processes, the KKL decomposition facilitates inferential procedures through the diagonalization of the covariance structure, thereby yielding independent components amenable to separate analysis. In the case of non-Gaussian processes possessing finite variance for all t T, the KKKL decomposition furnishes an optimal orthonormal basis for the representation of stochastic processes, with particular relevance to functional data analysis. This decomposition enables the expansion of random functions into mutually uncorrelated components (not necessarily independant). When the underlying process exhibits infinite variance, as exemplified by α-stable Lévy processes, we establish a non-linear generalization of the KKL expansion. This methodology employs a judiciously constructed homeomorphism that exploits the inherent Hilbertian structure, thereby facilitating the construction of a non-linear transformation of the canonical Hilbertian KKL expansion. We demonstrate that this transformation preserves isometric properties on the unit spheres of the function spaces $L^p(\Omega)$ and $L^2(\Omega)$.

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