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A SIMPLE PROOF OF THE EXISTENCE OF THE DICKMAN FUNCTION

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Abstract. In this article we give a simple proof of the existence of the Dickman's function related with smooth numbers. We only use the concept of integral of a continuous function.

1. Introduction and Main Results

Let $0 < \alpha \leq 1$ be a fixed real number and consider the number of numbers not exceeding x such that their greatest prime factor does not exceed x^α. These numbers are called smooth numbers. We denote the number of these numbers $N(x, \alpha)$. It is well-known [1] that

$$N(x, \alpha) = \phi(\alpha)x + O\left(\frac{x}{\log x}\right), \tag{1.1}$$

where $\phi(\alpha)$ is called Dickman's function. This function of α is positive, strictly increasing and continuous on the interval $(0, 1]$. Clearly $\phi(1) = 1$. The proof of (1) use the Riemann-Stieltjes integral and the prime number theorem is not necessary (see [1]).

In this article using only the concept of integral of a continuous function we give a simple proof of the weaker result

$$N(x, \alpha) = \phi(\alpha)x + o(x).$$

In this article p denotes a positive prime and $\lfloor . \rfloor$ denotes the integer-part function.

We shall need the following well-known theorems.

Theorem 1.1. *The following asymptotic formula holds*

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + O\left(\frac{1}{\log x}\right),$$

where M is Mertens's constant.

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Theorem 1.2. *Let $\pi(x)$ be the prime counting function. The following asymptotic formula holds*

$$\pi(x) = O\left(\frac{x}{\log x}\right).$$

Theorem 1.3. *Let $f(x)$ be a continuous function on the closed interval $[a, b]$, then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)\ dx = f(c)(b - a).$$

Theorem 1.4. *Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Given any number $\epsilon > 0$ there exists δ_ϵ such that if $|x' - x''| < \delta_\epsilon$ then $|f(x') - f(x'')| < \epsilon$. That is, $f(x)$ is uniformly continuous on $[a, b]$.*

Now, we can to prove our main theorem.

Theorem 1.5. *If $0 < \alpha \leq 1$ then the following asymptotic formula holds*

$$N(\alpha, x) = \phi(\alpha)x + o(x) = \phi(\alpha)x + f(x)x, \tag{1.2}$$

where $\lim_{x\to\infty} f(x) = 0$ and $\phi(\alpha)$ is positive, strictly increasing and continuous on the interval $(0, 1]$. Note that $f(x)$ depends of α. Besides

$$\phi(\alpha) = 1 - \int_\alpha^1 \phi\left(\frac{x}{1-x}\right)\frac{1}{x}\ dx, \tag{1.3}$$

where we put $\phi\left(\frac{x}{1-x}\right) = 1$ if $x \epsilon\, [1/2, 1]$. Therefore if $1/2 \leq \alpha \leq 1$ then $\phi(\alpha) = 1 - \int_\alpha^1 \frac{1}{x}\ dx = 1 + \log\alpha$.

Proof. Let us consider the multiples of p not exceeding x. Namely

$$\left\{p.1, p.2, \ldots, p\left\lfloor\frac{x}{p}\right\rfloor\right\} \tag{1.4}$$

The number of multiples of p not exceeding x such that p is their greatest prime factor we denote $B(x, p)$. Therefore $B(x, p) \leq \left\lfloor\frac{x}{p}\right\rfloor$. Let $1/2 \leq \alpha < 1$ be, then $x^{1-\alpha} \leq x^\alpha$. On the other hand if $p > x^\alpha$ then we have $\frac{x}{p} < x^{1-\alpha}$. Therefore $\left\lfloor\frac{x}{p}\right\rfloor \leq \frac{x}{p} < x^{1-\alpha} \leq x^\alpha < p$. That is $\left\lfloor\frac{x}{p}\right\rfloor < p$, and consequently

$$B(x, p) = \left\lfloor\frac{x}{p}\right\rfloor. \tag{1.5}$$

We have (see (1.5))

$$\begin{aligned}
\lfloor x\rfloor - N(\alpha, x) &= \sum_{x^\alpha < p \leq x} B(x, p) = \sum_{x^\alpha < p \leq x}\left\lfloor\frac{x}{p}\right\rfloor \\
&= x\sum_{x^\alpha < p \leq x}\frac{1}{p} - \sum_{x^\alpha < p \leq x}\left(\frac{x}{p} - \left\lfloor\frac{x}{p}\right\rfloor\right).
\end{aligned} \tag{1.6}$$

Now (see Theorem 1.2)

$$0 \leq \sum_{x^\alpha < p \leq x} \left(\frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) \leq \sum_{p \leq x} 1 = \pi(x) < c \frac{x}{\log x}.$$

That is

$$\sum_{x^\alpha < p \leq x} \left(\frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) = O\left(\frac{x}{\log x} \right). \tag{1.7}$$

On the other hand, we have (see Theorem 1.1)

$$\sum_{x^\alpha < p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p} - \sum_{p \leq x^\alpha} \frac{1}{p} = -\log \alpha + O\left(\frac{1}{\log x} \right). \tag{1.8}$$

Substituting (1.7) and (1.8) into (1.6) we obtain

$$\lfloor x \rfloor - N(\alpha, x) = -x \log \alpha + O\left(\frac{x}{\log x} \right).$$

Hence

$$N(\alpha, x) = (1 + \log \alpha)x + O\left(\frac{x}{\log x} \right).$$

Therefore the theorem is true if $\alpha \in [1/2, 1]$.

Suppose that the theorem is true if $\alpha \in [1/j, 1]$, where j is a positive integer. That is, we have in this interval

$$N(\alpha, x) = \phi(\alpha)x + o(x) = \left(1 - \int_\alpha^1 \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right) x + o(x), \tag{1.9}$$

where the function

$$\phi(\alpha) = 1 - \int_\alpha^1 \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \tag{1.10}$$

is positive, strictly increasing and continuous on the interval $[1/j, 1]$.

Suppose that $\frac{1}{j+1} \leq \alpha < \frac{1}{j}$. Therefore we have

$$\lfloor x \rfloor - N(\alpha, x) = \sum_{x^\alpha < p \leq x} B(x,p) = \sum_{x^\alpha < p \leq x^{1/j}} B(x,p) + \sum_{x^{1/j} < p \leq x} B(x,p), \tag{1.11}$$

where (see (1.9))

$$\sum_{x^{1/j} < p \leq x} B(x,p) = \left(\int_{1/j}^1 \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right) x + o(x). \tag{1.12}$$

Note that (see (1.10)) on the interval $[\alpha, 1/j]$ the function $\phi\left(\frac{x}{1-x}\right)$ is positive, strictly increasing and uniformly continuous (see Theorem 1.4). Note that

$$\phi\left(\frac{1/(j+1)}{1 - (1/(j+1))} \right) = \phi(1/j).$$

Consequently the function $\phi\left(\frac{x}{1-x}\right)\frac{1}{x}$ is positive and uniformly continuous on the interval $[\alpha, 1/j]$. Therefore if we consider the positive number ϵ then there exists a partition of the interval $[\alpha, 1/j]$, namely $\alpha = \beta_1 < \beta_2 < \cdots < \beta_k = 1/j$, such that

$$\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) - \phi\left(\frac{\beta_i}{1-\beta_i}\right) < \frac{\epsilon}{j+1} \qquad (i = 1, 2, \dots, k-1) \tag{1.13}$$

and such that if $x', x'' \in [\beta_i, \beta_{i+1}]$ $(i = 1, 2, \dots, k-1)$ then we have

$$\left|\phi\left(\frac{x'}{1-x'}\right)\frac{1}{x'} - \phi\left(\frac{x''}{1-x''}\right)\frac{1}{x''}\right| < \frac{\epsilon}{j+1} \tag{1.14}$$

We have (see (1.11))

$$\sum_{x^\alpha < p \le x^{1/j}} B(x,p) = \sum_{i=1}^{k-1} \sum_{x^{\beta_i} < p \le x^{\beta_{i+1}}} B(x,p). \tag{1.15}$$

The inequality $x^{\beta_i} < p \le x^{\beta_{i+1}}$ implies the inequality $x^{1-\beta_{i+1}} \le \frac{x}{p} < x^{1-\beta_i}$. Therefore we have

$$p \le x^{\beta_{i+1}} = \left(x^{1-\beta_{i+1}}\right)^{\frac{\beta_{i+1}}{1-\beta_{i+1}}} \le \left(\frac{x}{p}\right)^{\frac{\beta_{i+1}}{1-\beta_{i+1}}} \tag{1.16}$$

and

$$p > x^{\beta_i} = \left(x^{1-\beta_i}\right)^{\frac{\beta_i}{1-\beta_i}} > \left(\frac{x}{p}\right)^{\frac{\beta_i}{1-\beta_i}}. \tag{1.17}$$

Let us consider (see (1.4)) the set $\left\{1, 2, 3, \dots, \left\lfloor \frac{x}{p} \right\rfloor\right\}$. Let $C(x,p)$ be the number of numbers in this set such that their greatest prime factor does not exceed p. Consequently $B(x,p) = C(x,p)$. Equations (1.2) and (1.16) give

$$\begin{aligned} B(x,p) = C(x,p) \le N\left(\frac{x}{p}, \frac{\beta_{i+1}}{1-\beta_{i+1}}\right) &= \phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{x}{p} + f\left(\frac{x}{p}\right)\frac{x}{p} \\ = \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) + \frac{\epsilon}{j+1}\right)\frac{x}{p} + \left(f\left(\frac{x}{p}\right) - \frac{\epsilon}{j+1}\right)\frac{x}{p} \\ \le \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) + \frac{\epsilon}{j+1}\right)\frac{x}{p}. \end{aligned} \tag{1.18}$$

Equations (1.2) and (1.17) give

$$\begin{aligned} B(x,p) = C(x,p) \ge N\left(\frac{x}{p}, \frac{\beta_i}{1-\beta_i}\right) &= \phi\left(\frac{\beta_i}{1-\beta_i}\right)\frac{x}{p} + f\left(\frac{x}{p}\right)\frac{x}{p} \\ = \left(\phi\left(\frac{\beta_i}{1-\beta_i}\right) - \frac{\epsilon}{j+1}\right)\frac{x}{p} + \left(f\left(\frac{x}{p}\right) + \frac{\epsilon}{j+1}\right)\frac{x}{p} \\ \ge \left(\phi\left(\frac{\beta_i}{1-\beta_i}\right) - \frac{\epsilon}{j+1}\right)\frac{x}{p}. \end{aligned} \tag{1.19}$$

Theorem 1.1 gives

$$\sum_{x^{\beta_i} < p \le x^{\beta_{i+1}}} \frac{1}{p} = \sum_{p \le x^{\beta_{i+1}}} \frac{1}{p} - \sum_{p \le x^{\beta_i}} \frac{1}{p} = (\log \beta_{i+1} - \log \beta_i) + o(1). \tag{1.20}$$

Equations (1.18), (1.20), (1.13) and the mean value theorem give

$$
\begin{aligned}
\sum_{x^{\beta_i}<p\leq x^{\beta_{i+1}}} B(x,p) &\leq \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\frac{\epsilon}{j+1}\right) x \sum_{x^{\beta_i}<p\leq x^{\beta_{i+1}}} \frac{1}{p} \\
&= \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\frac{\epsilon}{j+1}\right)(\log \beta_{i+1}-\log \beta_i)\, x + o(x) \\
&\leq \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\frac{\epsilon}{j+1}\right)\frac{1}{\beta_i}(\beta_{i+1}-\beta_i)\, x + o(x) \\
&\leq \left(\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)+2\frac{\epsilon}{j+1}\right)\frac{1}{\beta_i}(\beta_{i+1}-\beta_i)\, x + o(x) \\
&= \phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)\frac{1}{\beta_i}(\beta_{i+1}-\beta_i)\, x + 2\frac{\epsilon}{j+1}\frac{1}{\beta_i}(\beta_{i+1}-\beta_i)\, x + o(x). \qquad (1.21)
\end{aligned}
$$

Equations (1.19), (1.20), (1.13) and the mean value theorem give

$$
\begin{aligned}
\sum_{x^{\beta_i}<p\leq x^{\beta_{i+1}}} B(x,p) &\geq \left(\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)-\frac{\epsilon}{j+1}\right) x \sum_{x^{\beta_i}<p\leq x^{\beta_{i+1}}} \frac{1}{p} \\
&= \left(\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)-\frac{\epsilon}{j+1}\right)(\log \beta_{i+1}-\log \beta_i)\, x + o(x) \\
&\geq \left(\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)-\frac{\epsilon}{j+1}\right)\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i)\, x + o(x) \\
&\geq \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)-2\frac{\epsilon}{j+1}\right)\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i)\, x + o(x) \\
&= \phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i)\, x - 2\frac{\epsilon}{j+1}\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i)\, x + o(x) \qquad (1.22)
\end{aligned}
$$

Equations (1.15) and (1.21) give

$$
\begin{aligned}
\sum_{x^{\alpha}<p\leq x^{1/j}} B(x,p) &\leq \left(\sum_{i=1}^{k-1}\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)\frac{1}{\beta_i}(\beta_{i+1}-\beta_i)\right) x \\
&+ 2\frac{\epsilon}{j+1} x \sum_{i=1}^{k-1}\frac{1}{\beta_i}(\beta_{i+1}-\beta_i) + o(x). \qquad (1.23)
\end{aligned}
$$

Equations (1.15) and (1.22) give

$$
\begin{aligned}
\sum_{x^{\alpha}<p\leq x^{1/j}} B(x,p) &\geq \left(\sum_{i=1}^{k-1}\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i)\right) x \\
&- 2\frac{\epsilon}{j+1} x \sum_{i=1}^{k-1}\frac{1}{\beta_{i+1}}(\beta_{i+1}-\beta_i) + o(x). \qquad (1.24)
\end{aligned}
$$

We have $\frac{1}{\beta_i} \leq 1/\alpha$ $(i = 1, 2, \ldots, k)$, since $\beta_i \geq \alpha$ (see above). On the other hand $(1/\alpha) \leq j + 1$, since $\alpha \geq 1/(j + 1)$ (see above). Therefore we have

$$\sum_{i=1}^{k-1} \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) \leq \frac{1}{\alpha} \left(\frac{1}{2} - \alpha \right) \leq \frac{1}{\alpha} \leq j + 1 \tag{1.25}$$

and

$$\sum_{i=1}^{k-1} \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) \leq \frac{1}{\alpha} \left(\frac{1}{2} - \alpha \right) \leq \frac{1}{\alpha} \leq j + 1. \tag{1.26}$$

On the other hand we have (see Theorem 1.3 and (1.14))

$$\begin{aligned} & \left| \sum_{i=1}^{k-1} \phi\left(\frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) - \int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right| \\ = & \left| \sum_{i=1}^{k-1} \phi\left(\frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) - \sum_{i=1}^{k-1} \int_{\beta_i}^{\beta_{i+1}} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right| \\ = & \left| \sum_{i=1}^{k-1} \left(\phi\left(\frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} - \phi\left(\frac{c_i}{1 - c_i} \right) \frac{1}{c_i} \right) (\beta_{i+1} - \beta_i) \right| \leq \frac{\epsilon}{j+1} \left(\frac{1}{j} - \alpha \right) \\ \leq & \frac{\epsilon}{j+1}, \end{aligned} \tag{1.27}$$

where $c_i \in [\beta_i, \beta_{i+1}]$. In the same way we obtain

$$\left| \sum_{i=1}^{k-1} \phi\left(\frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) - \int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right| \leq \frac{\epsilon}{j+1}. \tag{1.28}$$

Equations (1.23), (1.25) and (1.27) give

$$\begin{aligned} & \sum_{x^\alpha < p \leq x^{1/j}} B(x, p) \leq \left(\int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right) x + \frac{\epsilon}{j+1} x + 2 \frac{\epsilon}{j+1} (j+1) x \\ + \ & o(x) \leq \left(\int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx + 4\epsilon \right) x. \end{aligned} \tag{1.29}$$

Equations (1.24), (1.26) and (1.28) give

$$\begin{aligned} & \sum_{x^\alpha < p \leq x^{1/j}} B(x, p) \geq \left(\int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right) x - \frac{\epsilon}{j+1} x - 2 \frac{\epsilon}{j+1} (j+1) x \\ + \ & o(x) \geq \left(\int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx - 4\epsilon \right) x. \end{aligned} \tag{1.30}$$

Now, ϵ is arbitrarily small. Therefore equations (1.29) and (1.30) give

$$\sum_{x^\alpha < p \leq x^{1/j}} B(x, p) = \left(\int_\alpha^{1/j} \phi\left(\frac{x}{1-x} \right) \frac{1}{x}\, dx \right) x + o(x). \tag{1.31}$$

Equations (1.31), (1.12) and (1.11) give equations (1.9) and (1.10), where these equations are true if $\alpha \in [1/(j+1), 1]$, since the theorem is true if $\alpha \in [1/2, 1]$. Now, $\lim_{j\to\infty}(1/j) = 0$ and consequently equations (1.9) and (1.10) are true for all $\alpha > 0$. The theorem is proved. □

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