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EXTENDED CENTROID OF HYPERRINGS

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Abstract. In this paper, the notation of extended centroid is applied to hyperring. We show that the extended centroid C of a hyperring is a hyperfield. Also, we show that if a, $b \in S$ such that $axb = bxa$ for all $x \in R$, then there exists $q \in C$ such that $qa = b$ where R is a prime hyperring and S be the central closure of R. Finally we give some relations between the extended centroid and derivation in hyperring.

1. Introduction

The concept of hyperstructures was introduced by Marty in 1934 at the 8th Congress of the Scandinavian Mathematicians [11] and has been studied in the following decades by many mathematicians. The theory of hypergroups appears in [3]. Krasner [9] introduced the notion of hyperrings and hyperfields. The basic results of hyperstructures and hyperrings are found in [4] and [5]. Also, we refer the reader to see [6, 7, 8, 12, 13, 14, 15, 16, 17, 18]. A well-known type of a hyperring is called the Krasner hyperring. Krasner hyperring is an essential ring with approximately modified axioms in which addition is hyperoperation, while the multiplication is an operation. This type of hyperrings has been studied by a variety of authors. The notion of Martindale rings of quotients, jointly with the notion of the extended centroid, play an important role in the study of prime rings satisfying a generalized polynomial identity, see [10]. The study of derivations in rings got interested after Posner [19], who give remarkable results on derivations of prime rings. Then the notion of derivations has been developed by many authors. There exist many results concerning the relationship between the quotient ring and the existence of certain specific types of the derivation of the ring. In [1], Asokkumar studied derivations in Krasner hyperrings and gives examples. In this paper, the notation of extended centroid is applied to hyperring. We show that the extended centroid C of a hyperring is a hyperfield. Also, we show that if a, $b \in S$ such that $axb = bxa$ for all $x \in R$, then there exists $q \in C$ such that $qa = b$ where R is a prime hyperring and S be the central closure of R. Finally we give some relations between the extended centroid and derivation in hyperring.

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2. Preliminaries

A mapping $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the set of nonempty subsets of H. An algebraic system $(H, \circ)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b\ ,\ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hyperoperation "$\circ$" is called associative if $a\circ(b\circ c) = (a\circ b)\circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u\in b\circ c} a \circ u = \bigcup_{v\in a\circ b} v \circ c.$$

A hypergroupoid with the associative hyperoperation is called a semihypergroup.

A hypergroupoid $(H, \circ)$ is a quasihypergroup, whenever $a \circ H = H = H \circ a$ for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called a hypergroup.

Definition 2.1. [5] A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:
(1) $(R, +)$ is a canonical hypergroup, i.e.,
(i) $(x + y) + z = x + (y + z)$ for every $x, y, z \in R$,
(ii) $x + y = y + x$, for every $x, y \in R$,
(iii) For all $x \in R$, there exists $0 \in R$ such that $0 + x = \{x\}$,
(iv) There exists an unique element denoted by $-x \in R$ for every $x \in R$ such that $0 \in x + (-x)$,
(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for every $x, y, z \in R$,
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e.,
(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for every $x, y, z \in R$,
(ii) For all $x \in R$, $x \cdot 0 = 0 \cdot x = 0$,
(3) The multiplication is distributive with respect to the hyperoperation $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

Since $-A$ is the defined by the set $\{-a \mid a \in A\}$, we have $-(-x) = x$ and $-(x + y) = -x - y$. In definition, for simplicity of notations we write sometimes xy instead of $x \cdot y$ and in (iii), $0 + x = x$ instead of $0 + x = \{x\}$.

If there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in A$ in a hyperring R, then the element 1 is called the identity element of the hyperring R. The hyperring R is called a commutative hyperring, if $ab = ba$ for every $a, b \in R$.

A hyperring R is called a hyperdomain if R does not have zero divisors. In other words, for $x, y \in R$ if $xy = 0$ then either $x = 0$ or $y = 0$.

A Krasner hyperring is called a Krasner hyperfield, if $(R\setminus \{0\}, \cdot)$ is a group.

Throughout this paper, by a hyperring we mean that Krasner hyperring.

Let R be a hyperring. A nonempty subset S of R is called a subhyperring of R, if $x - y \subseteq S$ and $xy \in S$ for all $x, y \in S$.

A subhyperring I of a hyperring R is a left (resp. right) hyperideal of R if $ra \in I$ (resp. $ar \in I$) for all $r \in R$, $a \in I$. A hyperideal of R is both a left and a right hyperideal.

Lemma 2.2. [5] *A nonempty subset A of a hyperrring R is a left (right) hyperideal if and only if*
(1) $a - b \subseteq A$, for all a, $b \in A$,
(2) $ra \in A$ ($ar \in A$) for all $a \in A$, $r \in R$.

Let A and B be nonempty subsets of a hyperring R

$$A + B = \{x \mid x \in a + b \text{ for some } a \in A,\, b \in B\}$$

and

$$AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i bi,\, a_i \in A,\, b_i \in B,\, n \in \mathbb{Z}^+ \right\}.$$

If A and B are hyperideals of R, then $A + B$ and AB are also hyperideals of R.

A hyperideal P of a hyperring R is called prime hyperideal, if for hyperideals I and J of R, satisfying $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. A hyperring R is called a prime hyperring if $aRb = 0$ for all $a, b \in R$ implies $a = 0$ or $b = 0$.

Example 2.3. Let $R = \{0, x, y\}$ with the hyperoperation and the multiplication given in the following tables:

$+$	0	x	y
0	0	x	y
x	x	x	R
y	y	R	y

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	y	x

Then, R is a prime hyperring.

Definition 2.4. [5] Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a good (strong) homomorphism if for all $a, b \in R_1$,

$$\varphi(a + b) = \varphi(a) + \varphi(b),\ \varphi(ab) = \varphi(a)\varphi(b) \text{ and } \varphi(0) = 0.$$

A good homomorphism φ is an isomorphism if φ is one to one and onto. If there exists isomorphism between hyperrings R_1 and R_2, we write $R_1 \cong R_2$.

Corollary 2.5. *Let φ be a good homomorphism from R_1 into R_2. Then φ is one to one if and only if* $\ker \varphi = \{0\}$.

Let R be a hyperring. A canonical hypergroup $(M, +)$ together with the map $\cdot : R \times M \to M$ is called a left hypermodule over R if for all $r_1, r_2 \in R$, $m_1, m_2 \in M$ the following axioms hold:

(1) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,
(2) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,
(3) $(r_1r_2)m_1 = r_1(r_2m_1)$,
(4) $0_Rm_1 = 0_M$.

A subhypermodule N of M is a subhypergroup of M which is closed under multiplication by elements of R.

Definition 2.6. Let M and N be two R-hypermodules. A function $f : M \to N$ that satisfies the conditions:
(1) $f(x + y) \subseteq f(x) + f(y)$,
(2) $f(xr) = f(x)r$, for all $r \in R$ and all $x, y \in M$,
is called to be an right R-homomorphism from M into N.

In definition, if the equality holds, then f is called a good (strong) right R-homomorphism.

Definition 2.7. Let R be a hyperring. A map $d : R \to R$ is said to be a derivation of R if d satisfies:

$$\begin{aligned} (i)\ d(x + y) &\subseteq d(x) + d(y) \\ (ii)\ d(xy) &\in d(x)y + xd(y) \text{ for all } x, y \in R. \end{aligned}$$

If the map d is such that $d(x + y) = d(x) + d(y)$ and $d(xy) \in d(x)y + xd(y)$ for all $x, y \in R$, then d is called a strong derivation of R.

Proposition 2.8. [1] *Let d be a derivation of a prime hyperring R and $a \in R$ such that $ad(u) = 0$ (or $d(u)a = 0$) for all $u \in R$. Then either $a = 0$ or $d = 0$.*

3. Extended Centroid

Let R is a prime hyperring and $H = H(R)$ be the set of all nonzero hyperideal of R by

$$H = H(R) = \{U \mid \{0_R\} \neq U \text{ is hyperideal of } R\}.$$

Let $F = \{f_U \mid f : U \to R \text{ is a good right } R\text{-homomorphism and } U \in H\}$. The relation $\approx$ is defined by:

$$f_U \approx g_V \Leftrightarrow \text{there exists } K \in H \text{ and } K \subseteq U \cap V \text{ such that } f = g \text{ on } K.$$

Let $U, V \in H$. Then there exists a $\{0_R\} \neq K \in H$, since R is a prime hyperring. As $U \subseteq U \cap U$ and $f = f$ on U, $f_U \approx f_U$ for all $U \in H$. Let $f_U, g_V \in F$ and we assume that $f_U \approx g_V$. Hence there exists $K \in H$ and $K \subseteq U \cap V$ such that $f = g$ on K. Thus, $K \in H$ and $K \subseteq V \cap U$ such that $g = f$ on K. That is, $g_V \approx f_U$. For $f_U, g_V, h_W \in F$, suppose that $f_U \approx g_V$ and $g_V \approx h_W$. Hence there exist $K_1, K_2 \in H$ and $K_1 \subseteq U \cap V$ and $K_2 \subseteq V \cap W$ such that $f = g$ on K_1 and $g = h$ on K_2. Then, $\{0_R\} \neq K = K_1 \cap K_2 \subseteq (U \cap V) \cap (V \cap W) \subseteq U \cap W$ and $f = h$ on K, i.e., $f_U \approx h_W$. As a result, we have $\approx$ an equivalence relation on F. We denote the equivalence class by $\bar{f}$ and denote by Q_r set of all equivalence classes. We define a hyperaddition "+" on Q_r as follows:

$$\bar{f}_U + \bar{g}_V := \overline{f + g}_{U \cap V}$$

for all $\bar{f}_U$, $\bar{g}_V \in Q_r$,where $f + g : U \cap V \to R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1 (\in F) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2 (\in F) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K = K_1 \cap K_2$ and so $K \in D$. For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) =$

$\bigcup\{t(x) \mid t(x) \in f_1(x) + g_1(x)\} = \bigcup\{t(x) \in f_2(x) + g_2(x)\} = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ on K. Therefore $f_1 + g_{1_{U_1 \cap V_1}} \approx f_2 + g_{2_{U_2 \cap V_2}}$, which means that the addition in Q_r is well-defined.

Now we will prove that Q_r is a canonical hypergroup. Let $\bar{f}_U$, $\bar{g}_V$, $\bar{h}_W$ be elements of Q_r. Since $U \cap (V \cap W) = (U \cap V) \cap W$, for all $x \in U \cap (V \cap W)$

$$\begin{aligned}
[(f+g)+h](x) = (f+g)(x) + h(x) &= \bigcup_{t(x) \in (f+g)(x)} t(x) + h(x) \\
&= \bigcup_{t(x) \in (f+g)(x)} \{k(x) \mid k(x) \in t(x) + h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in (f(x) + g(x)) + h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in f(x) + (g(x) + h(x))\} \\
&= \bigcup_{p(x) \in g(x)+h(x)} \{k(x) \mid k(x) \in f(x) + p(x)\} \\
&= \bigcup_{p(x) \in (g+h)(x)} f(x) + p(x) = f(x) + (g+h)(x) \\
&= [f + (g+h)](x)\ .
\end{aligned}$$

Hence $(f+g)+h = f+(g+h)$ on $U \cap (V \cap W)$. That is $\left(\bar{f}_U + \bar{g}_V\right) + \bar{h}_W = \bar{f}_U + \left(\bar{g}_V + \bar{h}_W\right)$. One can easily check that $\bar{f}_U + \bar{g}_V = \bar{g}_V + \bar{f}_U$. Taking $\bar{\theta}_R \in Q_r$ where $\theta : R \to R$, $x \mapsto 0$ for all $x \in R$. Since $U \subseteq U \cap R$, $(\theta + f)(x) = \theta(x) + f(x) = 0 + f(x) = f(x)$ for all $x \in U$. Then we have $\bar{\theta}_R + \bar{f}_U = \bar{f}_U$ and similarly $\bar{f}_U + \bar{\theta}_R = \bar{f}_U$ for all $\bar{f}_U \in Q_r$. Hence $\bar{\theta}_R$ is the additive identity in Q_r. Let $-\bar{f}_U \in Q_r$, where $-f : U \to R$, $x \mapsto -f(x) = (-f)(x)$ for all $x \in U$. Since $-f(x)$ is the unique inverse of $f(x)$ in R, we have $\theta(x) \in f(x) - f(x) = f(x) + (-f)(x)$ for all $x \in U$. So $\bar{\theta}_R \in \bar{f}_U + (-\bar{f}_U)$. Finally, let $\bar{f}_U$, $\bar{g}_V$, $\bar{h}_W$ be elements of Q_r and $\bar{h}_W \in \bar{f}_U + \bar{g}_V$. So there exists a $f_1 \in \bar{f}_U$ and a $g_1 \in \bar{g}_V$ such that $h = f_1 + g_1$. For any $x \in K(\in F) \subseteq U \cap V$, we get $h(x) = (f_1 + g_1)(x) = f_1(x) + g_1(x) \subseteq f(x) + g(x)$. Since R is a hyperring, $h(x) \in f(x) + g(x)$ implies $g(x) \in -f(x) + h(x)$ and $f(x) \in h(x) - g(x)$. Thus we have $g(x) \in (-f+h)(x)$ and $f(x) \in (h-g)(x)$. That is, $\bar{g}_V \in -\bar{f}_U + \bar{h}_W$ and $\bar{f}_U \in \bar{h}_W - \bar{g}_V$. Therefore $(Q_r, +)$ a canonical hypergroup.

Now we define a multiplication "." on Q_r as follows: for all $\bar{f}_U, \bar{g}_V \in Q_r$

$$\bar{f}_U \bar{g}_V := \overline{fg}_{VU}$$

where $fg : VU \to R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1(\in F) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and

$\exists K_2(\in F) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Also $V_1U_1 \cap V_2U_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2)$ and there exists $\{0_R\} \neq K \in H$ such that $K \subseteq V_1U_1 \cap V_2U_2$. For any $x \in K$, $x \in V_1U_1 \cap V_2U_2$. Thus $x \in V_1U_1$ and $x \in V_2U_2$. Then $x = \sum\limits_{finite} a_ib_i$, $a_i \in V_1 \cap V_2$, $b_i \in U_1 \cap U_2$. Therefore,

$$\begin{aligned}
(f_1g_1)(x) = f_1(g_1(x)) &= f_1\left(g_1\left(\sum_{finite} a_ib_i\right)\right) \\
&= f_1\left(\left(\sum_{finite} g_1(a_i)b_i\right)\right) = f_1\left(\left(\sum_{finite} g_2(a_i)b_i\right)\right) \\
&= f_2\left(\left(\sum_{finite} g_2(a_i)b_i\right)\right) = f_2\left(g_2\left(\sum_{finite} a_ib_i\right)\right) \\
&= f_2(g_2(x)) = (f_2g_2)(x).
\end{aligned}$$

Thus $f_1g_1 = f_2g_2$ on K. Hence "." is well-defined. Now we will prove that $(Q_r, .)$ is a semigroup having zero as a bilaterally element. Let $\bar{f}_U, \bar{g}_V, \bar{h}_W \in Q_r$. Since $W(VU) = (WV)U$, for all $x \in W(VU)$,

$$\begin{aligned}
[(fg)h](x) = (fg)(h(x)) &= f(g(h(x))) \\
&= f((gh)(x)) = (f(gh))(x).
\end{aligned}$$

Then $(fg)h = f(gh)$ on $W(VU)$. Thus, $\left(\bar{f}_U\bar{g}_V\right)\bar{h}_W = \bar{f}_U\left(\bar{g}_V\bar{h}_W\right)$. Now we prove that $\bar{f}_U\ \bar{\theta}_R = \bar{\theta}_R = \bar{\theta}_R\bar{f}_U$ for all $\bar{f}_U \in Q_r$. Since $RU \subseteq RU \cap R$ and $f\theta = \theta$ on RU, we get $\bar{f}_U\bar{\theta}_R = \bar{\theta}_R$. Similarly $\bar{\theta}_R\bar{f}_U = \bar{\theta}_R$.

Let $\bar{f}_U, \bar{g}_V, \bar{h}_W$ be elements of Q_r. Since $(V \cap W)U \subseteq VU \cap WU$, we get for all $x \in (V \cap W)U$,

$$\begin{aligned}
[f(g+h)](x) = f((g+h)(x)) &= f(g(x) + h(x)) \\
&= f(g(x)) + f(h(x)) = (fg + fh)(x).
\end{aligned}$$

Then $f(g+h) = fg+fh$ on $(V \cap W)U$. That is, $\bar{f}_U\left(\bar{g}_V + \bar{h}_W\right) = \bar{f}_U\bar{g}_V + \bar{f}_U\bar{h}_W$. Similarly, $\left(\bar{f}_U + \bar{g}_V\right)\bar{h}_W = \bar{f}_U\bar{h}_W + \bar{g}_V\bar{h}_W$.

Therefore, $(Q_r, +, .)$ be a Krasner hyperring.

Let $\bar{1}_R \in Q_r$, by $1 : R \to R$, $x \mapsto x$ for all $x \in R$. Since $RU \subseteq U$, we have $(f1)(x) = f(1(x)) = f(x)$ and $(1f)(x) = 1(f(x)) = f(x)$ for all $x \in RU$ and $\bar{f}_U \in Q_r$. Then, $\bar{f}_U\bar{1}_R = \bar{1}_R\bar{f}_U = \bar{f}_U$. That is, $\bar{1}_R$ is identity element of Q_r.

Theorem 3.1. *Let R be prime hyperring. Then R may be embedded in Q_r as a subhyperring.*

Proof. Let $a \in R$. We consider the mapping $\lambda_a : R \to R$, $\lambda_a(r) = ar$ for all $r \in R$. Clearly, λ_a is a good right R -homomorphism. Thus, λ_a defines $\overline{\lambda}_{a_R}$of Q_r. Now, we define $\lambda : R \to Q_r$ by $\lambda(a) = \overline{\lambda}_{a_R}$ for all $a \in R$. Then λ is well-defined and λ is a good homomorphism. Also,

$$\begin{aligned}
\ker \lambda &= \left\{ a \in R \mid \lambda(a) = \bar{\theta}_R \right\} = \left\{ a \in R \mid \overline{\lambda}_{a_R} = \bar{\theta}_R \right\} \\
&= \{ a \in R \mid \lambda_{a_R} \approx \theta_R \} = \{ a \in R \mid aU = 0 \} \\
&= \{ a \in R \mid a = 0 \} = \{ 0 \} .
\end{aligned}$$

That is, λ is injective. Thus, R is a subhyperring of Q_r. □

We call Q_r the right quotient hyperring of R. We use q instead of $\overline{q}_U \in Q_r$ for purpose of convenience.

Lemma 3.2. *Let R be a prime hyperring. For each nonzero $q \in Q_r$, there is nonzero hyperideal U of R such that $q(U) \subseteq R$.*

Proof. It is obvious. □

Lemma 3.3. *Let R is a prime hyperring. Then the quotient hyperring Q_r of R is a prime hyperring.*

Proof. Let $p, q \in Q_r$ and $pQ_rq = \theta$. If $p \neq \theta$ and $q \neq \theta$, then there exist nonzero hyperideals U and V of R such that $p(U) \subseteq R$, $q(V) \subseteq R$. Since $p \neq \theta$and $q \neq \theta$, there exist nonzero elements $u \in U$ and $v \in V$ such that $p(u) \neq 0_R \neq q(v)$. Since R is a subhyperring of Q_r,

$$p(u)Rq(v) \subseteq p(u)Q_rq(v) = \{0_R\} .$$

Hence, $p(u)Rq(v) = \{0_R\}$. Since R is a prime hyperring, we have a contradiction. Then, $p = \theta$ or $q = \theta$. Thus Q_r is a prime hyperring. □

Definition 3.4. The set

$$C := \{ g \in Q_r \mid \ gf = fg \text{ for all } f \in Q_r \}$$

is called the extended centroid of a hyperring R.

Obviously, C is a subhyperring of Q_r. Let $\theta \neq c \in C$. Suppose that $cf = \theta$ for any $f \in C$. Thus, $gcf = \theta$ for any $g \in Q_r$. Hence $cgf = \theta$ and so $cQ_rf = \theta$. Since $c \neq \theta$, we have $f = \theta$. Therefore, C is a hyperdomain.

Theorem 3.5. *C is a hyperfield.*

Proof. Let $\theta \neq c \in C$. Since $c \in Q_r$, there exists $\{0\} \neq U \in H$ such that $cU \subseteq R$. Since R is a prime hyperring, $cU \neq \{0\}$. Let $\{0\} \neq V = cU$. Then V is a hyperideal of R. Also, $d : V \to R$, $ca \mapsto a$ is a good R-homomorphism. Then, $d \in Q_r$. Since $dc(a) = a = 1(a)$ for all $a \in U$, $dc = 1$. Thus, $\theta \neq c \in C$ is inverse. Moreover, $1 \in Q_r$ is the multiplicative identity in C. Therefore, C be a hyperfield. □

Let $S = RC$, a subhyperring of Q_r. We will call S the central closure of R. One can easily show that S is prime.

Theorem 3.6. *Let R is a prime hyperring and S be the central closure of R. If $a, b \in S$ such that $axb = bxa$ for all $x \in R$, then there exists $q \in C$ such that $qa = b$.*

Proof. We may suppose that $a \neq \theta$, $b \neq \theta$. Let U be a nonzero hyperideal of R such that $aU \subseteq R$ and $bU \subseteq R$ and $V = UaU$. We define a mapping $f : V \to R$, $f(\sum_i x_i a y_i) = \sum_i x_i b y_i$, x_i, $y_i \in U$. Suppose $\sum_i x_i a y_i = 0$. Then

$$\begin{aligned} 0 &= br\sum_i x_i a y_i = \sum_i (br)x_i a y_i = \sum_i b(rx_i)ay_i \\ &= \sum_i a(rx_i)by_i = \sum_i (ar)x_i b y_i = ar\sum_i x_i b y_i. \end{aligned}$$

Thus $aUR(\sum_i x_i b y_i) \subseteq aR(\sum_i x_i b y_i) = 0$, i.e., $(aU)R(\sum_i x_i b y_i) = 0$ and so $\sum_i x_i b y_i = 0$ since R is prime. This show that f is well-defined. Clearly, f is a good right R -homomorphism. Let q is the element of Q_r determined by f and let p be any element of Q_r, with $pK \subseteq R$ for some nonzero hyperideal K of R. For $x, y \in U$ and $z \in K$, $(qp)(zxay) = q((pz)xay) = (pz)xby = p(zxby) = p(q(zxay)) = (pq)(zxay)$. Since $KU \subseteq K \cap U$ and $pq = qp$ on KU, we have $qp = pq$ for all $p \in Q_r$. This implies that $q \in C$. In particular,

$$0 \in xby - xby = q(xay) - xby = xqay - xby = x(qa - b)y.$$

Then $U(qa - b)U = 0$. Since $V = UaU$, we get

$$V(qa - b)V = UaU(qa - b)UaU = 0.$$

Since R is prime hyperring, we obtain $qa = b$. □

Proposition 3.7. *Let d_1, d_2 two non-zero derivation of a prime hyperring R. If*

$$d_1(x)d_2(y) = d_2(x)d_1(y) \text{ for all } x, y \in R \tag{3.1}$$

then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ for all $x \in R$.

Proof. For $x, y \in R$ we have

$$0 \in d_1(x)d_2(y) - d_2(x)d_1(y). \tag{3.2}$$

Replacing x by xz in (3.2), we get $0 \in d_1(x)zd_2(y) - d_2(x)zd_1(y)$ for all $x, y \in R$. Thus,

$$d_1(x)zd_2(y) = d_2(x)zd_1(y) \text{ for all } x, y \in R. \tag{3.3}$$

Hence, there exists $\lambda(x) \in C$ such that $d_2(x) = \lambda(x)d_1(x)$ for all $x \in R$ by Theorem 3.6. Thus by (3.3),

$$0 \in (\lambda(x) - \lambda(y))d_1(x)zd_1(y). \tag{3.4}$$

So, there is $t \in \lambda(x) - \lambda(y)$ such that $0 = td_1(x)zd_1(y)$. Since R is a prime hyperring and $d_1 \neq 0$, we conclude by using Proposition 2.8 that $t = 0$. This means that $\lambda(x) = \lambda(y)$ for all $x, y \in R$. Therefore, we show that there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ for all $x \in R$. This completes the proof. □

Proposition 3.8. *Let R be a prime hyperring, d, g, h and f be non-zero derivations of R. Let $d(x)g(y) = h(x)f(y)$ for all $x,\ y \in R$. If $d \neq 0$ and $f \neq 0$, then there exists $\lambda \in C$ such that $g(x) = \lambda f(x)$ and $h(x) = \lambda d(x)$ for all $x \in R$.*

Proof. Suppose that $d(x)g(y) = h(x)f(y)$ for all $x,\ y \in R$ and $d \neq 0,\ f \neq 0$. Hence,

$$0 \in d(x)g(y) - h(x)f(y). \tag{3.4}$$

For every $z \in R$, replace y by yz in (3.4),

$$\begin{aligned} 0 &\in d(x)g(yz) - h(x)f(yz) \in d(x)(g(y)z + yg(z)) - h(x)(f(y)z + yf(z)) \\ &= (d(x)g(y) - h(x)f(y))z + d(x)yg(z) - h(x)yf(z). \end{aligned}$$

Thus,

$$0 \in d(x)yg(z) - h(x)yf(z) \tag{3.5}$$

for all $x, y, z \in R$. Replacing y by $yf(t)$ for all $t \in R$ in (3.5), we get

$$\begin{aligned} 0 &\in d(x)yf(t)g(z) - h(x)yf(t)f(z) = d(x)yf(t)g(z) - d(x)yg(t)f(z) \\ &= d(x)y(f(t)g(z) - g(t)f(z)). \end{aligned}$$

Since $d \neq 0$ and from Proposition 2.8, we have $f(t)g(z) = g(t)f(z)$ for all $z, t \in R$. Therefore, it follows from Proposition 3.7 that for all $z \in R$, $g(z) = \lambda f(z)$. Then,

$$0 \in d(x)y\lambda f(z) - h(x)yf(z) = (\lambda d(x) - h(x))yf(z).$$

Since $f \neq 0$, we have $h(x) = \lambda d(x)$ for all $x \in R$ by Proposition 2.8. □

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