ON $\alpha$-SEMIDERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

KYUNG HO KIM

ABSTRACT. In this paper, we introduce the notion of an $\alpha$-semiderivation on prime rings, and we try to extend some results for derivations of rings or near-rings to a more general case for $\alpha$-semiderivations of prime rings.

1. Introduction and Preliminaries

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring $R$ and certain specific types of derivations of $R$. The first result in this direction is due to E. C. Posner [9] who proved that if a ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. This result was subsequently, refined and extended by a number of authors. In [6], Bresar and Vukman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. Furthermore, Bresar and Vukman [5] studied the notions of a $*$-derivation and a Jordan $*$-derivation of $R$. In this paper, we introduce the notion of an $\alpha$-semiderivation on prime rings, and we try to extend some results for derivations of rings or near-rings to a more general case for $\alpha$-semiderivations of prime rings.

Let $R$ is a ring. Then $R$ is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Also, we make use of the following two basic identities without any specific mention:

\[
x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z
\]
\[
(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]
\]
\[
[x, y, z] = x[y, z] + [x, z]y \text{ and } [x, y]z = y[x, z] + [x, y]z
\]
for all $x, y, z \in R$.

Definition 1.1. Let $R$ be a prime ring and $\alpha$ be an automorphism on $R$. An additive mapping $d : R \to R$ is called a $\alpha$-semiderivation associated with an epimorphism $g : R \to R$ if
(i) \( d(xy) = d(x)g(y) + \alpha(x)d(y) = d(x)\alpha(y) + g(x)d(y), \)
(ii) \( d(g(x)) = g(d(x)), \) for all \( x, y \in R.\)

**Definition 1.2.** Let \( R \) be a prime ring and \( \alpha \) be an automorphism on \( R. \) An additive mapping \( d : R \rightarrow R \) is called a reverse \( \alpha \)-semiderivation associated with an epimorphism \( g : R \rightarrow R \) if

(i) \( d(xy) = d(y)g(x) + \alpha(y)d(x) = d(y)\alpha(x) + g(y)d(x), \)
(ii) \( d(g(x)) = g(d(x)), \) for all \( x, y \in R.\)

2. \( \alpha \)-Semiderivations and Commutativity of Prime Rings

**Lemma 2.1.** Let \( R \) be a prime ring and let \( d \) be a nonzero \( \alpha \)-semiderivation associated with \( g \) and \( a \in R. \) If \( ad(R) = 0, \) then \( a = 0. \)

**Proof.** By hypothesis, we have

\[
\text{ad}(xy) = 0, \forall x, y \in R, \tag{2.1}
\]

which implies that \( \text{ad}(x)g(y) + a\alpha(x)d(y)) = 0 \) for all \( x, y \in R. \) By the hypothesis, we have \( a\alpha(x)d(y) = 0 \) for all \( x, y \in R. \) Replacing \( x \) by \( \alpha^{-1}(x) \) in this relation, we get \( axd(y) = 0 \) for all \( x, y \in R, \) which implies that \( aRd(y) = 0 \) for all \( y \in R. \) Since \( R \) is prime and \( d \neq 0, \) we have \( a = 0. \)

**Theorem 2.2.** Let \( R \) be a prime ring and \( g \) be an epimorphism on \( R. \) If \( R \) admits an \( \alpha \)-semiderivation \( d \) associated with \( g \) such that \( d([x, y]) = 0 \) for all \( x, y \in R, \) then \( d = 0 \) or \( R \) is commutative.

**Proof.** By hypothesis, we have

\[
d([x, y]) = 0, \forall x, y \in R. \tag{2.2}
\]

Replacing \( y \) by \( yx \) in (2), we have

\[
d([x, yx]) = d([x, y]x) = d([x, y])g(x) + \alpha([x, y])d(x) = 0
\]

for all \( x, y \in R. \) By the hypothesis, we get

\[
\alpha([x, y])d(x) = 0, \forall x, y \in R. \tag{2.3}
\]

Taking \( \alpha^{-1}([x, y]) \) instead of \([x, y]\) in this relation, we have \([x, y]d(x) = 0\) for all \( x, y \in R. \) Taking \( zy \) instead of \( y \) with \( z \in R \) in this relation, we obtain \([x, z]yd(x) = 0\) for all \( x, y, z \in R. \) This implies that \([x, z]Rd(x) = \{0\}\) for all \( x, z \in R. \) Since \( R \) is prime, we have \([x, z] = 0 \) or \( d(x) = 0 \) for all \( x, z \in R. \) Let \( K = \{x \in R|d(x) = 0\} \) and \( L = \{x \in R|[x, z] = 0, \forall z \in R\}. \) Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R, \) but \((R, +)\) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R. \) In the former case, we have \( d(x) = 0 \) for all \( x \in R, \) that is, \( d = 0. \) If \( L = R, \) then we get \([x, z] = 0 \) for all \( x, y \in R, \) which implies that \( R \) is commutative.

**Theorem 2.3.** Let \( R \) be a prime ring and \( g \) be an epimorphism on \( R. \) If \( R \) admits an \( \alpha \)-semiderivation \( d \) associated with \( g \) such that \( d(x \circ y) = 0 \) for all \( x, y \in R, \) then \( d = 0 \) or \( R \) is commutative.
Proof. By hypothesis, we have
\[ d(x \circ y) = 0, \forall x, y \in R. \quad (2.4) \]
Replacing \( y \) by \( yx \) in (4), we have \( d(x \circ yx) = d((x \circ y)x) = d(x \circ y)g(x) + \alpha(x \circ y)d(x) = 0 \) for all \( x, y \in R \). By the hypothesis, we get
\[ \alpha(x \circ y)d(x) = 0, \forall x, y \in R. \quad (2.5) \]
Taking \( \alpha^{-1}(x \circ y) \) instead of \( x \circ y \) in the last relation, we have \( (x \circ y)d(x) = 0 \) for all \( x, y \in R \). Taking \( yx \) instead of \( y \) in this relation, we obtain \( (x \circ y)xzd(x) = 0 \) for all \( x, y \in R \). This implies that \( (x \circ y)Rd(x) = \{0\} \) for all \( x, y \in R \). Since \( R \) is prime, we have \( x \circ y = 0 \) or \( d(x) = 0 \) for all \( x, y \in R \). Let \( K = \{x \in R | d(x) = 0\} \) and \( L = \{x \in R | x \circ y = 0, \forall y \in R\} \). Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R \), but \( (R, +) \) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R \). In the former case, we have \( d(x) = 0 \) for all \( x \in R \), that is, \( d = 0 \). If \( L = R \), then we get \( x \circ y = 0 \) for all \( x, y \in R \), which implies that \( xz = xy \) for all \( x, y \in R \). Again, replacing \( x \) by \( xz \) in the last relation, we have \( xzy = yxz = yxz \), that is, \( x[z, y] = 0 \) for all \( x, y, z \in R \). This implies that \( R[z, y] = \{0\} \) for all \( x, z \in R \). Hence \( tR[z, y] = \{0\} \) for all \( t \neq y, z \in R \). Since \( R \) is prime, we have \( [z, y] = 0 \) for all \( y, z \in R \), which implies that \( R \) is commutative.

Theorem 2.4. Let \( R \) be a prime ring and \( g \) be an epimorphism on \( R \). If \( R \) admits an \( \alpha \)-semiderivation \( d \) associated with \( g \) such that \([d(x), y] = 0 \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have
\[ [d(x), y] = 0, \forall x, y \in R. \quad (2.6) \]
Replacing \( x \) by \( xz \) in (6) and using (6), we have
\[
0 = [d(xz), y] = [d(x)g(z) + \alpha(x)d(z), y]
= [d(x)g(z), y] + [\alpha(x)d(z), y]
= d(x)[g(z), y] + [d(x), y]g(z) + [\alpha(x)[d(z), y] + [\alpha(x), y]d(z)
= d(x)[g(z), y] + [\alpha(x), y]d(z)
\]
for all \( x, y, z \in R \). Taking \( \alpha(z) \) instead of \( y \) in (7), we have \([\alpha(x), g(z)]d(z) = 0 \) for all \( x, z \in R \). Substituting \( \alpha^{-1}(x) \) for \( x \) in this relation, we get \( d(x)[x, g(z)] = 0 \) for all \( x, z \in R \). Again, replacing \( x \) by \( yx \) in the last relation, we obtain \( d(x)y[x, g(z)] = 0 \) for all \( x, y, z \in R \). Hence \( d(x)R[x, g(z)] = 0 \) for all \( x, y, z \in R \). Since \( R \) is prime, we have \( d(x) = 0 \) or \([x, g(z)] = 0 \) for all \( x, y \in R \). Let \( K = \{x \in R | d(x) = 0\} \) and \( L = \{x \in R | [x, g(z)] = 0, \forall y \in R\} \). Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R \), but \((R, +)\) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R \). In the former case, we have \( d(x) = 0 \) for all \( x \in R \), that is, \( d = 0 \). If \( L = R \), then we get \([y, g(x)] = 0 \) for all \( x, y \in R \). Since \( g \) is onto, we have \([y, x] = 0 \) for all \( x, y \in R \), which implies that \( R \) is commutative.

\[ \square \]
Theorem 2.5. Let \( R \) be a prime ring and \( g \) be an epimorphism on \( R \). If \( R \) admits an \( \alpha \)-semiderivation \( d \) associated with \( g \) such that \( d(x) \circ y = 0 \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have
\[
d(x) \circ y = 0, \ \forall \ x, y \in R. \tag{2.8}
\]
Replacing \( x \) by \( xz \) in (8) and using (8), we have
\[
0 = d(xz) \circ y = d(x)g(z) \circ y + \alpha(x)d(z) \circ y
= (d(x) \circ y)g(z) + d(x)[g(z), y] + \alpha(x)(d(z) \circ y) - [\alpha(x), y]d(z)
= d(x)[g(z), y] - [\alpha(x), y]d(z) \tag{2.9}
\]
for all \( x, y, z \in R \). Taking \( g(z) \) instead of \( y \) in (9), we have \([\alpha(x), g(z)]d(z) = 0\) for all \( x, z \in R \). Substituting \( \alpha^{-1}(x) \) for \( x \) in this relation, we get \([x, g(z)]d(z) = 0\) for all \( x, z \in R \). Again, replacing \( x \) by \( yx \) in the last relation, we obtain \([y, g(z)]x d(z) = 0\) for all \( x, y, z \in R \). Hence \([y, g(z)]x d(z) = 0\) for all \( x, y \in R \).

Since \( R \) is prime, we have \( d(z) = 0 \) or \([y, g(z)] = 0\) for all \( y, z \in R \). Let \( K = \{z \in R | d(z) = 0\} \) and \( L = \{y \in R | [y, g(z)] = 0, \forall z \in R\} \). Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R \), but \((R, +)\) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R \). In the former case, we have \( d(z) = 0 \) for all \( z \in R \), that is, \( d = 0 \). If \( L = R \), then we get \([y, g(z)] = 0\) for all \( x, y \in R \). Since \( g \) is onto, we have \([y, z] = 0\) for all \( y, z \in R \), which implies that \( R \) is commutative.

\( \square \)

Theorem 2.6. Let \( R \) be a prime ring and let \( g \) be an epimorphism on \( R \). If \( d \) is an \( \alpha \)-semiderivation associated with \( g \) such that \( d(xy) = d(x)d(y) \) for all \( x, y \in R \), then \( d = 0 \).

Proof. For any \( x, y \in R \), we have
\[
d(xy) = d(x)g(y) + \alpha(x)d(y) = d(x)d(y), \ \forall \ x, y \in R. \tag{2.10}
\]
Replacing \( x \) by \( xw \) in (10), we obtain \( d(xw)g(y) + \alpha(xw)d(y) = d(xw)d(y) \) for all \( x, y, w \in R \). Hence \( d(x)d(w)g(y) + \alpha(w)\alpha(x)d(y) = d(x)d(w)d(y) = d(x)d(xy) \) for all \( x, y, w \in R \), and hence \( d(x)d(w)g(y) + \alpha(x)\alpha(w)d(y) = d(x)d(w)g(y) + d(x)\alpha(w)d(y) \) for all \( x, y, w \in R \). This implies that \((\alpha(x) - d(x))\alpha(w)d(y) = 0\) for all \( x, y, w \in R \). Substituting \( \alpha^{-1}(w) \) for \( w \) in the last relation, we have \((\alpha(x) - d(x))Rd(y) = 0\) for all \( x, y \in R \). Since \( R \) is prime, we have \( d(x) = \alpha(x) \) or \( d(y) = 0\) for all \( x, y \in R \). Let us assume that \( d(x) = \alpha(x) \) for all \( x \in R \). Substituting \( xy \) for \( x \) in the last relation, we have \( d(x)g(y) + \alpha(x)d(y) = \alpha(x)\alpha(y) = \alpha(x)d(y) \) for all \( x, y \in R \), that is, \( d(x)g(y) = 0 \) for all \( x, y \in R \). Since \( g \) is onto, we have \( d(x)y = 0 \), which implies that \( d(x)R = \{0\} \) for all \( x \in R \). Thus we obtain \( d(x) = 0 \) for all \( x \in R \) in any case.

\( \square \)

Theorem 2.7. Let \( R \) be a prime ring and let \( g \) be an epimorphism on \( R \). If \( d \) is an \( \alpha \)-semiderivation associated with \( g \) such that \( d(xy) = d(y)d(x) \) for all \( x, y \in R \) and \( \alpha(y) \neq d(y) \) for all \( y \in R \), then \( d = 0 \).
Proof. For any \( x, y \in R \), we have
\[
d(xy) = d(x)g(y) + \alpha(x)d(y) = d(y)d(x), \quad \forall \ x, y \in R.
\]
Replacing \( x \) by \( xy \) in (11), we obtain \( d(xy)g(y) + \alpha(xy)d(y) = d(y)d(xy) \) for all \( x, y \in R \). Hence we have
\[
d(y)d(x)g(y) + \alpha(y)\alpha(x)d(y) = d(y)d(x)g(y) + d(y)\alpha(x)d(y)
\]
for all \( x, y \in R \), and hence \((\alpha(y) - d(y))\alpha(x)d(y) = 0\) for all \( x, y \in R \). Substituting \( \alpha^{-1}(x) \) for \( x \) in the last relation, we get \((\alpha(y) - d(y))xd(y) = 0\) for all \( x, y \in R \). This implies that \((\alpha(y) - d(y))Rd(y) = 0\) for all \( y \in R \). Since \( R \) is prime, we have \( \alpha(y) - d(y) = 0 \) or \( d(y) = 0 \) for all \( y \in R \). But \( \alpha(y) \neq d(y) \) for all \( y \in R \), and so \( d(y) = 0 \) for all \( y \in R \).

\[\square\]

**Theorem 2.8.** Let \( R \) be a prime ring and let \( g \) be an epimorphism on \( R \). If \( d \) is an \( \alpha \)-semiderivation associated with \( g \) such that \( \alpha(xy) = \alpha(y)\alpha(x) \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have
\[
d(xy) = d(x)g(y) + \alpha(x)d(y), \quad \forall \ x, y \in R.
\]
Replacing \( y \) by \( yz \) in (12), we have
\[
d(x(yz)) = d(x)g(yz) + \alpha(x)d(yz)
= d(x)g(y)g(z) + \alpha(x)d(y)g(z) + \alpha(x)\alpha(y)d(z)
\]
for all \( x, y, z \in R \). On the other hand, we get
\[
d(xyz) = d((xy)z)
= d(xy)g(z) + \alpha(xy)d(z)
= d(x)g(y)g(z) + \alpha(x)d(y)g(z) + \alpha(y)\alpha(x)d(z)
\]
for all \( x, y, z \in R \). Comparing (13) and (14), we have \( d(x)[\alpha(y), \alpha(z)] = 0 \) for all \( x, y, z \in R \). Substituting \( \alpha^{-1}(y) \) for \( y \) and \( \alpha^{-1}(z) \) for \( z \) in the last relation. Since \( g \) is onto, we have \( d(x)[y, z] = 0 \) for all \( x, y, z \in R \). Replacing \( z \) by \( zr \), in this relation, we obtain
\[
d(x)[y, z] = 0, \quad \forall \ r, x, z \in R.
\]
This implies \( d(x)R[y, r] \) for all \( r, x, y \in R \). Since \( R \) is prime, we have \( d(x) = 0 \) or \( [y, z] = 0 \) for all \( x, y, z \in R \). Let \( K = \{ x \in R | d(x) = 0 \} \) and \( L = \{ z \in R | [y, z] = 0, \forall y \in R \} \). Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R \), but \((R, +)\) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R \). In the former case, we have \( d = 0 \). If \( L = R \), then \( [g(y), z] = 0 \) for all \( y, z \in R \). Since \( g \) is onto, we obtain \( [y, z] = 0 \), which implies that \( R \) is commutative.

\[\square\]

**Theorem 2.9.** Let \( R \) be a prime ring and let \( g \) be an epimorphism on \( R \). If \( d \) is a reverse \( \alpha \)-semiderivation associated with \( g \) such that \( g(xy) = g(y)g(x) \) for all \( x, y \in R \), then \( [d(x), z] = 0 \) for all \( x, z \in R \) or \( d = 0 \).
Proof. By hypothesis, we have

\[ d(xy) = d(y)g(x) + \alpha(y)d(x), \forall \, x, y \in R. \]  

(2.16)

Replacing \( x \) by \( xz \) in (16), we have

\[
d(xzy) = d(y)g(xz) + \alpha(y)d(xz)
= d(y)g(z)g(x) + \alpha(y)d(z)g(x) + \alpha(y)\alpha(z)d(x)
\]  

(2.17)

for all \( x, y, z \in R \). On the other hand,

\[
d(xzy) = d(x(zy)) = d(zy)g(x) + \alpha(zy)d(x)
= d(y)g(z)g(x) + \alpha(y)d(z)g(x) + \alpha(z)\alpha(y)d(x)
\]  

(2.18)

Comparing (17) with (18), we get \([\alpha(z), \alpha(y)]d(x)\) for all \( x, y, z \in R \). Again, replacing \( y \) by \( \alpha^{-1}(y) \) and \( z \) by \( \alpha^{-1}(z) \) in this relation, we have \([z, y]d(x) = 0\) for all \( x, y, z \in R \). Taking \( d(x)z \) instead of \( z \) in this relation, we have

\[
0 = [d(x)z, z]d(x)
= d(x)[z, z]d(x) + [d(x), z]zd(x)
= [d(x), z]zd(x)
\]  

(2.19)

This implies that \([d(x), z]Rd(x) = \{0\}\) for all \( x, z \in R \). Since \( R \) is prime, we get either \([d(x), z] = 0\) or \( d(x) = 0\) for all \( x, z \in R \). 

\[\Box\]

Theorem 2.10. Let \( R \) be a prime ring and let \( g \) be an epimorphism on \( R \). If \( d \) is an \( \alpha \)-semiderivation associated with \( g \) such that \( d(x) \circ \alpha(y) = 0 \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have

\[
d(x) \circ \alpha(y) = 0, \forall \, x, y \in R.
\]  

(2.20)

Replacing \( x \) by \( yx \) in (20), we have

\[
0 = d(yx) \circ \alpha(y)
= (d(y)g(x) + \alpha(y)d(x)) \circ \alpha(y)
= d(y)g(x) \circ \alpha(y) + \alpha(y)d(x) \circ \alpha(y)
= (d(y) \circ \alpha(y))g(x) + d(y)[g(x), \alpha(y)] + \alpha(y)(d(x) \circ \alpha(y)) - [\alpha(y), \alpha(y)]d(x)
= d(y)[g(x), \alpha(y)]
\]  

(2.21)

for every \( x, y \in R \). Since \( g \) is onto, we get \( d(y)[x, \alpha(y)] = 0 \) for all \( x, y \in R \). Taking \( xz \) instead of \( x \) in this relation, we obtain \( d(y)x[z, \alpha(y)] = 0 \) for all \( x, y, z \in R \).

This implies that \( d(y)R[z, \alpha(y)] = \{0\} \) for all \( y, z \in R \). Since \( R \) is prime, we have \( d(y) = 0 \) or \([z, \alpha(y)] = 0\) for all \( y, z \in R \). Let \( K = \{y \in R | d(y) = 0\} \) and \( L = \{y \in R | [z, \alpha(y)] = 0, \forall z \in R\} \). Then \( K \) and \( L \) are both additive subgroups and \( K \cup L = R \), but \( (R, +) \) is not union of two its proper subgroups, which implies that either \( K = R \) or \( L = R \). In the former case, we have \( d = 0 \). If \( L = R \), then \([z, \alpha(y)] = 0 \) for all \( y, z \in R \). Again, replacing \( y \) by \( \alpha^{-1}(y) \) in the last relation, we get \([z, y] = 0\) for all \( y, z \in R \), which implies that \( R \) is commutative. 

\[\Box\]
Theorem 2.11. Let $R$ be a prime ring and let $g$ be an epimorphism on $R$. If $d$ is an $\alpha$-semiderivation associated with $g$ such that $[d(x), \alpha(y)] = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

Proof. By hypothesis, we have

$$[d(x), \alpha(y)] = 0, \forall x, y \in R. \quad (2.22)$$

Replacing $x$ by $yx$ in (22), we have

$$0 = [d(yx), \alpha(y)] = [d(y)g(x) + \alpha(y)d(x), \alpha(y)] = [d(y)g(x), \alpha(y)] + [\alpha(y)d(x), \alpha(y)] = d(y)[g(x), \alpha(y)] + [d(y), \alpha(y)]g(x) + \alpha(y)[d(x), \alpha(y)] + [\alpha(y), \alpha(y)]d(x) = d(y)[g(x), \alpha(y)] \quad (2.23)$$

for every $x, y \in R$. Since $g$ is onto, we get $d(y)[x, \alpha(y)] = 0$ for all $x, y \in R$. Taking $xz$ instead of $x$ in this relation, we obtain $d(y)[x, \alpha(y)] = 0$ for all $x, y, z \in R$. This implies that $d(y)R[z, \alpha(y)] = \{0\}$ for all $y, z \in R$. Since $R$ is prime, we have $d(y) = 0$ or $\alpha(y) = 0$ for all $y, z \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, \alpha(y)] = 0, \forall z \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, \alpha(y)] = 0$ for all $y, z \in R$. Again, replacing $y$ by $\alpha^{-1}(y)$ in the last relation, we get $[z, y] = 0$ for all $y, z \in R$, which implies that $R$ is commutative.

\[\square\]

Theorem 2.12. Let $R$ be a prime ring and let $g$ be an epimorphism on $R$. If $d$ is an $\alpha$-semiderivation associated with $g$ such that $d(x)d(y) = 0$ for all $x, y \in R$, then $d = 0$.

Proof. By hypothesis, we have

$$d(x)d(y) = 0, \forall x, y \in R. \quad (2.24)$$

Replacing $y$ by $yz$ in (24), we have $d(x)d(yz) = d(x)(d(y)g(z) + \alpha(y)d(z)) = 0$ for all $x, y, z \in R$. Hence $d(x)\alpha(y)d(z) = 0$ for all $x, y, z \in R$. Taking $\alpha^{-1}(y)$ instead of $y$ in the last relation, we get $d(x)yd(z) = 0$ for all $x, y, z \in R$, which implies that $d(x)Rd(z) = \{0\}$ for all $x, z \in R$. Since $R$ is prime, we have $d(x) = 0$ or $d(z) = 0$ for all $x, z \in R$. That is, $d = 0$.

\[\square\]

Theorem 2.13. Let $R$ be a prime ring and let $g$ be an epimorphism on $R$. If $d$ is an $\alpha$-semiderivation associated with $g$ such that $[d(x), g(y)] = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

Proof. By hypothesis, we have

$$[d(x), g(y)] = 0, \forall x, y \in R. \quad (2.25)$$
Replacing $x$ by $xz$ in (25), we have

$$0 = [d(xz), g(y)]$$
$$= [d(x)g(y) + \alpha(x)d(y), g(y)]$$
$$= [d(x)g(y), g(y)] + [\alpha(x)d(y), g(y)]$$
$$= d(x)[g(y), g(y)] + [d(x), g(y)]g(y) + \alpha(x)[d(y), g(y)] + [\alpha(x), g(y)]d(y)$$
$$= [\alpha(x), g(y)]d(y)$$ (2.26)

for all $x, y \in R$. Also, replacing $x$ by $\alpha^{-1}(x)$ in the last relation, we have $[x, g(y)]d(y) = 0$ for all $x, y \in R$. Taking $zx$ instead of $x$ in this relation, we have $[z, g(y)]xd(y) = 0$ for all $x, y, z \in R$. This implies that $[z, g(y)]Rd(y) = \{0\}$ for all $y, z \in R$. Since $R$ is prime, we get $d(y) = 0$ or $[z, g(y)] = 0$ for all $y, x \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, g(y)] = 0, \forall z \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, g(y)] = 0$ for all $y, z \in R$. Since $g$ is onto, we get $[z, y] = 0$ for all $y, z \in R$, which implies that $R$ is commutative.

\[\Box\]

**Theorem 2.14.** Let $R$ be a prime ring and let $g$ be an epimorphism on $R$. If $d$ is an $\alpha$-semiderivation associated with $g$ such that $d(x) \circ g(y) = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

**Proof.** By hypothesis, we have

$$d(x) \circ g(y) = 0, \forall x, y \in R.$$ (2.27)

Replacing $x$ by $xz$ in (27), we have

$$0 = d(xz) \circ g(y)$$
$$= (d(x)g(y) + \alpha(x)d(y)) \circ g(y)$$
$$= d(x)g(y) \circ g(y) + \alpha(x)d(y) \circ g(y)$$
$$= (d(x) \circ g(y))g(y) + d(x)[g(y), g(y)] + \alpha(x)(d(y) \circ g(y)) - [\alpha(x), g(y)]d(y)$$
$$= [\alpha(x), g(y)]d(y)$$ (2.28)

for all $x, y \in R$. Also, replacing $x$ by $\alpha^{-1}(x)$ in the last relation, we have $[x, g(y)]d(y) = 0$ for all $x, y \in R$. Taking $zx$ instead of $x$ in this relation, we have $[z, g(y)]xd(y) = 0$ for all $x, y, z \in R$. This implies that $[z, g(y)]Rd(y) = \{0\}$ for all $y, z \in R$. Since $R$ is prime, we get $d(y) = 0$ or $[z, g(y)] = 0$ for all $y, x \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, g(y)] = 0, \forall z \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, g(y)] = 0$ for all $y, z \in R$. Since $g$ is onto, we get $[z, y] = 0$ for all $y, z \in R$, which implies that $R$ is commutative.

\[\Box\]
References


1 Department of Mathematics, Korea National University of Transportation, Chungju, 27469, Korea.

Email address: ghkim@ut.ac.kr