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ON α-SEMIDERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

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ABSTRACT. In this paper, we introduce the notion of an α-semiderivation on prime rings, and we try to extend some results for derivations of rings or near-rings to a more general case for α-semiderivations of prime rings.

1. INTRODUCTION AND PRELIMINARIES

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [9] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently, refined and extended by a number of authors. In [6], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. Furthermore, Bresar and Vukman [5] studied the notions of a $*$-derivation and a Jordan $*$-derivation of R. In this paper, we introduce the notion of an α-semiderivation on prime rings, and we try to extend some results for derivations of rings or near-rings to a more general case for α-semiderivations of prime rings.

Let R is a ring. Then R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Also, we make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$
$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$

for all $x, y, z \in R$.

Definition 1.1. Let R be a prime ring and α be an automorphism on R. An additive mapping $d : R \to R$ is called a *α-semiderivation* associated with an epimorphism $g : R \to R$ if

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(i) $d(xy) = d(x)g(y) + \alpha(x)d(y) = d(x)\alpha(y) + g(x)d(y)$,
(ii) $d(g(x)) = g(d(x))$, for all $x, y \in R$.

Definition 1.2. Let R be a prime ring and α be an automorphism on R. An additive mapping $d : R \to R$ is called a *reverse α-semiderivation* associated with an epimorphism $g : R \to R$ if
(i) $d(xy) = d(y)g(x) + \alpha(y)d(x) = d(y)\alpha(x) + g(y)d(x)$,
(ii) $d(g(x)) = g(d(x))$, for all $x, y \in R$.

2. α-SEMIDERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

Lemma 2.1. *Let R be a prime ring and let d be a nonzero α-semiderivation associated with g and $a \in R$. If $ad(R) = 0$, then $a = 0$.*

Proof. By hypothesis, we have

$$ad(xy) = 0, \ \forall\ x, y \in R, \tag{2.1}$$

which implies that $ad(x)g(y) + a\alpha(x)d(y)) = 0$ for all $x, y \in R$. By the hypothesis, we have $a\alpha(x)d(y) = 0$ for all $x, y \in R$. Replacing x by $\alpha^{-1}(x)$ in this relation, we get $axd(y) = 0$ for all $x, y \in R$, which implies that $aRd(y) = 0$ for all $y \in R$. Since R is prime and $d \neq 0$, we have $a = 0$.

□

Theorem 2.2. *Let R be a prime ring and g be an epimorphism on R. If R admits an α-semiderivation d associated with g such that $d([x, y]) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d([x, y]) = 0, \ \forall\ x, y \in R. \tag{2.2}$$

Replacing y by yx in (2), we have

$$d([x, yx]) = d([x, y]x) = d([x, y])g(x) + \alpha([x, y])d(x) = 0$$

for all $x, y \in R$. By the hypothesis, we get

$$\alpha([x, y])d(x) = 0, \ \forall\ x, y \in R. \tag{2.3}$$

Taking $\alpha^{-1}([x, y])$ instead of $[x, y]$ in this relation, we have $[x, y]d(x) = 0$ for all $x, y \in R$. Taking zy instead of y with $z \in R$ in this relation, we obtain $[x, z]yd(x) = 0$ for all $x, y, z \in R$. This implies that $[x, z]Rd(x) = \{0\}$ for all $x, z \in R$. Since R is prime, we have $[x, z] = 0$ or $d(x) = 0$ for all $x, z \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | [x, z] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $[x, z] = 0$ for all $x, y \in R$, which implies that R is commutative.

□

Theorem 2.3. *Let R be a prime ring and g be an epimorphism on R. If R admits an α-semiderivation d associated with g such that $d(x \circ y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x \circ y) = 0, \ \forall \ x, y \in R. \tag{2.4}$$

Replacing y by yx in (4), we have $d(x \circ yx) = d((x \circ y)x) = d(x \circ y)g(x) + \alpha((x \circ y))d(x) = 0$ for all $x, y \in R$. By the hypothesis, we get

$$\alpha(x \circ y)d(x) = 0, \ \forall \ x, y \in R. \tag{2.5}$$

Taking $\alpha^{-1}(x \circ y)$ instead of $x \circ y$ in the last relation, we have $(x \circ y)d(x) = 0$ for all $x, y \in R$. Taking yx instead of y in this relation, we obtain $(x \circ y)xd(x) = 0$ for all $x, y \in R$. This implies that $(x \circ y)Rd(x) = \{0\}$ for all $x, y \in R$. Since R is prime, we have $x \circ y = 0$ or $d(x) = 0$ for all $x, y \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | x \circ y = 0, \forall y \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $x \circ y = 0$ for all $x, y \in R$, which implies that $xy = -yx$ for all $x, y \in R$. Again, replacing x by xz in the last relation, we have $xzy = -yxz = xyz$, that is, $x[z, y] = 0$ for all $x, y, z \in R$. This implies that $R[z, y] = \{0\}$ for all $x, z \in R$. Hence $tR[z, y] = \{0\}$ for all $0 \neq t, y, z \in R$. Since R is prime, we have $[z, y] = 0$ for all $y, z \in R$, which implies that R is commutative. □

Theorem 2.4. *Let R be a prime ring and g be an epimorphism on R. If R admits an α-semiderivation d associated with g such that $[d(x), y] = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$[d(x), y] = 0, \ \forall \ x, y \in R. \tag{2.6}$$

Replacing x by xz in (6) and using (6), we have

$$\begin{aligned} 0 &= [d(xz), y] = [d(x)g(z) + \alpha(x)d(z), y] \\ &= [d(x)g(z), y] + [\alpha(x)d(z), y] \\ &= d(x)[g(z), y] + [d(x), y]g(z) + \alpha(x)[d(z), y] + [\alpha(x), y]d(z) \\ &= d(x)[g(z), y] + [\alpha(x), y]d(z) \end{aligned} \tag{2.7}$$

for all $x, y, z \in R$. Taking $g(z)$ instead of y in (7), we have $[\alpha(x), g(z)]d(z) = 0$ for all $x, z \in R$. Substituting $\alpha^{-1}(x)$ for x in this relation, we get $d(x)[x, g(z)] = 0$ for all $x, z \in R$. Again, replacing x by yx in the last relation, we obtain $d(x)y[x, g(z)] = 0$ for all $x, y, z \in R$. Hence $d(x)R[x, g(z)] = 0$ for all $x, y, z \in R$. Since R is prime, we have $d(x) = 0$ or $[x, g(z)] = 0$ for all $x, y\, z \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | [x, g(z)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $[y, g(x)] = 0$ for all $x, y \in R$. Since g is onto, we have $[y, x] = 0$ for all $x, y \in R$, which implies that R is commutative.

□

Theorem 2.5. *Let R be a prime ring and g be an epimorphism on R. If R admits an α-semiderivation d associated with g such that $d(x) \circ y = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x) \circ y = 0, \ \forall \ x, y \in R. \tag{2.8}$$

Replacing x by xz in (8) and using (8), we have

$$\begin{aligned} 0 &= d(xz) \circ y = d(x)g(z) \circ y + \alpha(x)d(z) \circ y \\ &= (d(x) \circ y)g(z) + d(x)[g(z), y] + \alpha(x)(d(z) \circ y) - [\alpha(x), y]d(z) \\ &= d(x)[g(z), y] - [\alpha(x), y]d(z) \end{aligned} \tag{2.9}$$

for all $x, y, z \in R$. Taking $g(z)$ instead of y in (9), we have $[\alpha(x), g(z)]d(z) = 0$ for all $x, z \in R$. Substituting $\alpha^{-1}(x)$ for x in this relation, we get $[x, g(z)]d(z) = 0$ for all $x, z \in R$. Again, replacing x by yx in the last relation, we obtain $[y, g(z)]xd(z) = 0$ for all $x, y, z \in R$. Hence $[y, g(z)]xd(z) = 0$ for all $x, y \in R$. Since R is prime, we have $d(z) = 0$ or $[y, g(z)] = 0$ for all $y, z \in R$. Let $K = \{z \in R | d(z) = 0\}$ and $L = \{y \in R | [y, g(z)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(z) = 0$ for all $z \in R$, that is, $d = 0$. If $L = R$, then we get $[y, g(z)] = 0$ for all $x, y \in R$. Since g is onto, we have $[y, z] = 0$ for all $y, z \in R$, which implies that R is commutative.

□

Theorem 2.6. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $d(xy) = d(x)d(y)$ for all $x, y \in R$, then $d = 0$.*

Proof. For any $x, y \in R$, we have

$$d(xy) = d(x)g(y) + \alpha(x)d(y) = d(x)d(y), \ \forall \ x, y \in R. \tag{2.10}$$

Replacing x by xw in (10), we obtain $d(xw)g(y) + \alpha(xw)d(y) = d(xw)d(y)$ for all $x, y, w \in R$. Hence $d(x)d(w)g(y) + \alpha(w)\alpha(x)d(y) = d(x)d(w)d(y) = d(x)d(wy)$ for all $x, y, w \in R$, and hence $d(x)d(w)g(y) + \alpha(x)\alpha(w)d(y) = d(x)d(w)g(y) + d(x)\alpha(w)d(y)$ for all $x, y, w \in R$. This implies that $(\alpha(x) - d(x))\alpha(w)d(y) = 0$ for all $x, y, w \in R$. Substituting $\alpha^{-1}(w)$ for w in the last relation, we have $(\alpha(x) - d(x))Rd(y) = 0$ for all $x, y \in R$. Since R is prime, we have $d(x) = \alpha(x)$ or $d(y) = 0$ for all $x, y \in R$. Let us assume that $d(x) = \alpha(x)$ for all $x \in R$. Substituting xy for x in the last relation, we have $d(x)g(y) + \alpha(x)d(y) = \alpha(x)\alpha(y) = \alpha(x)d(y)$ for all $x, y \in R$, that is, $d(x)g(y) = 0$ for all $x, y \in R$. Since g is onto, we have $d(x)y = 0$, which implies that $d(x)R = \{0\}$ for all $x \in R$. Thus we obtain $d(x) = 0$ for all $x \in R$ in any case.

□

Theorem 2.7. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $d(xy) = d(y)d(x)$ for all $x, y \in R$ and $\alpha(y) \neq d(y)$ for all $y \in R$, then $d = 0$.*

Proof. For any $x, y \in R$, we have

$$d(xy) = d(x)g(y) + \alpha(x)d(y) = d(y)d(x),\ \forall\ x, y \in R. \tag{2.11}$$

Replacing x by xy in (11), we obtain $d(xy)g(y) + \alpha(xy)d(y) = d(y)d(xy)$ for all $x, y \in R$. Hence we have

$$d(y)d(x)g(y) + \alpha(y)\alpha(x)d(y) = d(y)d(x)g(y) + d(y)\alpha(x)d(y)$$

for all $x, y \in R$, and hence $(\alpha(y)-d(y))\alpha(x)d(y) = 0$ for all $x, y \in R$. Substituting $\alpha^{-1}(x)$ for x in the last relation, we get $(\alpha(y) - d(y))xd(y) = 0$ for all $x, y \in R$. This implies that $(\alpha(y)-d(y))Rd(y) = 0$ for all $y \in R$. Since R is prime, we have $\alpha(y) - d(y) = 0$ or $d(y) = 0$ for all $y \in R$. But $\alpha(y) \neq d(y)$ for all $y \in R$, and so $d(y) = 0$ for all $y \in R$.

□

Theorem 2.8. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $\alpha(xy) = \alpha(y)\alpha(x)$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(xy) = d(x)g(y) + \alpha(x)d(y),\ \forall\ x, y \in R. \tag{2.12}$$

Replacing y by yz in (12), we have

$$\begin{aligned} d(x(yz)) &= d(x)g(yz) + \alpha(x)d(yz) \\ &= d(x)g(y)g(z) + \alpha(x)d(y)g(z) + \alpha(x)\alpha(y)d(z) \end{aligned} \tag{2.13}$$

for all $x, y, z \in R$. On the other hand, we get

$$\begin{aligned} d(xyz) &= d((xy)z) \\ &= d(xy)g(z) + \alpha(xy)d(z) \\ &= d(x)g(y)g(z) + \alpha(x)d(y)g(z) + \alpha(y)\alpha(x)d(z) \end{aligned} \tag{2.14}$$

for all $x, y, z \in R$. Comparing (13) and (14), we have $d(x)[\alpha(y), \alpha(z)] = 0$ for all $x, y, z \in R$. Substituting $\alpha^{-1}(y)$ for y and $\alpha^{-1}(z)$ for z in the last relation Since g is onto, we have $d(x)[y, z] = 0$ for all $x, y, z \in R$. Replacing z by zr, in this relation, we obtain

$$d(x)z[y, r]) = 0,\ \forall\ r, x, z \in R. \tag{2.15}$$

This implies $d(x)R[y, r]$ for all $r, x, y \in R$. Since R is prime, we have $d(x) = 0$ or $[y, z] = 0$ for all $x, y, z \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{z \in R | [y, z] = 0, \forall y \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[g(y), z] = 0$ for all $y, z \in R$. Since g is onto, we obtain $[y, z] = 0$, which implies that R is commutative.

□

Theorem 2.9. *Let R be a prime ring and let g be an epimorphism on R. If d is a reverse α-semiderivation associated with g such that $g(xy) = g(y)g(x)$ for all $x, y \in R$, then $[d(x), z] = 0$ for all $x, z \in R$ or $d = 0$.*

Proof. By hypothesis, we have

$$d(xy) = d(y)g(x) + \alpha(y)d(x),\ \forall\ x, y \in R. \tag{2.16}$$

Replacing x by xz in (16), we have

$$\begin{aligned} d(xzy) &= d(y)g(xz) + \alpha(y)d(xz) \\ &= d(y)g(z)g(x) + \alpha(y)d(z)g(x) + \alpha(y)\alpha(z)d(x) \end{aligned} \tag{2.17}$$

for all $x, y, z \in R$. On the other hand,

$$\begin{aligned} d(xzy) &= d(x(zy)) = d(zy)g(x) + \alpha(zy)d(x) \\ &= d(y)g(z)g(x) + \alpha(y)d(z)g(x) + \alpha(z)\alpha(y)d(x) \end{aligned} \tag{2.18}$$

Comparing (17) with (18), we get $[\alpha(z), \alpha(y)]d(x)$ for all $x, y, z \in R$. Again, replacing y by $\alpha^{-1}(y)$ and z by $\alpha^{-1}(z)$ in this relation, we have $[z, y]d(x) = 0$ for all $x, y, z \in R$. Taking $d(x)z$ instead of z in this relation, we have

$$\begin{aligned} 0 &= [d(x)z, z]d(x) \\ &= d(x)[z, z]d(x) + [d(x), z]zd(x) \\ &= [d(x), z]zd(x) \end{aligned} \tag{2.19}$$

This implies that $[d(x), z]Rd(x) = \{0\}$ for all $x, z \in R$. Since R is prime, we get either $[d(x), z] = 0$ or $d(x) = 0$ for all $x, z \in R$.

□

Theorem 2.10. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $d(x) \circ \alpha(y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x) \circ \alpha(y) = 0,\ \forall\ x, y \in R. \tag{2.20}$$

Replacing x by yx in (20), we have

$$\begin{aligned} 0 &= d(yx) \circ \alpha(y) \\ &= (d(y)g(x) + \alpha(y)d(x)) \circ \alpha(y) \\ &= d(y)g(x) \circ \alpha(y) + \alpha(y)d(x) \circ \alpha(y) \\ &= (d(y) \circ \alpha(y))g(x) + d(y)[g(x), \alpha(y)] + \alpha(y)(d(x) \circ \alpha(y)) - [\alpha(y), \alpha(y)]d(x) \\ &= d(y)[g(x), \alpha(y)] \end{aligned} \tag{2.21}$$

for every $x, y \in R$. Since g is onto, we get $d(y)[x, \alpha(y)] = 0$ for all $x, y \in R$. Taking xz instead of x in this relation, we obtain $d(y)x[z, \alpha(y)] = 0$ for all $x, y, z \in R$. This implies that $d(y)R[z, \alpha(y)] = \{0\}$ for all $y, z \in R$. Since R is prime, we have $d(y) = 0$ or $[z, \alpha(y)] = 0$ for all $y, z \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, \alpha(y)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, \alpha(y)] = 0$ for all $y, z \in R$. Again, replacing y by $\alpha^{-1}(y)$ in the last relation, we get $[z, y] = 0$ for all $y, z \in R$, which implies that R is commutative.

□

Theorem 2.11. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $[d(x),\alpha(y)]=0$ for all $x,y\in R$, then $d=0$ or R is commutative.*

Proof. By hypothesis, we have

$$[d(x),\alpha(y)]=0,\ \forall\ x,y\in R. \tag{2.22}$$

Replacing x by yx in (22), we have

$$\begin{aligned}0&=[d(yx),\alpha(y)]\\&=[d(y)g(x)+\alpha(y)d(x),\alpha(y)]\\&=[d(y)g(x),\alpha(y)]+[\alpha(y)d(x),\alpha(y)]\\&=d(y)[g(x),\alpha(y)]+[d(y),\alpha(y)]g(x)+\alpha(y)[d(x),\alpha(y)]+[\alpha(y),\alpha(y)]d(x)\\&=d(y)[g(x),\alpha(y)]\end{aligned} \tag{2.23}$$

for every $x,y\in R$. Since g is onto, we get $d(y)[x,\alpha(y)]=0$ for all $x,y\in R$. Taking xz instead of x in this relation, we obtain $d(y)x[z,\alpha(y)]=0$ for all $x,y,z\in R$. This implies that $d(y)R[z,\alpha(y)]=\{0\}$ for all $y,z\in R$. Since R is prime, we have $d(y)=0$ or $[z,\alpha(y)]=0$ for all $y,z\in R$. Let $K=\{y\in R|d(y)=0\}$ and $L=\{y\in R|[z,\alpha(y)]=0,\forall z\in R\}$. Then K and L are both additive subgroups and $K\cup L=R$, but $(R,+)$ is not union of two its proper subgroups, which implies that either $K=R$ or $L=R$. In the former case, we have $d=0$. If $L=R$, then $[z,\alpha(y)]=0$ for all $y,z\in R$. Again, replacing y by $\alpha^{-1}(y)$ in the last relation, we get $[z,y]=0$ for all $y,z\in R$, which implies that R is commutative.

□

Theorem 2.12. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $d(x)d(y)=0$ for all $x,y\in R$, then $d=0$.*

Proof. By hypothesis, we have

$$d(x)d(y)=0,\ \forall\ x,y\in R. \tag{2.24}$$

Replacing y by yz in (24), we have $d(x)d(yz)=d(x)(d(y)g(z)+\alpha(y)d(z))=0$ for all $x,y,z\in R$. Hence $d(x)\alpha(y)d(z)=0$ for all $x,y,z\in R$. Taking $\alpha^{-1}(y)$ instead of y in the last relation, we get $d(x)yd(z)=0$ for all $x,y,z\in R$, which implies that $d(x)Rd(z)=\{0\}$ for all $x,z\in R$. Since R is prime, we have $d(x)=0$ or $d(z)=0$ for all $x,z\in R$. That is, $d=0$.

□

Theorem 2.13. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $[d(x),g(y)]=0$ for all $x,y\in R$, then $d=0$ or R is commutative.*

Proof. By hypothesis, we have

$$[d(x),g(y)]=0,\ \forall\ x,y\in R. \tag{2.25}$$

Replacing x by xz in (25), we have

$$\begin{aligned}0 &= [d(xz), g(y)]\\ &= [d(x)g(y) + \alpha(x)d(y), g(y)]\\ &= [d(x)g(y), g(y)] + [\alpha(x)d(y), g(y)]\\ &= d(x)[g(y), g(y)] + [d(x), g(y)]g(y) + \alpha(x)[d(y), g(y)] + [\alpha(x), g(y)]d(y)\\ &= [\alpha(x), g(y)]d(y) \qquad (2.26)\end{aligned}$$

for all $x, y \in R$. Also, replacing x by $\alpha^{-1}(x)$ in the last relation, we have $[x, g(y)]d(y) = 0$ for all $x, y \in R$. Taking zx instead of x in this relation, we have $[z, g(y)]xd(y) = 0$ for all $x, y, z \in R$. This implies that $[z, g(y)]Rd(y) = \{0\}$ for all $y, z \in R$. Since R is prime, we get $d(y) = 0$ or $[z, g(y)] = 0$ for all $y, x \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, g(y)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, g(y)] = 0$ for all $y, z \in R$. Since g is onto, we get $[z, y] = 0$ for all $y, z \in R$, which implies that R is commutative.

□

Theorem 2.14. *Let R be a prime ring and let g be an epimorphism on R. If d is an α-semiderivation associated with g such that $d(x) \circ g(y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x) \circ g(y) = 0, \ \forall\, x, y \in R. \qquad (2.27)$$

Replacing x by xz in (27), we have

$$\begin{aligned}0 &= d(xy) \circ g(y)\\ &= (d(x)g(y) + \alpha(x)d(y)) \circ g(y)\\ &= d(x)g(y) \circ g(y) + \alpha(x)d(y) \circ g(y)\\ &= (d(x) \circ g(y))g(y) + d(x)[g(y), g(y)] + \alpha(x)(d(y) \circ g(y)) - [\alpha(x), g(y)]d(y)\\ &= [\alpha(x), g(y)]d(y) \qquad (2.28)\end{aligned}$$

for all $x, y \in R$. Also, replacing x by $\alpha^{-1}(x)$ in the last relation, we have $[x, g(y)]d(y) = 0$ for all $x, y \in R$. Taking zx instead of x in this relation, we have $[z, g(y)]xd(y) = 0$ for all $x, y, z \in R$. This implies that $[z, g(y)]Rd(y) = \{0\}$ for all $y, z \in R$. Since R is prime, we get $d(y) = 0$ or $[z, g(y)] = 0$ for all $y, x \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, g(y)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, g(y)] = 0$ for all $y, z \in R$. Since g is onto, we get $[z, y] = 0$ for all $y, z \in R$, which implies that R is commutative.

□

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