Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 1 (2025) 330-337 https://doi.org/10.56947/gjom.v20i.2934

FIXED POINT CURVES FOR NON-EXPANSIVE MAPPINGS IN 2-BANACH SPACES

RIYAS P[1] and SHIJU GEORGE[2] and HARIKRISHNAN P.K.[3*]

Abstract. This paper explores the existence and properties of fixed point curves for non-expansive mappings in 2-Banach spaces. We introduce the concept of a fixed point curve associated with weakly inward non-expansive mappings and provide sufficient conditions for its existence in 2-Banach spaces. Furthermore, we establish an analogue of Browder's fixed point theorem within the context of 2-Banach spaces, thereby extending classical fixed point results to this broader framework.

1. Introduction

Fixed point theory is a fundamental area of nonlinear analysis with wide-ranging applications in optimization, differential equations, and dynamical systems [12]. The study of fixed points for non-expansive mappings has been extensively developed in Banach spaces, culminating in landmark results such as the Browder–Göhde–Kirk fixed point theorem [12]. In recent years, attention has shifted towards extending this theory to 2-Banach spaces, a generalization of classical Banach spaces characterized by a bivariate norm structure [5, 16, 1].

This paper focuses on fixed point curves associated with weakly inward non-expansive mappings in 2-Banach spaces. We investigate the existence of such curves and examine their convergence properties. Our findings contribute to the broader understanding of non-expansive mappings in generalized normed spaces and enhance the theoretical foundation for fixed point approximation methods [13, 14].

The concept of a 2-Banach space was first introduced by S. Gähler [5, 7, 8] and later developed by K. Iseki, a developing body of literature in this area. In 1988, Sam B. Nadler and K. Ushijima introduced the notion of fixed point curves in the context of linear interpolations of contraction mappings, employing the Banach contraction principle . Building upon this foundation, our recent work has focused on weakly inward contractions in 2-normed spaces, leading to the formulation of fixed point curves in that setting [13, 14]. The notion of accretive operators and the analogues of the Banach–Alaoglu theorem in linear 2-normed spaces was explored by Harikrishnan et al. [10].

Date: Received: April 30, 2025; Accepted: May 25, 2025.
2020 *Mathematics Subject Classification.* 41A65, 41A15.
Key words and phrases. Fixed points, 2-norm, 2-Banach spaces.

In this paper, we extend these ideas to the realm of weakly inward non-expansive mappings in 2-Banach spaces. We provide a detailed analysis of the associated fixed point structures, offering new insights into their behavior and further generalizing classical fixed point results.

2. Preliminaries

In [7], S. Gähler introduced the concept of a 2-normed space, which forms the foundational basis for our investigations.

Definition 2.1. [7] *Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real valued function on $X \times X$ satisfying the following conditions: For $x, y, z \in X$ and $\alpha \in \mathbb{F}$*

(1) $\|x, y\| = 0$ *if and only if x and y are linearly dependent*
(2) $\|x, y\| = \|y, x\|$
(3) $\|\alpha x, y\| = |\alpha| \|x, y\|$
(4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

Then $\|.,.\|$ is called a 2-norm *on X and the pair $(X, \|.,.\|)$ is called a linear* 2-normed space *or 2-normed space.*

Definition 2.2. [9] *Let X be a 2-normed space and C be a nonempty subset of X. Then a mapping $T : C \to X$ is said to be non-expansive if for each $x,\ y \in C$*

$$\|Tx - Ty, z\| \leq \|x - y, z\|, \quad \forall\, z \in X.$$

Definition 2.3. [13] *Let X be a 2-normed space and C be a nonempty subset of X. A sequence $\{x_n\}$ in C is a b-approximating fixed point sequence of $T : C \to X$ if*

$$\lim_{n\to\infty} ||x_n - Tx_n, b|| = 0 \quad ......(1)$$

and it becomes an approximating fixed point sequence of T if (1) *is true for all $b \in X$.*

Definition 2.4. [12] *Let K be a closed convex subset of a Banach space X and let $C \subset K$. Then a mapping $T : C \to X$ is said to be* inward mapping (respectively, weakly inward) *on C relative to K if $Tx \in I_K(x)$ (respectively,$Tx \in \overline{I_K(x)}$) for $x \in C$ where*

$$I_K(x) = \{(1-t)x + ty \colon y \in K \text{ and } t \geq 0\}.$$

and $\overline{I_K(x)}$ is a closure of the $I_K(x)$ in X

Lemma 2.5. [12] $\overline{I_K(x)}$ *is a closed convex set containing K for each $x \in K$*

Definition 2.6. *Let K be a closed convex set. A map $T \colon C \subseteq K \to X$ is said to be* generalized inward *on C relative to K if the following condition is satisfied*

$$d\,(Tx, K) < \|x - Tx\|,\ \forall\, x \in C \text{ with } Tx \notin K$$

where $d\,(y, K) = \inf\{\|y - u\| \colon u \in K\}$.

Theorem 2.7. [13] *Let K be a non empty closed convex subset of a 2-Banach space X and $T : K \to X$, a weakly inward contraction mapping. Then T has a unique fixed point in K.*

3. Fixed point curve for Weakly inward non expansive mapping

In this section, we explore the fixed point theory in the context of 2-Banach spaces, which generalizes classical Banach space theory to spaces endowed with a 2-norm. A key focus is the study of nonexpansive mappings, weakly inward mappings, and conditions that guarantee the existence of fixed points for such mappings.

We begin by defining the concept of weak convergence and b-weak convergence in 2-Banach spaces, followed by the notion of a demiclosed mapping, which plays a crucial role in establishing fixed point results. The 2-Opial condition, a significant property of 2-Banach spaces, is then introduced, which aids in the analysis of convergence behavior for sequences in these spaces.

The section also includes an analogue of Browder's theorem in the setting of 2-Banach spaces. Specifically, we establish that every nonexpansive mapping on a uniformly convex 2-Banach space has a fixed point, provided the space is nonempty, closed, convex, and bounded. Furthermore, we show that if the mapping $I - T$ is closed, the nonexpansive mapping T has a fixed point in the given subset.

Finally, we derive key results such as the demiclosedness of certain mappings, the approximate fixed point property, and the convergence of sequences under weakly inward nonexpansive mappings.

Theorem 3.1. *Let K be a non empty closed convex subset of a 2-Banach space X and let $T\colon K \to X$ be a weakly inward non expansive mapping. Then, for any $u \in K$ and $\lambda \in (0,1)$, there exists exactly one point $x_\lambda \in K$ such that*

$$x_\lambda = (1-\lambda)u + \lambda T x_\lambda.$$

If $diam(K) = \sup\{||x-y,z|| : x, y \in K, z \in X \text{ with } z \neq 0\} < \infty$, then $x_\lambda - Tx_\lambda \to 0$ as $\lambda \to 1$.

Proof. Let $\lambda \in (0,1)$ and $u \in K$. Define the mapping $T_\lambda : K \to X$ by

$$T_\lambda(x) = (1-\lambda)u + \lambda T(x).$$

Then for any $x, y \in K$ and for all $z \in X$, we have

$$\|T_\lambda(x) - T_\lambda(y), z\| = |\lambda| \cdot \|T(x) - T(y), z\| \leq \lambda \|x - y, z\|.$$

This shows that T_λ is a contraction with Lipschitz constant $\lambda < 1$. Hence, by the Banach contraction principle, there exists a unique fixed point $x_\lambda \in K$ such that

$$x_\lambda = T_\lambda(x_\lambda) = (1-\lambda)u + \lambda T(x_\lambda).$$

Furthermore, if $\mathrm{diam}(K) < \infty$, then

$$\|x_\lambda - T(x_\lambda), z\| = (1-\lambda)\|u - T(x_\lambda), z\| \leq (1-\lambda) \cdot \mathrm{diam}(K),$$

which tends to zero as $\lambda \to 1$.

□

'Theorem 3.1 shows that, even if T does not have a fixed point, the family $\{x_\lambda\}$ approximates a fixed point in the limit.

Corollary 3.2. *Let K be a non-empty closed convex and bounded subset of a 2-Banach space X and $T\colon K \to K$ a nonexpansive mapping. Then there exists a sequence $\{x_n\}$ in K such that for any $z \in X$,*

$$\lim_{n\to\infty} \|x_n - Tx_n, z\| = 0.$$

Proof. For each $n = 2, 3, \ldots$, define $\lambda = 1 - \frac{1}{n} \in (0, 1)$ and fix $y \in K$. Consider the mapping $T_n : K \to K$ given by

$$T_n(x) = \frac{1}{n}y + \left(1 - \frac{1}{n}\right)T(x).$$

This map is a contraction with Lipschitz constant $1 - \frac{1}{n}$. By the contraction mapping theorem, T_n has a unique fixed point $x_n \in K$ for each n. Moreover,

$$\lim_{n\to\infty} \|x_n - Tx_n, z\| = 0.$$

□

Theorem 3.3. *Let X be a 2-Banach space, and let $K \subset X$ be a nonempty, closed, and convex subset. Suppose $T_1, T_2\colon K \to X$ are two weakly inward non-expansive mappings. Then there exists a family of weakly inward nonexpansive maps*

$$\{\phi_t\colon K \to X \mid t \in [0, 1]\}$$

such that for every $t \in (0, 1)$ and for any $u \in K$, there exists a unique point $x_t \in K$ satisfying

$$x_t = (1 - t)u + (t - t^2)T_1(x_t) + t^2T_2(x_t).$$

Proof. Define the map $\phi\colon K \times [0, 1] \to X$ by

$$\phi(x, t) = (1 - t)T_1(x) + tT_2(x).$$

Then,

$$\begin{aligned}\|\phi(x, t) - \phi(y, t), z\| &= \|(1 - t)\left(T_1(x) - T_1(y)\right) + t\left(T_2(x) - T_2(y)\right), z\| \\ &\le (1 - t)\|T_1(x) - T_1(y), z\| + t\|T_2(x) - T_2(y), z\| \\ &\le (1 - t)\|x - y, z\| + t\|x - y, z\| \\ &= \|x - y, z\| \quad \text{for all } z \in X.\end{aligned}$$

Hence, ϕ is nonexpansive uniformly over $[0, 1]$.

Since T_1 and T_2 are weakly inward, we have $T_1(x), T_2(x) \in \overline{I_K(x)}$ for all $x \in K$. Then, by Lemma[2.5],

$$(1 - t)T_1(x) + tT_2(x) \in (1 - t)\overline{I_K(x)} + t\overline{I_K(x)} = \overline{I_K(x)}.$$

That is,

$$\phi(x, t) \in \overline{I_K(x)} \quad \text{for all } x \in K \text{ and } t \in [0, 1].$$

This shows that ϕ is weakly inward and nonexpansive on $[0, 1]$.

Now define $\phi_t\colon K \to X$ by

$$\phi_t(x) = \phi(x, t), \quad \text{for } t \in [0, 1].$$

Then for all $x, y \in K$, $z \in X$, and $t \in [0,1]$,

$$\|\phi_t(x) - \phi_t(y), z\| \leq \|x - y, z\|.$$

That is, for each $t \in [0,1]$, the map ϕ_t is weakly inward and nonexpansive on the closed convex subset $K \subset X$.

Therefore, by Theorem 3.1, for each $t \in (0,1)$ and $u \in K$, there exists a unique point $x_t \in K$ such that

$$\begin{aligned} x_t = (1-t)u + t\phi_t(x_t) &= (1-t)u + t[(1-t)T_1(x_t) + tT_2(x_t)] \\ &= (1-t)u + (t-t^2)T_1(x_t) + t^2T_2(x_t). \end{aligned}$$

□

Definition 3.4. ***(Fixed Point Curve)*** *Let T_1 and T_2 be two weakly inward nonexpansive mappings on a closed, convex, and bounded subset K of a 2-Banach space X. Let $F_W(T_1, T_2)$ denote the set of all solutions of the equation*

$$x = (1-t)u + (t-t^2)T_1(x) + t^2T_2(x).$$

for $t \in (0,1)$ and $u \in K$. Then the map

$$G\colon (0,1) \to F_W(T_1, T_2), \quad G(t) = x_t,$$

is called the fixed point curve *associated with T_1 and T_2.*

Example 3.5. *Consider the 2-Banach space $(\mathbb{R}^2, \|\cdot,\cdot\|)$, where the standard 2-norm is defined by*

$$\|x, y\| = |x_1y_2 - x_2y_1| \quad \textit{for } x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2.$$

Define the mappings $T_1, T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_1(x,y) = \left(\frac{x}{2}, y\right) \quad \textit{and} \quad T_2(x,y) = \left(x, \frac{y}{2}\right).$$

Both T_1 and T_2 are nonexpansive mappings on $\mathbb{R}^2$, and for each $t \in (0,1)$, the convex combination

$$\phi_t = (1-t)T_1 + tT_2$$

is also nonexpansive.

For $t = 0.5$ and $u = (0,0)$, the fixed point equation

$$x_t = (1-t)u + (t-t^2)T_1(x_t) + t^2T_2(x_t)$$

has a solution $x_t = (0,0)$. Therefore, the fixed point curve satisfies

$$G(0.5) = (0,0).$$

Definition 3.6. *Let X be a 2-normed space and $T\colon X \to X$. Then T is said to have the approximate fixed point property (a.f.p.p.) if for every $\epsilon > 0$, the set of approximate fixed points $F_\epsilon(T)$ is non-empty, i.e.,*

$$F_\epsilon(T) = \{x \in X : \ \|x - T(x), z\| < \epsilon\} \neq \emptyset.$$

Definition 3.7. *Let X be a 2-normed space. A mapping $T\colon X \to X$ is said to be asymptotically regular if*

$$\|T^n(x) - T^{n+1}(x), z\| \to 0 \quad \textit{as} \quad n \to \infty, \quad \forall\, x, z \in X.$$

Lemma 3.8. *Let X be a 2-normed space and $T\colon X \to X$ be such that T is asymptotically regular. Then, T has the approximate fixed point property.*

For each nonzero element b in a 2-Banach Space X , let $< b >$ denote the subspace of X generated by b consisting of all scalar multiples of b. The relation $\sim$ defined by $x \sim y$ if and only if $x - y \in < b >$, is an equivalence on X and this relation partitions X into equivalence classes. The equivalence class containing x is denoted by $(x)_b$. Let X_b be the collection of all such classes. Then X_b is a Banach space with respect to the operations:

$$(x)_b + (y)_b = (x + y)_b$$

$$\alpha(x)_b = (\alpha x)_b$$

and the norm on X_b is given by $||(x)_b|| = ||x, b||$.
Let X_b^* be the dual of X_b, that is the collection of all bounded linear functionals $T : X_b \to \mathbb{R}$ such that $|T((x)_b)| \leq K||x, b||$ for all $(x)_b \in X_b$. Thus if X is a 2-Banach Space, both X_b and X_b^* are Banach Spaces.

Definition 3.9. *A sequence $\{x_n\}$ in a 2-Banach space X is said to b-weakly converges to $x \in X$ if for every bounded linear functional $T \in X_b^*$, we have*

$$\lim_{n\to\infty} T(x_n) = T(x).$$

If this condition holds for all $b \in X$, we say that the sequence $\{x_n\}$ weakly converges to x in X.

Definition 3.10. *Let K be a nonempty convex subset of a 2-Banach space X, and let $T\colon K \to X$ be a mapping. Then T is said to be demiclosed at $v \in X$ if for any sequence $\{x_n\}$ in K such that $x_n \to u \in K$ and $T(x_n) \to v$, it follows that $T(u) = v$.*

Definition 3.11. *A 2-Banach space is said to satisfy the 2-Opial condition if every sequence that converges weakly to some $x \in X$ satisfies*

$$\liminf_{n\to\infty} \|x_n - x, z\| < \liminf_{n\to\infty} \|x_n - y, z\| \quad \textit{for any } y \neq x \textit{ and for all } z \in X.$$

Theorem 3.12. *Let X be a 2-Banach space that satisfies the 2-Opial condition, K a nonempty weakly compact subset of X, and $T\colon K \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed.*

Proof. Suppose that $\{x_n\}$ converges to u in K, and that

$$\lim_{n\to\infty} \|(I - T)(x_n) - v, z\| = 0 \quad \text{for some } v \in X \text{ and for all } z \in X.$$

By the 2-Opial condition, we have

$$\liminf_{n\to\infty} \|x_n - u, z\| < \liminf_{n\to\infty} \|x_n - y, z\| \quad \text{for any } y \neq u \text{ and for all } z \in X.$$

If $(I - T)(u) \neq v$, then

$$\|x_n - Tu - v, z\| \leq \|x_n - T(x_n) - v, z\| + \|T(x_n) - Tu, z\|$$

would imply that

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - Tu - v, z\| &\le \lim_{n\to\infty} \|x_n - T(x_n) - v, z\| + \liminf_{n\to\infty} \|x_n - u, z\| \\ &= \liminf_{n\to\infty} \|x_n - u, z\|,\end{aligned}$$

which contradicts the 2-Opial condition. Therefore, $(I - T)(u) = v$, and hence the mapping $I - T$ is demiclosed. □

Theorem 3.13. *Let K be a nonempty closed convex bounded subset of a 2-Banach space X, and let $T\colon K \to X$ be a nonexpansive mapping that is weakly inward. If $I - T$ is closed, then T has a fixed point in K.*

Proof. By Theorem 3.1, $x_\lambda - Tx_\lambda \to 0$ as $\lambda \to 1$. This implies that 0 lies in the closure of $(I-T)(C)$. Since $I - T$ is closed, there exists a point $x \in K$ such that $(I - T)(x) = 0$, or equivalently, $Tx = x$. □

This following result provides an analogue of Browder's theorem in 2-Banach spaces, which can be derived directly from Theorem 3.13.

Theorem 3.14. ***(Analogue of Browder's Theorem in 2-Banach Space)**:* *Let X be a uniformly convex 2-Banach space satisfying the 2-opial condition and let K be a nonempty bounded closed convex and weakly compact subset of X. Then, every nonexpansive mapping $T\colon K \to K$ has a fixed point in K.*

Proof. Let $y \in \overline{(I - T)(K)}$. Then there exist a sequence $\{x_n\}$ in K such that $x_n \to x$ in K and $(I - T)(x_n) \to y$ in the 2-normed space X. By Theorem 3.12, $I - T$ is demiclosed, which implies $(I - T)(x)) = y$. Hence, $(I - T)(K)$ is closed for any nonexpansive map $T : K \to X$. Therefore, by Theorem 3.13, the nonexpansive self map $T : K \to K$ has fixed point in K. □

Conclusion

In this paper, we have extended the theory of fixed points by investigating the existence and properties of fixed point curves for weakly inward non-expansive mappings in the setting of 2-Banach spaces. By introducing the notion of fixed point curves in this generalized framework, we have identified sufficient conditions under which such curves exist. Additionally, we have established a 2-Banach space analogue of Browder's fixed point theorem, thereby broadening the scope of classical fixed point theory. These results not only deepen the understanding of non-expansive mappings in 2-Banach spaces but also pave the way for further studies on the geometric and topological properties of such spaces in the context of fixed point theory.

Acknowledgment

The authors gratefully acknowledge the reviewers for their valuable comments and constructive suggestions, which have greatly helped to improve the quality of this paper. The first and second author gratefully acknowledges Krishna Menon Memorial Govt. Women's College, Kannur University for their kind support. The third author thank Manipal Institute of Technology, MAHE Manipal and

ANRF (SERB), DST, Government of India (TAR/2022/000219) for the TARE Fellowship Grant.

References

[1] Y.J. Cho, P.C.S. Lin, S.S. Kim, and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science, New York, 2001.
[2] C. Constantinescu, *C*-Algebra*, Volume 1: Banach Spaces, Elsevier, 2001.
[3] F. Lael and K. Nouruzi, *Compact operators defined on 2-normed and 2-probabilistic normed spaces*, Mathematical Problems in Engineering, Volume 2009, Article ID 950234.
[4] S. Gähler, A.H. Siddiqi, and S.C. Gupta, Contributions to non-Archimedean functional analysis, *Math. Nachr.*, **69** (1975), 162–171.
[5] S. Gähler, Über 2-Banach-Räume, *Math. Nachr.*, **42** (1969), 335–347.
[6] M. Gürdal, A. Şahiner, and I. Acik, Approximation theory in 2-Banach spaces, *Nonlinear Analysis*, **71** (2009), 1654–1661.
[7] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, *Math. Nachr.*, **26** (1963), 115–148.
[8] S. Gähler, *Lineare 2-normierte Räume*, *Math. Nachr.*, **28** (1964), 1–43.
[9] P.K. Harikrishnan and K.T. Ravindran, *Some properties of accretive operators in linear 2-normed spaces*, *International Mathematical Forum*, **6** (2011), no. 59, 2941–2947.
[10] P.K. Harikrishnan, B. Lafuerza Guillén, and K.T. Ravindran, *Accretive operators and Banach–Alaoglu theorem in linear 2-normed spaces*, *Proyecciones*, **30** (2011), no. 3, 319–327.
[11] F.R. Watson and C.Y. Je, *Geometry of Linear 2-Normed Spaces*, Nova Science Publishers, Inc., Hauppauge, NY, 2001.
[12] R.P. Agarwal, D. O'Regan, and D.R. Sahu, *Fixed Point Theory for Lipschitzian Type Mappings with Applications*, Springer, 2009.
[13] P. Riyas and K.T. Ravindran, *Unique fixed point theorem for weakly inward contractions and fixed point curve in 2-Banach space*, *Mathematical Journal of Interdisciplinary Sciences*, **3** (2005), no. 2, 173–182.
[14] P. Riyas and K.T. Ravindran, *Fixed point curve for weakly inward contractions and approximate fixed point property*, *Analysis in Theory and Applications*, **29** (2013), no. 4, 342–347.
[15] A.H. Siddiqi, 2-normed spaces, *Aligarh Bull. Math.*, (1980), 53–70.
[16] A.G. White, Jr., 2-Banach spaces, *Math. Nachr.*, **42** (1969), 43–60.

[1, 2] Department of Mathematics, Krishna Menon Govt. Women's College, Pallikunnu, Kannur, India.
Email address: riyasmankadavu@gmail.com,shijugeorgem@gmail.com

[3] Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, 576104, Karnataka, India.
Email address: pk.harikrishnan@manipal.edu