Gulf Journal of Mathematics
Vol 20, Issue 1 (2025) 338-359
ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i.2957

LOCAL LINEAR ESTIMATION OF CONDITIONAL PROBABILITY DENSITY AND MODE UNDER RIGHT CENSORING AND LEFT TRUNCATION: DEPENDENT DATA CASE

H. DABANA[1*], K. AGBOKOU[2] and K. E. GNEYOU[3]

Abstract. In this work, we propose a local linear estimator of the conditional density and mode when the covariate is functional, using the right-censored and left-truncated data model. Under conditions of weak dependence, we study the almost sure pointwise convergence of the proposed estimators and then evaluate their performance on simulated datasets.

1. Introduction

Nonparametric estimation is a field of statistics that focuses on modeling data without making prior assumptions about their underlying distribution. In many cases, the available data can be quite incomplete, meaning that part of the information is missing or inaccessible (possible presence of censorship and/or truncation). For example, one studies the lifespans after retirement of subjects who enter the survey following a random selection from a retirement fund. A subject is therefore only observed if their lifespan after retirement exceeds the time between their retirement and the moment of the survey. The lifespan after retirement is thus left-truncated by this time. It can also be right-censored if the end of the survey occurs while the subject is still alive. These phenomena are often encountered in survival analysis. In survival analysis, one often needs to model and analyze the probabilistic relationships between a variable representing a lifespan and a number of factors (covariates or explanatory variables) to make predictions based on these relationships. For example, calculating the probability that a random variable takes a certain value based on the observed value of another variable. In this sense, the conditional density is frequently used, and its applications are quite varied (in econometrics, signal processing, image processing, machine learning, and environmental sciences to name a few). For more precision in the field of signal processing, the conditional density can be used to estimate the probability that an event occurs in a signal based on the observation of another signal. This can be useful for event detection, noise reduction, data compression, or other

Date: Received: May 1, 2025; Accepted: May 25, 2025.
* Corresponding author.
2020 *Mathematics Subject Classification.* 62G05; 62G07; 62N02; 62R10.
Key words and phrases. Nonparametric estimation; conditional density; censored data; truncated data; functional random variable.

signal processing operations. In image processing, conditional density plays a crucial role by modeling the probabilistic relationships between the pixels of an image and using this information to perform various image processing operations more accurately and efficiently, such as object detection, image segmentation, image restoration, and pattern recognition.

Many authors have focused on nonparametric estimation of conditional density. Historically, nonparametric estimation of conditional density was first introduced in the statistical literature by [30]; [29] through the kernel method and the associated bandwidth selection problem, which concerns the choice of the appropriate smoothing quality ([31]; [18]; [8]). The subject has been developed on incomplete data with several methods by other authors, including for censored data [26]; [20]; [35]; [32]; [33]; [21]; [1] and [11]. Regarding left-truncated data, [27] studied the conditional mode estimator in the i.i.d case. The authors also studied the convergence rate of this estimator in the α-mixing case (see [28]). [23] examine the asymptotic normality of the Nadaraya-Watson type estimator and the local linear estimator of conditional density with this type of data under weak dependence conditions (α-mixing data) in the presence of true covariates. [24] establish the asymptotic normality of the conditional mode estimator in the presence of left-truncated data in the dependent case. The LTRC data model has also received attention in the literature. To this end, [3] studied a strong representation and strong results of uniform convergence of the conditional hazard function estimator and its maximum. [4] extended this work on the study of asymptotic normality. [25] defined a kernel estimator of the conditional density for the left-truncated and right-censored model in the presence of a multivariate covariate. The authors investigated the uniform convergence and asymptotic normality of the proposed estimator in the case of dependent data. [9] adapted the work of [25] to the case of the real covariate. Regarding the mode, authors such as [12] or for example in [16]; [17]; [2]; [4] and [11] for the case of censored data; [27]; [28] and [24] for left-truncated data.

The main objective of this work is to propose a local linear estimator of the conditional density and mode when the response variable is real and subject to right censoring and left truncation (LTRC) in the presence of a functional covariate. We will study the asymptotic properties of these estimators under strong mixing conditions. This work constitutes a generalization of the work of [1] on the LTRC model. It is also a generalization of the work of [25] and [9]. This article is organized as follows. We present the estimators in Section (2); Section (3) will discuss the assumptions that allowed us to obtain the results. Section (4) will be reserved for the presentation of the main results such as the almost sure pointwise convergence of the conditional density estimator and the almost sure uniform convergence of the mode estimator. We then evaluate the performance of these proposed estimators on simulated datasets in Section (5) and in Section (6), we will find the proofs of the stated lemmas.

2. Notations and definitions of estimators

2.1. **Notations.** Let (X, T, Z, C) be a random vector where T is the random variable (r.v.) of interest taking values in $\mathbb{R}$ (lifetime); Z is the left truncation; C is the right censoring and X is a covariate taking values in the semi-metric space $(\mathcal{F}, d(.;.))$ related to Y. Let us denote $Y = \min(T, C)$ and $\delta = \mathbb{I}_{\{T \leq C\}}$. Assuming that T, Z, and C are pairwise independent conditional on X, the r.v.s actually observed are (X, Y, Z, δ) if $Y \geq Z$. When $Y < Z$, nothing is observed.
Consider $(X_i, T_i, Z_i, C_i)_{1 \leq i \leq n}$ with $n \in \mathbb{N}$, an i.i.d sample from the vector (X, T, Z, C) and $(X_i, Y_i, Z_i, \delta_i)_{1 \leq i \leq N}$ with $N \in \mathbb{N}$ such that $N \leq n$, the actually observed sample, meaning $Z_i \leq Y_i \ \forall i \in \{1, \cdots, N\}$. The size n is deterministic but unknown; the sample of size n that is actually observed is a random variable distributed according to the Binomial law with parameters n and μ where $\mu = \mathbb{P}(Z \leq Y)$ $(\mu \neq 0)$. In this sense, to avoid confusion, we denote:

$\mathbb{P}$: the probability related to the n-sample,
P: the probability related to the N-sample.

$\mathbb{E}$ and E denote the expectations related to $\mathbb{P}$ and P respectively.

We also consider in this work:

a) $F(u) = \mathbb{P}(T \leq u)$ the cumulative distribution function of the variable T,
b) $M(u) = \mathbb{P}(X \leq u)$ the cumulative distribution function of the variable X,
c) $L(u) = \mathbb{P}(Z \leq u)$ the cumulative distribution function of the truncation variable Z,
d) $H(u) = \mathbb{P}(Y \leq u)$ the cumulative distribution function of the variable Y,
e) $H_1(u) = \mathbb{P}(Y \leq u, \delta = 1)$ the cumulative distribution function of the uncensored observation,
f) $C(u) = \mathbb{P}(Z \leq u \leq Y | Z \leq Y) = P(Z \leq u \leq Y)$ the cumulative risk.

For any cumulative distribution function W, let us define

$$a_W = \inf\{x : W(x) > 0\} \quad et \quad b_W = \sup\{x : W(x) < 1\} \tag{2.1}$$

the lower limit and the upper limit respectively of the support of W.

Considering these notations, we define the following estimators.

2.2. **Definitions of Estimators.** By the strong law of large numbers, we have as $n \to \infty$ $\hat{\mu}_n = \dfrac{N}{n} \to \mu \ \mathbb{P} - a.s.$

$H_1^*(u)$ is defined by:

$$\begin{aligned} H_1^*(u) &= P(Y \leq u, \delta = 1) \\ H_1^*(u) &= \mu^{-1} \int_{-\infty}^{u} L(U)(1 - G(U^-))dF(U) \end{aligned} \tag{2.2}$$

and the consistent estimator of $H_1^*(u)$ is given by:

$$H_{1n}^*(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{Y_i \leq u, \delta_i = 1\}}. \tag{2.3}$$

We also have $C(u) = \mu^{-1}L(u)(1-F(u))(1-G(u))$ and by replacing F, L, and G with their estimators, we get:

$$\mu_n = \frac{L_n(u)\big(1-\hat{F}_n(u^-)\big)\big(1-G_n(u)\big)}{C_n(u)}, \tag{2.4}$$

$\forall y : C_n(u) > 0$ where $\hat{F}_n(u^-)$ is the lower limit of $\hat{F}_n(u)$ at y. L_n is the limit product estimator of Lynden-Bell of L defined by:

$$L_n(u) = \prod_{\{i:Z_i>u\}} \Big(1 - \frac{1}{nC_{n(Z_i)}}\Big) \tag{2.5}$$

and G_n, the limit product estimator of Kaplan and Meier of G defined as in [9] by:

$$G_n(u) = 1 - \prod_{\{i:T_i\leq u\}} \Big(1 - \frac{1-\delta_i}{n-i+1}\Big). \tag{2.6}$$

Given the quadruplets of observed random variables $(X_i, Y_i, Z_i, \delta_i),\ i = 1, \cdots, n$ where $(X_i)_{i\in\{1,\cdots,n\}}$ are values in $\mathcal{F}$ with $(\mathcal{F}, d)$, a semi-metric space of infinite dimension. Adopting the ideas of [9], the kernel estimator of the conditional density $f(t|x)$ for the LTRC model in the presence of functional covariate is given by:

$$\hat{f}_n(t|x) = \frac{\dfrac{\mu_n}{nh_n^2}\displaystyle\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)L_n(Y_i)}K\big(h_K^{-1}d(x,X_i)\big)H\Big(\frac{t-Y_i}{h_H}\Big)}{\dfrac{\mu_n}{nh_n}\displaystyle\sum_{i=1}^{n}\frac{1}{L_n(Y_i)}K\big(h_K^{-1}d(x,X_i)\big)}. \tag{2.7}$$

Note that this estimator was introduced for the first time in the literature by [9] for the case of the real covariate. We now define a new estimator for the conditional density for this model based on the linear local method. To do this, we combine the ideas of [6], [35], and [7]. Thus, the local linear estimator of $f(t|x)$ can be seen as the solution to the following minimization problem:

$$\min_{(a,b)\in\mathbb{R}^2}\sum_{i=1}^{n}\Big(\delta_i\bar{G}_n^{-1}(Y_i)h_H^{-1}H(h_H^{-1}(t-Y_i)) - a - b\beta(X_i,x)\Big)^2 \times$$
$$K(h_K^{-1}\delta(x,X_i))L_n^{-1}(Y_i), \tag{2.8}$$

where $\beta(.,.)$ is a known function of $\mathcal{F}^2$ in $\mathbb{R}$, such that $\forall\xi\in\mathcal{F}, \beta(\xi,\xi) = 0$, where K and H are kernels, $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers, and $\delta(.,.)$ is a function from $\mathcal{F}\times\mathcal{F}$, such that $d(.,.) = |\delta(.,.)|$. Here, we denote by a, $f_n(t|x)$.
According to (2.8) we can write (see [35] equation (2.2))

$$\hat{f}_n(t|x) = e_1^t(P^tWP)^{-1}P'WQ \tag{2.9}$$

with

$$P = \begin{bmatrix} 1 & \beta(X_1, x) \\ \vdots & \vdots \\ 1 & \beta(X_n, x) \end{bmatrix}, \qquad Q = \begin{bmatrix} \delta_1 \bar{G}_n^{-1}(Y_1) h_H^{-1} H(h_H^{-1}(t - Y_1)) \\ \vdots \\ \delta_n \bar{G}_n^{-1}(Y_n) h_H^{-1} H(h_H^{-1}(t - Y_n)) \end{bmatrix},$$

$$W = diag\Big(K(h_K^{-1}\delta(X_i, x))L_n^{-1}(Z_i)\Big) \quad \text{where} \quad diag(a) = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

and

$$e_1 \quad = \quad (1, 0) \in \mathbb{R}^2.$$

e_1^t and P^t are respectively the transposes of e_1 and P.
By simple algebraic calculations, we can find the following formula for our estimator:

$$\hat{f}_n(t/x) \quad = \quad \frac{\displaystyle\sum_{i,j=1}^{n} \delta_i \bar{G}_n^{-1}(Y_i) W_{ij}(x) H(h_H^{-1}(t - Y_i))}{\displaystyle h_H \sum_{i,j=1}^{n} W_{ij}(x)},$$

where

$$W_{ij} = \beta(X_i, x)\Big[\beta(X_i, x) - \beta(X_j, x)\Big]K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j))L_n^{-1}(Y_i)L_n^{-1}(Y_j).$$

We will note Δ_{ij} by:

$$\Delta_{ij}(x) \quad = \quad \beta(X_i, x)\Big[\beta(X_i, x) - \beta(X_j, x)\Big]K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j)),$$

to obtain:

$$\hat{f}_n(t/x) \quad = \quad \frac{\displaystyle\sum_{i,j=1}^{n} \delta_i \bar{G}_n^{-1}(Y_i) L_n^{-1}(Y_i) L_n^{-1}(Y_j) \Delta_{ij}(x) H(h_H^{-1}(t - Y_i))}{\displaystyle h_H \sum_{i,j=1}^{n} L_n^{-1}(Y_i) L_n^{-1}(Y_j) \Delta_{ij}(x)}. \qquad (2.10)$$

We specify that $\bar{G}_n^{-1}$ and L_n^{-1} are the inverses of $\bar{G}_n$ and L_n respectively.
We assume that there exists a certain compact set $\mathcal{S} \subset \mathbb{R}$ such that $f(t|x)$) has a unique mode $\theta(x)$ on $\mathcal{S}$. The estimator of mode $\hat{\theta}_n(x)$ is a random variable such that:

$$\hat{\theta}_n(x) \quad = \quad \arg\max_{t \in \mathcal{S}} \hat{f}_n(t|x). \qquad (2.11)$$

3. Assumptions

Let N_x be a fixed neighborhood of x. To achieve the results, we introduce the following assumptions.
Considering the bounds given by equality (2.1), suppose that $a_L \leq a_H,\ b_L \leq b_H$ and $\mathcal{S} = [a, b]$ is a compact such that $\mathcal{S} \subset \mathcal{S}_0 = \{t : t \in [a_H, b_H]\}$. Let us note $B(x, r) = \{x' \in \mathcal{F} : |\delta(x', x)| \leq r\}$ and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$.
Also note that our nonparametric method is quite general in the sense that we only need the following assumptions:

(H1) for all $r > 0,\ \phi_x(r) := \phi_x(-r, r) > 0$

(H2) (i) The conditional density $f(t|x)$ is such that: $\exists\, b_1 > 0,\ b_2 > 0, \forall (t_1, t_2) \in \mathcal{S}^2$ and $\forall (x_1, x_2) \in N_x \times N_x$:

$$|f(t_1|x_1) - f(t_2|x_2)| \leq C_x\Big(|\delta^{b_1}(x_1, x_2)| + |t_1 - t_2|^{b_2}\Big),$$

where C_x is a positive constant depending on x.

(ii) $f(t|x)$ is twice differentiable, its second derivative $f^{(2)}(t|x)$ is continuous in the neighborhood of $\theta(x)$ and $f^{(2)}(\theta(x)|x) < 0$.

(H3) The function $\beta(.,.)$ is such that:

$$\forall x' \in \mathcal{F},\ C_1|\delta(x, x') \leq |\beta(x, x')| \leq C_2|\delta(x, x')|.$$

(H4) The sequence $(X_i, T_i)_{i\in\mathbb{N}}$ satisfies : $\exists a > 0, \exists c > 0,\ \forall n \in \mathbb{N}\ \alpha(n) \leq cn^{-a}$ and:

$$\max_{i\neq j} \mathbb{P}\Big((X_i, X_j) \in B(x, h) \times B(x, h)\Big) = \varphi_x(h) > 0$$

(H5) The conditional density of (T_i, T_j) given (X_i, X_j) exists and is bounded.

(H6) K is a positive differentiable function with support $[-1, 1]$.

(H7) H is a continuous positive function, bounded, Lipschitzian, such that:

$$\int |u|^{b_2} H(u)du < \infty \text{ and } \int H^2(u)du < \infty.$$

(H8) The bandwidth h_K satisfies: there exists an integer n_0, such that:

$$\forall n > n_0, -\frac{1}{\phi_x(h_K)} \int_{-1}^{1} \phi_x(uh_K, h_K) \frac{d}{du}\big(u^2 K(u)\big)du\ > C_3 > 0,$$

and

$$h_K \int_{B(x,h_K)} \beta(u, x)dP(u) = \mathcal{O}\Big(\int_{B(x,h_K)} \beta(u, x)^2 dP(u)\Big),$$

where $P(u)$ is the cumulative distribution.

(H9) $\lim\limits_{n\to\infty} h_H = 0,$ and $\exists \beta_1 > 0$, such that: $\lim\limits_{n\to\infty} n^{\beta_1} h_H = \infty.$

(H10) (i) $\lim\limits_{n\to\infty} h_K = 0,\ \lim\limits_{n\to\infty} \dfrac{\chi_x^{1/2}(h_K)\log(n)}{nh_H\phi_x^2(h_K)} = 0.$

(ii) $Cn^{\frac{(3-a)}{a+1}+\frac{3\beta_1+1}{a+1}} \log n[\log\log n]^{6/(a+1)} \leq h_H\chi_x^{1/2}(h_K)$, where $\chi_x(h) = \max(\phi_x^2(h), \varphi_x(h))$.

(H11) $f(t|x) < f(\theta(x)|x)\ \forall t \neq \theta(x),\ t \in \mathcal{S}$.

Remark 3.1 (Comments on assumptions). Most of the hypotheses are common in the context of non-parametric functional data analysis (NFDA). More specifically, hypothesis (H1) is typically used in NFDA and is related to the topological structure of the functional space $\mathcal{F}$, of the explanatory variable X (see [15] for further discussion). Furthermore, hypotheses (H2) and (H3) are moderate regularity hypotheses on the conditional density function. These hypotheses are also common in the literature (see, for example, [1]). Finally, conditions (H4), (H5), and (H10) are technical hypotheses (see [15] for the case of the local constant method).

4. Main Results

4.1. Almost Sure Pointwise Convergence of the Conditional Density.

In this Subsection, we examine the asymptotic behavior of the estimator (2.10) for a fixed point x in $\mathcal{F}$.
The following theorem provides the almost sure pointwise convergence of the estimator (2.10).

Theorem 4.1. *Under the hypotheses (H1), (H2i), (H3)-(H10), we have:*

$$\sup_{t\in\mathcal{S}}\left|\hat{f}_n(t|x) - f(t|x)\right| = \mathcal{O}_{a.s.}\left(h_K^{b_1}\right) + \mathcal{O}_{a.s.}\left(h_H^{b_2}\right) + \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{4.1}$$

Proof of theorem 4.1. For the demonstration of theorem 4.1, we need to determine the following pseudo-estimators:

$$\hat{f}_n(x,t) = \frac{\mu_n^2}{n(n-1)h_H E(\Delta_{12}(x))}\sum_{i,j=1}^{n}\frac{\delta_i\Delta_{ij}(x)H(h_H^{-1}(t-Y_i))}{\bar{G}_n(Y_i)L_n(Y_j)L_n(Y_i)}; \tag{4.2}$$

$$\tilde{f}_n(x,t) = \frac{\mu^2}{n(n-1)h_H E(\Delta_{12}(x))}\sum_{i,j=1}^{n}\frac{\delta_i\Delta_{ij}(x)H(h_H^{-1}(t-Y_i))}{\bar{G}(Y_i)L(Y_j)L(Y_i)}; \tag{4.3}$$

$$\hat{m}_n(x) = \frac{\mu_n^2}{n(n-1)E(\Delta_{12}(x))}\sum_{i,j=1}^{n}L_n^{-1}(Y_j)L_n^{-1}(Y_i)\Delta_{ij}(x) \tag{4.4}$$

and

$$\tilde{m}_n(x) = \frac{\mu^2}{n(n-1)E(\Delta_{12}(x))}\sum_{i,j=1}^{n}L^{-1}(Y_j)L^{-1}(Y_i)\Delta_{ij}(x). \tag{4.5}$$

Thus, the proof follows on the one hand from the decomposition

$$\begin{aligned}\hat{f}_n(t|x) - f(t|x) &= \frac{1}{\hat{m}_n(x)}\Bigg(\Big[\hat{f}_n(x,t) - \tilde{f}_n(x,t)\Big] + \Big[\tilde{f}_n(x,t) - E\tilde{f}_n(x,t)\Big] \\ &\quad + \Big[E\tilde{f}_n(x,t) - f(t|x)\Big]\Bigg) + \frac{f(t|x)}{\hat{m}_n(x)}\Bigg(\Big[\tilde{m}_n(x) - \hat{m}_n(x)\Big] \\ &\quad + \Big[E\tilde{m}_n(x) - \tilde{m}_n(x)\Big] + \Big[1 - E\tilde{m}_n(x)\Big]\Bigg)\end{aligned}$$

and on the other hand from the following lemmas:

Lemma 4.2. *Under the hypotheses (H1), (H6) and (H7) we have:*

$$\big|\hat{m}_n(x) - \tilde{m}_n(x)\big| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log(\log n)}{n}}\right). \tag{4.6}$$

$$\sup_{t\in\mathcal{S}}\Big|\hat{f}_n(x,t) - \tilde{f}_n(x,t)\Big| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log(\log n)}{n}}\right), \tag{4.7}$$

Lemma 4.3. *Under the hypotheses (H2i) and (H7), we have:*

$$\sup_{t\in\mathcal{S}}\Big|E\tilde{f}_n(x,t) - f(t|x)\Big| = \mathcal{O}_{a.s.}\Big(h_K^{b_1}\Big) + \mathcal{O}_{a.s.}\Big(h_H^{b_2}\Big).$$

Lemma 4.4. *Under the hypotheses of the theorem (4.1) we have:*

$$\sup_{t\in\mathcal{S}}\Big|\tilde{f}_n(x,t) - E\tilde{f}_n(x,t)\Big| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right), \tag{4.8}$$

$$\big|1 - \tilde{m}_n(x)\big| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{4.9}$$

□

4.2. **Almost Sure Convergence of the Mode Estimator.** The rate of convergence of the local linear mode estimator is a direct consequence of the previous results.

Corollary 4.5. *Under the hypotheses (H1)-(H11), we have:*

$$\big|\hat{\theta}_n(x) - \theta(x)\big| = \mathcal{O}_{a.s.}\Big(h_K^{b_1/2} + h_H^{b_2/2}\Big) + \mathcal{O}_{a.s.}\left(\left(\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}\right)^{\frac{1}{4}}\right).$$

Proof of corollary 4.5. Following the same approach as the proof of theorem 2 by [2] and that of theorem 3.7 by [1], we will have:

$$
\begin{aligned}
\Big|f\big(\hat{\theta}_n(x)|x\big)-f\big(\theta(x)|x\big)\Big| &\leq \Big|f\big(\hat{\theta}_n(x)|x\big)-\hat{f}_n\big(\hat{\theta}_n(x)|x\big)\Big|+\Big|\hat{f}_n\big(\hat{\theta}_n(x)|x\big)-f\big(\theta(x)|x\big)\Big| \\
&\leq \sup_{t\in\mathcal{S}}\Big|\hat{f}_n(t|x)-f(t|x)\Big|+\sup_{t\in\mathcal{S}}\Big|\hat{f}_n(t|x)-f(t|x)\Big| \\
&\leq 2\sup_{t\in\mathcal{S}}\Big|\hat{f}_n(t|x)-f(t|x)\Big|. \qquad (4.10)
\end{aligned}
$$

Using a Taylor expansion of the function $f(.|x)$ we obtain:

$$
f\big(\hat{\theta}_n(x)|x\big)-f\big(\theta(x)|x\big)=\frac{1}{2}\big(\hat{\theta}_n(x)-\theta(x)\big)^2 f^{(2)}\big(\theta^*(x)|x\big) \qquad (4.11)
$$

where $\theta^*(x)$ is between $\theta(x)$ and $\hat{\theta}_n(x)$.
By combining (4.10) and (4.11) we find:

$$
\Big|\hat{\theta}_n(x)-\theta(x)\Big|\leq 2\sqrt{\frac{\sup_{t\in\mathcal{S}}\Big|\hat{f}_n(t|x)-f(t|x)\Big|}{f^{(2)}\big(\theta^*(x)|x\big)}}.
$$

Thus, with the theorem 4.1 the proof of the corollary 4.5 is completed. □

5. Numerical study

In this Section, we illustrate the behavior of our estimators $\hat{f}_n(t|x)$ and $\hat{\theta}_n$ defined in Subsection 2.2 respectively by (2.10) and (2.11) and we examine their convergence. We consider the Epanechnikov kernel defined by: $K(x)=H(x)=\frac{3}{4}\big(1-x^2\big)\mathbb{I}_{[-1;1]}(x)$. The smoothing parameters h_H and h_K, the semi-metric d are obtained by the same principle as in [11]. We generate the functional covariate $(X_i)_{1\leq i\leq n}$ assumed to be a diffusion process on [0,1] generated by the following equation (see [1]):

$$
X(t)=A\big(2-\cos(\pi tW)\big)+(1-A)\cos(\pi tW),\ t\in[0,1]
$$

where $W\rightsquigarrow\mathcal{N}(0,1)$ and A is a Bernoulli random variable with parameter $p=0,5$. To control the effect of censorship and truncation on the efficiency of our estimators, we simulate the n i.i.d. censorship variables $(C_i)_{1\leq i\leq n}$ and truncation variables $(Z_i)_{1\leq i\leq n}$ respectively through the exponential distributions $\mathcal{E}(\lambda_1)$ and $\mathcal{E}(\lambda_2)$ adapted in order to obtain different censorship rate (CR) and truncation rate (TR). The real response variable T is defined by the same principle as in [11]. With this model, the conditional density of T given $X=x$ can be defined by relation (13) of [11]).

We determine and illustrate under this model the performance of the estimators (2.10) and (2.11) by following the following steps:

Step 1: We set the sample size n (remember that n is deterministic but unknown) and we generate the random variables $X_i,\ T_i,\ C_i$ and Z_i such that $1\leq i\leq n$

Step 2: We determine the actually observed sample as follows:

✓ First, we determine Y_i such that $Y_i = \min(T_i; C_i)$ to obtain the sample $(X_i, Y_i, Z_i, \delta_{(i)})_{1\le i\le n}$ with $\delta_i = \mathbb{I}_{\{T_i \le C_i\}}$.

✓ To obtain the actually observed sample, we perform the following test:

We initialize N=0, i=0. While $i \le n$, we get $N = N + 1$ and we test: if $Y_i < Z_i$, we reject the quadruplet $(X_i, Y_i, Z_i, \delta_{(i)})$. Otherwise, we keep it. At the end of the counts, we obtain N which allows us to get $\mu = \dfrac{N}{n}$ (the proportion of absence of truncation). The truncation rate (TR) is thus given by $TR = (1 - \mu) \times 100$.

Step 3: To verify the convergence of our estimators, we set $\lambda_1 = 0.01$ and $\lambda_2 = 0.15$ and determine the mean squared errors (MSE) of the estimators for each of following sizes n: 100; 200 and 300.

The mean squared error of each estimator is calculated over a grid of 200 points using the following formulas:

$$\begin{aligned} MSE(f_n, f) &= \frac{1}{200}\sum_{j=1}^{200}\Big(\hat{f}_{nj}(t|x) - f_j(t|x)\Big)^2 \\ &\mathit{and} \\ MSE(\theta_n, \theta) &= \frac{1}{200}\sum_{j=1}^{200}\Big(\hat{\theta}_{nj}(x) - \theta_j(x)\Big)^2. \end{aligned}$$

The obtained results are presented in Table 1.

Step 4: To observe the effect of the censorship rate (CR) and the truncation rate (TR) on the convergence speed of our estimators, we first set $\lambda_2 = 0.25$ (approximately 30% truncation rate since the variation of CR impacts TR) and we vary λ_1 on 0.0001; 0.06 and 0.1 corresponding to censorship rates of approximately 0%; 25% and 35% respectively. In a second step, we set $\lambda_1 = 0.15$ (CR = 25% approximately) and we vary λ_2 on 0.15; 0.2 and 0.4 corresponding to truncation rates of approximately 55%; 40% et 20% respectively. We compare the MSE with the variation of these rates for each of the considered sample sizes n. The obtained results are presented in tables 2 and 3.

Step 5: We illustrate the performance of these estimators as stated in steps 3 and 4 by visualizing, on one hand, the real conditional density as well as its estimator and on the other hand, the mode of each of the two functions in each graph. (see Figures 1 to 5).

The following table (Table 1) presents the result of the simulation of the MSE of the estimation for the covariate $X = x_5$.

From Table 1, we can see that the MSE decreases as the size n increases or as the size N of the actually observed sample increases, which shows the convergence of our estimators. In particular, the conditional density estimator performs better than the mode estimator. The performance of the latter improves when n=300 for reasonable censorship and truncation rates. This is confirmed by Table 3.

TABLE 1. Evolution of MSE according to size n

$\lambda_1 = 0,01,\ \lambda_2 = 0,15$							
n	$MSE(f_n, f)$	$MSE(\theta_n, \theta)$	h_K	h_H	N	TR	CR
100	0.0249571	3.9599	0.005504181	1.9899	58	58%	2%
200	0.01798389	2.505619	0.001080219	1.9899	98	51%	2%
300	0.01405676	0.8181612	0.1175594	1.9899	163	45.66%	3.67%

TABLE 2. Evolution of MSE according to the values of λ_1 and n for $\lambda_2 = 0.025$

$\lambda_2 = 0,25,\ X = X_5$								
λ_1	n	$MSE(f_n, f)$	$MSE(\theta_n, \theta)$	h_K	h_H	N	TR	CR
0.0001	100	0.0244568	4.518295	0.005504181	1.9899	77	23%	0%
	200	0.01565475	1.720184	0.001080219	1.9899	146	27%	0%
	300	0.0104913	1.1664	0.1175594	1.77568	226	24.67%	0%
0.06	100	0.02081927	3.78195	0.005504181	1.9899	73	27%	22%
	200	0.02106466	2.505619	0.005504181	1.9899	124	38%	26.5%
	300	0.01580761	1.703025	0.1175594	1.989851	201	33%	24,67%
0.1	100	0.02370787	6.645514	0.005504181	1.9899	67	33%	31%
	200	0.02574877	16.93798	0.001080219	1.9899	116	42%	41%
	300	0.02178438	7.862529	0.1175594	1.9899	190	36.67%	39.33%

TABLE 3. Evolution of MSE according to the values of λ_2 and n for $\lambda_1 = 0.06$

$\lambda_1 = 0.06,\ X = X_5$								
λ_2	n	$MSE(f_n, f)$	$MSE(\theta_n, \theta)$	h_K	h_H	N	TT	TC
0.15	100	0.01919839	2.650842	0.005504181	1.9899	50	50%	22%
	200	0.02148863	1.720184	0.001080219	1.9899	90	55%	26.5%
	300	0.01572742	0.3120571	0.1175594	1.9899	146	51.33%	24.67%
0.2	100	0.02021675	3.272645	0.005504181	1.9899	65	35%	22%
	200	0.02269892	9.457943	0.001080219	1.9899	104	48%	26.5%
	300	0.01611427	0.2094226	0.1175594	1.9899	178	40.67%	24.67%
0.4	100	0.02161651	5.113507	0.005504181	1.9899	81	19%	22%
	200	0.02143723	14.09078	0.001080219	1.9899	156	22%	26.5%
	300	0.01670175	0.03730037	0.1175594	1.9899	234	22%	24.67%

From Table 2, we can see that (i) the MSE increases with the censorship rate for each of the considered sample sizes. This is predictable since we are moving towards situations with much more complete observations; (ii) we also observe an improvement in the performance of our estimators when the censorship rate is fixed and the size n increases. This confirms the result of Table 1. This is illustrated by Figures 2 and 3.

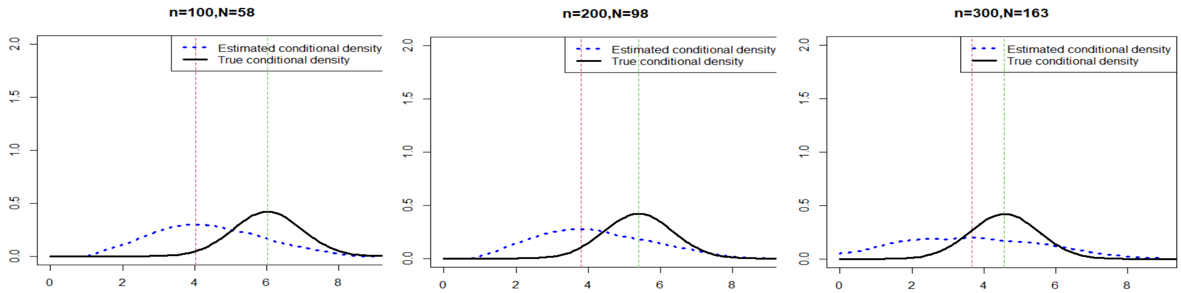

FIGURE 1. Graphs corresponding to $\lambda_1 = 0.01$ and $\lambda_2 = 0.15$ according to size n

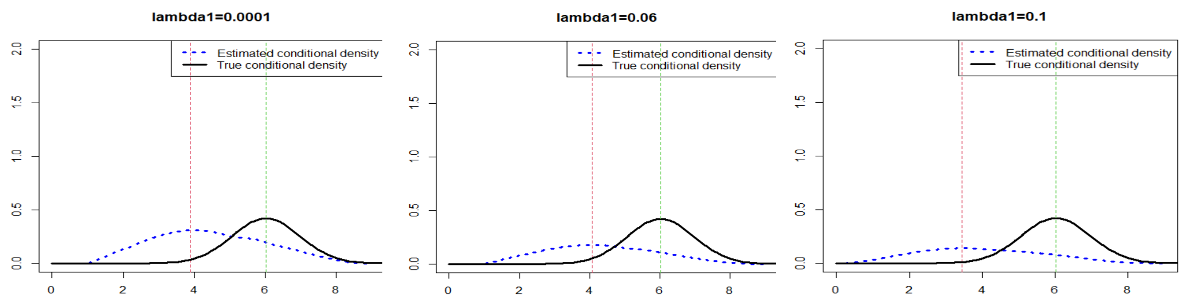

FIGURE 2. Graphs corresponding to n=100 and $\lambda_2 = 0.25$ according to λ_1 (CR= 0%; 22% and 31%)

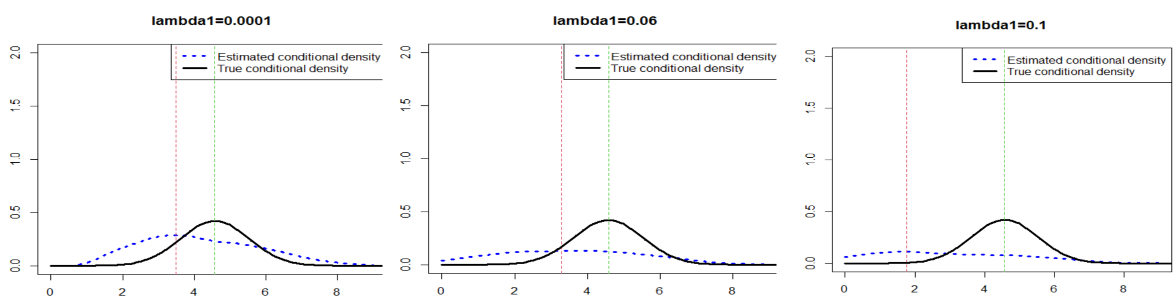

FIGURE 3. Graphs corresponding to n=300 and $\lambda_2 = 0.25$ according to λ_1 (CR= 0%; 24.67% and 39.33%)

From Table 3, we can see that (i) the MSE increases when the truncation rate decreases for sizes n=100 and n=200. The opposite situation is noted for size n=300, meaning that the MSE decreases with the truncation rate. (ii) The convergence of the estimators is not noted in this table due to the instability of the truncation rate with the evolution of the sample size n. All these findings are illustrated by Figures 4 and 5.

From Tables 2 and 3, we also mention that for each fixed size n, when λ_1 increases, the size N of the actually observed sample decreases, while λ_2 increases with this size N.

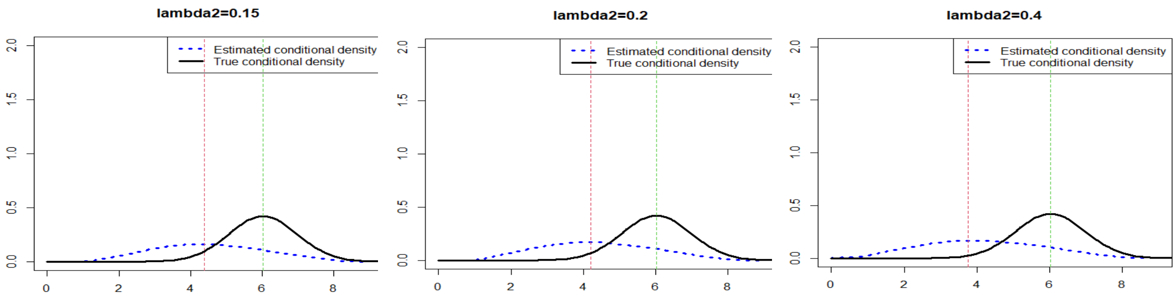

FIGURE 4. Graphs corresponding to n=100 and $\lambda_1 = 0.06$ (CR=22%) according to λ_2 (TR=50%; 35% and 19%)

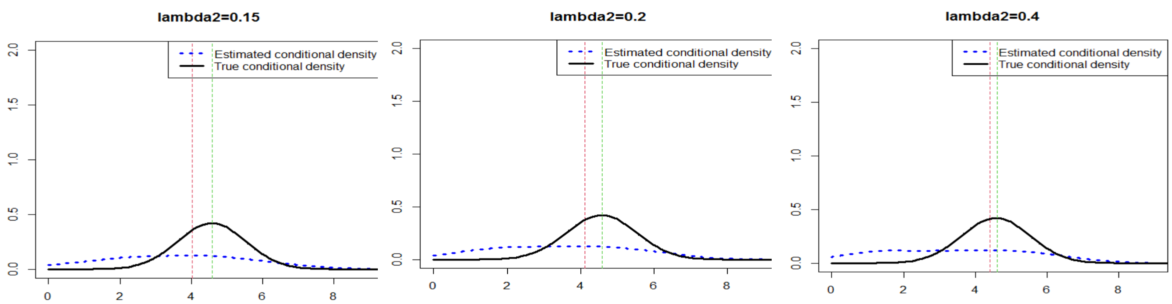

FIGURE 5. Graphs corresponding to n=300 and $\lambda_1 = 0.06$ (CR=24.67%) according to λ_2 (TR=51.33%; 40.67% and 22%)

6. APPENDIX

In this Section, we present the proofs of the lemmas stated in subsection 4.1. In what follows, when no confusion is possible, we will denote the strictly positive generic constants by C and C'. Furthermore, for all $x \in \mathcal{F}$ and for all $i = 1, \cdots, n$:

$$K_i(x) = K(h_K^{-1}\delta(x, X_i)), \;\; \beta_i(x) = \beta(X_i, x) \text{ and } H_i(t) = H(h_H^{-1}(t - Y_i)).$$

Proof of lemma 4.2. Proof of relation (4.6). We have:

$$\begin{aligned}
\left|\hat{m}_n(x) - \tilde{m}_n(x)\right| &= \left|\frac{\mu_n^2}{n(n-1)E(\Delta_{12}(x))}\sum_{i,j=1}^{n} L_n^{-1}(Y_j)L_n^{-1}(Y_i)\Delta_{ij}(x)\right.\\
&- \left.\frac{\mu^2}{n(n-1)E(\Delta_{12}(x))}\sum_{i,j=1}^{n} L^{-1}(Y_j)L^{-1}(Y_i)\Delta_{ij}(x)\right|\\
&\leq \left[\frac{|\mu_n^2 - \mu^2|}{L_n^2(a)} + \mu^2\frac{\sup_{t\in\mathcal{S}}\left|L_n^2(t) - L^2(t)\right|}{L^2(a)L_n^2(a)}\right]\left|m_{n0}(x)\right|.
\end{aligned}$$

with

$$m_{n0}(x) = \frac{1}{n(n-1)E(\Delta_{12}(x))}\sum_{i,j=1}^{n}\Delta_{ij}(x). \tag{6.1}$$

According to (3.2) of [19] , we have: $|\mu_n - \mu| = \mathcal{O}_{a.s}(n^{-1/2})$ while remark (6) of [34] gives $\big|L_n(a) - L(a)\big| = \mathcal{O}_{a.s}(n^{-1/2})$ who is insignificant compared to $\mathcal{O}_{a.s.}\left(\sqrt{\dfrac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)$ and $m_{n0}(x)$ being defined and treated in [5] (the second part of lemma 2 for $Y=1$)

Let's now prove the second part of the lemma.

Proof of relation (4.7). Thanks to the boundedness of H (according to (H7)), we can obtain

$$\begin{aligned}\left|\hat{f}_n(x,t) - \tilde{f}_n(x,t)\right| &= \left|\sum_{i,j=1}^{n}\frac{\delta_i\Delta_{ij}(x)H_i(t)}{n(n-1)h_H E(\Delta_{12}(x))}\Big(\frac{\mu_n^2}{\bar{G}_n(Y_i)L_n(Y_j)L_n(Y_i)} - \right.\\ &\qquad \left.\frac{\mu^2}{\bar{G}(Y_i)L(Y_j)L(Y_i)}\Big)\right|\\ &\leq \frac{M}{h_H}\left[\bar{G}_n(a)\sup_{t\in\mathcal{S}}\left|\Big(\frac{1-F_n(t)}{C_n(t)}\Big)^2 - \Big(\frac{1-F(t)}{C(t)}\Big)^2\right|\right.\\ &+ \left.\Big(\frac{1-F(a)}{C(a)}\Big)^2\sup_{t\in\mathcal{S}}\left|\bar{G}_n(t)-\bar{G}(t)\right|\right]\left|m_{n0}(x)\right|\end{aligned}$$

Considering that

$\sup_{t\in\mathcal{S}}\left|\dfrac{1-F_n(t)}{C_n(t)} - \dfrac{1-F(t)}{C(t)}\right| = \mathcal{O}_{a.s.}\left(\sqrt{\dfrac{\log\log n}{n}}\right)$ (See lemma 4.2 of [10]),

$\sup_{t\in\mathcal{S}}\left|\bar{G}_n(t)-\bar{G}(t)\right| = \mathcal{O}_{a.s.}\left(\sqrt{\dfrac{\log\log n}{n}}\right)$ (see formula (4.28) in [13]), therefore, we have

$$\sup_{t\in\mathcal{S}}\left|\hat{f}_n(x,t) - \tilde{f}_n(x,t)\right| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log(\log n)}{n}}\right).$$

□

Proof of lemma 4.3.

$$\begin{aligned}E\big[\tilde{f}_n(x,t)\big] &= E\left[\frac{\mu^2}{n(n-1)h_H E(\Delta_{12}(x))}\sum_{i,j=1}^{n}\delta_i\bar{G}^{-1}(Y_i)L^{-1}(Y_j)L^{-1}(Y_i)\Delta_{ij}(x)H_i(t)\right]\\ &= \frac{\mu^2}{h_H E(\Delta_{12}(x))}E\Big[\Delta_{12}(x)\gamma\Big] \qquad (6.2)\end{aligned}$$

with

$$\gamma = E\Big(\delta_1\bar{G}^{-1}(Y_1)L^{-1}(Y_2)L^{-1}(Y_1)H_1(t)|X_1\Big).$$

By using the property of conditional expectation and the fact that:

$$\mathbb{I}_{\{T_1\leq C_1\}}\varphi(Z_1) = \mathbb{I}_{\{T_1\leq C_1\}}\varphi(T_1) \text{ for any measurable function } \varphi, \qquad (6.3)$$

since T is independent of (C, Z) and assuming that $X_1 \in B(x, h_K)$ we have:

$$\gamma = \frac{1}{\mu^2} \int_{\mathbb{R}} H\Big(\frac{t-u}{h_H}\Big) f\big(u|X_1\big) du. \tag{6.4}$$

From equalities (6.2) and (6.4), we can write:

$$\Big|E\tilde{f}_n(x,t) - f(t|x)\Big| = \frac{1}{h_H}\Big|E\Big[\int_{\mathbb{R}} H\Big(\frac{t-u}{h_H}\Big) f\big(u|X_1\big) du - f(t|x)\Big]\Big|.$$

By posing $r = \dfrac{t-u}{h_H}$ and applying the assumptions (H2(i)) and (H7), we have:

$$\begin{aligned}
\Big|E\tilde{f}_n(x,t) - f(t|x)\Big| &\leq \int_{\mathbb{R}} H(r)\Big|f\big(t - rh_H|X_1\big) - f(t|x)\Big|dr \\
&\leq C\int_{\mathbb{R}} H(r)\Big(h_K^{b_1} + |r|^{b_2}h_H^{b_2}\Big)dr \\
&= \mathcal{O}_{a.s.}\big(h_K^{b_1}\big) + \mathcal{O}_{a.s.}\big(h_H^{b_2}\big).
\end{aligned}$$

This proves lemma 4.3 □

Proof of lemma 4.4. Relation (4.8) is obtained by following a similar approach to that of proof of lemma 3.5 of [1]

Thus, the compactness of $\mathcal{S}$ allows it to be covered by an infinite number s_n of disjoint intervals of length l_n at certain points $(y_k)_{k=1,\cdots,n}$, that is to say:

$$\mathcal{S} \subset \cup_{k=1}^{s_n}]y_k - l_n, y_k + l_n[$$

with $l_n = n^{-\frac{3\gamma}{2}-\frac{1}{2}}$ and $s_n = \mathcal{O}_{a.s.}(l_n^{-1})$. Let

$$y_t = \arg\min_{y\in\{y_1\cdots y_n\}} |t-y|, \tag{6.5}$$

and consider the following decomposition:

$$\begin{aligned}
\sup_{t\in\mathcal{S}}\Big|\tilde{f}_n(x,t) - E[\tilde{f}_n(x,t)]\Big| &\leq \sup_{t\in\mathcal{S}}\Big|\tilde{f}_n(x,t) - \tilde{f}_n(x,y_t)\Big| + \sup_{t\in\mathcal{S}}\Big|\tilde{f}_n(x,y_t) - E[\tilde{f}_n(x,y_t)]\Big| \\
&+ \sup_{t\in\mathcal{S}}\Big|E[\tilde{f}_n(x,y_t)] - E[\tilde{f}_n(x,t)]\Big| \\
&:= \mathcal{F}_{1,n} + \mathcal{F}_{2,n} + \mathcal{F}_{3,n}.
\end{aligned}$$

From equality (6.5), we have $|t - y_t| \leq l_n$, then under (H7), (H9) and the lemma 4.3, we have:

$$\begin{aligned}
\mathcal{F}_{1,n} &\leq C\sup_{t\in\mathcal{S}}\Big|\frac{\mu^2|t-y_t|}{h_H^2 n(n-1)E\Delta_{12}(x)}\sum_{i,j=1}^{n}\delta_i\bar{G}^{-1}(Y_i)L^{-1}(Y_i)L^{-1}(Y_j)\Delta_{ij}(x)\Big| \\
&\leq C'\frac{l_n}{h_H^2}\Big|m_{n0}(x)\Big|.
\end{aligned}$$

$m_{n0}(x)$ being defined and treated in [5] (the second part of lemma 2 for $Y = 1$), considering the fact that $l_n = n^{-\frac{3\gamma}{2}-\frac{1}{2}}$ and as almost complete convergence implies almost sure convergence, we have:

$$\mathcal{F}_{1,n} = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{6.6}$$

Considering that $|\mathcal{F}_{3,n}| \leq E[\mathcal{F}_{1,n}]$, we find:

$$\mathcal{F}_{3,n} = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{6.7}$$

Let us then prove that:

$$\mathcal{F}_{2,n} = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right).$$

$\forall\eta > 0$, we have:

$$\mathbb{P}\left(\sup_{t\in\mathcal{S}}\left|\tilde{f}_n(x,y_t) - E[\tilde{f}_n(x,y_t)]\right| > \eta\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)$$
$$\leq s_n\mathbb{P}\left(\left|\tilde{f}_n(x,y_t) - E[\tilde{f}_n(x,y_t)]\right| > \eta\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right).$$

It is then necessary to show that:

$$s_n\mathbb{P}\left(\left|\tilde{f}_n(x,y_t) - E[\tilde{f}_n(x,y_t)]\right| > \eta\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) < \infty, \forall y_t \in \{y_1\cdots y_n\}.$$

For this, we consider the following decomposition:

$$\begin{aligned}\tilde{f}_n(x,y_t) &= \underbrace{\frac{n^2h_K^2\phi_x^2(h_K)}{n(n-1)E[\Delta_{12}(x)]}}_{T_1}\Bigg[\underbrace{\Big(\frac{1}{n}\sum_{j=1}^n\frac{\mu\delta_jK_j(x)H_j(y_t)}{\bar{G}(Y_j)L(Y_j)h_H\phi_x(h_K)}\Big)}_{T_2}\\
&\times \underbrace{\Big(\frac{1}{n}\sum_{i=1}^n\frac{\mu K_i(x)\beta_i^2(x)}{L(Y_i)h_K^2\phi_x(h_K)}\Big)}_{T_3} - \underbrace{\Big(\frac{1}{n}\sum_{j=1}^n\frac{\mu\delta_jK_j(x)\beta_j(x)H_j(y_t)}{\bar{G}(Y_j)L(Y_j)h_Hh_K\phi_x(h_K)}\Big)}_{T_4}\\
&\times \underbrace{\Big(\frac{1}{n}\sum_{i=1}^n\frac{\mu K_i(x)\beta_i(x)}{L(Y_i)h_K\phi_x(h_K)}\Big)}_{T_5}\Bigg],\end{aligned}$$

which implies that:

$$\tilde{f}_n(x,y_t) - E[\tilde{f}_n(x,y_t)] = T_1\big((T_2T_3 - E[T_2T_3]) - (T_4T_5 - E[T_4T_5])\big) \tag{6.8}$$

where

$$\begin{aligned}T_2T_3 - E[T_2T_3] &= \big(T_2 - E[T_2]\big)\big(T_3 - E[T_3]\big) + \big(T_3 - E[T_3]\big)E[T_2]\\
&+ \big(T_2 - E[T_2]\big)E[T_3] + E[T_2]E[T_3] - E[T_2T_3],\end{aligned}$$

and

$$\begin{aligned} T_4T_5 - E[T_4T_5] &= \big(T_4 - E[T_4]\big)\big(T_5 - E[T_5]\big) + \big(T_5 - E[T_5]\big)E[T_4] \\ &+ \big(T_4 - E[T_4]\big)E[T_5] + E[T_4]E[T_5] - E[T_4T_5]. \end{aligned}$$

Therefore, our claimed result is a direct consequence of the following statements:

$$s_n\mathbb{P}\left(\Big|T_i - E[T_i]\Big| > \eta\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) < \infty, \text{for } i \in \{2,3,4,5\} \tag{6.9}$$

$$T_1 = \mathcal{O}(1) \textit{ and } E[T_i] = \mathcal{O}(1), \text{ for } i \in \{2,3,4,5\}, \tag{6.10}$$

$$Cov(T_2,T_3) = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) \tag{6.11}$$

and

$$Cov(T_4,T_5) \;= \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{6.12}$$

Regarding statement (6.9), observe that the result for $i \in \{3,5\}$ was obtained in lemma 4.2. We focus only on cases $i \in \{2,4\}$. For this, we use proposition 5.1 of [1].
For $1 \le i \le n$ and $k \in \{0,1\}$, let:

$$\nabla_i^k = \frac{1}{h_K^k}\left(\frac{\mu\delta_i}{\bar{G}(Y_i)L(Y_i)}K_i(x)H_i(y_t)\beta_i^k(x) - E\Big[\frac{\mu\delta_i}{\bar{G}(Y_i)L(Y_i)}K_i(x)H_i(y_t)\beta_i^k(x)\Big]\right) \tag{6.13}$$

So we see that:

$$T_{2(k+1)} - E[T_{2(k+1)}] \;=\; \frac{1}{nh_H\phi_x(h_K)}\sum_{i=1}^n \nabla_i^k, \text{ for } k \in \{0,1\}.$$

Under the assumptions (H1), (H3), (H5) and (H7), we have:

$$\frac{1}{h_K^k}\frac{\mu\delta_i}{\bar{G}(Y_i)L(Y_i)}K_i(x)H_i(y_t)\beta_i^k(x) \;\le\; C'\mathbb{I}_{B(x,h_K)}(X_i). \tag{6.14}$$

Then let's calculate

$$S_n^2 \;=\; \sum_{i=1}^n\sum_{j=1}^n Cov(\nabla_i^k,\nabla_j^k) = S_n^{2*} + nVar(\nabla_1^k),$$

where

$$S_n^{2*} \;=\; \sum_{i=1}^n\sum_{i\neq j} Cov(\nabla_i^k,\nabla_j^k).$$

Using equalitie (6.14) and conditional expectation properties, we have:

$$\begin{aligned} Var(\nabla_1^k) &\le C'E\Big[\mathbb{I}_{B(x,h_K)}(X_i)E\big[H_i^2(y_t)|X_1\big]\Big] + CE^2\Big[\mathbb{I}_{B(x,h_K)}(X_i)E\big[H_i(y_t)|X_1\big]\Big] \\ &\le C'\chi_x^{1|2}(h_K)E\Big[H_i^2(y_t)|X_1\Big] + C\chi_x(h_K)E^2\Big[H_i(y_t)|X_1\Big]. \end{aligned}$$

Finally, using the fact that:

$$E\Big[H_i^2(y_t)|X_1\Big] = \mathcal{O}_{a.s.}(h_H) \quad and \quad E\Big[H_i(y_t)|X_1\Big] = \mathcal{O}_{a.s.}(h_H),$$

we have:

$$nVar(\nabla_1^k) \;\; = \;\; \mathcal{O}_{a.s.}(nh_H\chi_x(h_K)).$$

Then, from equation (6.13), again using the conditional expectation, and the fact that

$$\mathbb{E}[\delta_i\delta_j|T_iT_j] = \bar{G}(T_i)\bar{G}(T_j) \quad and \quad \mathbb{E}[\mathbb{I}_{\{T_i\geq Y_i\}}\mathbb{I}_{\{T_j\geq Y_j\}}|T_iT_j] = L(T_i)L(T_j),$$

we find under the assumptions (H4) and (H5), for $i \neq j$:

$$\begin{aligned} Cov(\nabla_i^k, \nabla_j^k) \;\; &+ \;\; E\Big[\mathbb{I}_{B(x,h_K)}(X_i)E\big[H_i(y_t)|X_i\big]\Big]E\Big[\mathbb{I}_{B(x,h_K)}(X_j)E\big[H_j(y_t)|X_j\big]\Big]\Big] \\ &\leq \;\; Ch_H^2\Big(\varphi_x(h_K) + \phi_x^2(h_K)\Big) \\ &\leq \;\; Ch_H^2\chi_x(h_K). \end{aligned} \tag{6.15}$$

The continuation of this proof is given by that of lemma 3.5 of [1] from relation (15).
Finally, we find:

$$s_n\mathbb{P}\left\{|T_{2(k+1)} - E[T_{2(k+1)}]| > \eta\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{n\phi_x^2(h_K)}}\right\} \;\; = \;\; < \infty.$$

Concerning (6.10), from [14], we have $T_1 = \mathcal{O}(1)$.

Now, we need to study the cases $E[T_i] = \mathcal{O}(1)$ where $i \in \{2, 3, 4, 5\}$.
For $i \in \{3, 5\}$ we have to evaluate:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\mu K_i(x)\beta_i^p(x)}{L(Y_i)h_K^p\phi_x(h_K)} \quad \text{for} \quad p \in \{0, 1\}.$$

By applying the lemma A1 i) of [6], we have for $p \in \{0, 1\}$:

$$\begin{aligned} E\Big[\frac{1}{n\phi_x(h_K)}\sum_{i=1}^{n}\frac{\mu K_i(x)\beta_i^p(x)}{L(Y_i)h_K^p}\Big] \;\; &= \;\; \mu\phi_x^{-1}(h_K)h_K^{-p}E\Big[E\Big(K_1(x)\beta_1^p(x)\mathbb{E}\Big(\frac{\mathbb{I}_{\{T_1\geq Z_1\}}}{\mu L(Y_1)}|T_1\Big)|X_1\Big)\Big] \\ &= \;\; h_K^{-p}\phi_x^{-1}(h_K)EK_1(x)\beta_1^p(x) \\ &\leq \;\; C \end{aligned} \tag{6.16}$$

Thus, $E[T_i] = \mathcal{O}(1)$ for $i \in \{3, 5\}$.
For $i \in \{2, 4\}$, because the couples (X_i, Y_i), $i = 1, \cdots n$ are identically distributed, we obtain:

$$E[T_2] \;\; = \;\; \frac{E[\mu\delta_1\bar{G}^{-1}(Y_1)L^{-1}(Y_1)K_1(x)H_1(y_t)]}{h_H\phi_x(h_K)}$$

and

$$E[T_4] \quad = \quad \frac{E[\mu\delta_1\bar{G}^{-1}(Y_1)L^{-1}(Y_1)K_1(x)\beta_1(x)H_1(y_t)]}{h_K h_H \phi_x(h_K)}.$$

So, we have to evaluate:

$$E\Big[\frac{\mu\delta_1 K_1(x)\beta_1^l(x)H_1(y_t)}{\bar{G}(Y_1)L(Y_1)}\Big], \text{ for } l \in \{0,1\}.$$

Using the boundedness of H, the conditional expectation properties, equation (6.3) and the fact that:

$$\mathbb{E}[\delta_i|T_i] = \bar{G}(T_i) \quad \text{and} \quad \mathbb{E}[\mathbb{I}_{\{T_i \geq Y_i\}}|T_i] = L(T_i),$$

we have for $l \in \{0,1\}$

$$E\Big[\frac{\mu\delta_1 K_1(x)\beta_1^l(x)H_1(y_t)}{\bar{G}(Y_1)L(Y_1)}\Big] \quad = \quad \mathcal{O}_{a.s.}(h_H E[K_1(x)\beta_1^l(x)]).$$

By using lemma 3 in [5], we obtain:

$$E\Big[\frac{\mu\delta_1 K_1(x)\beta_1^l(x)H_1(y_t)}{\bar{G}(Y_1)L(Y_1)}\Big] \quad = \quad \mathcal{O}_{a.s.}(h_H h_K^l \phi_x(h_K)). \tag{6.17}$$

From equation (6.17) we obtain:

$$E[T_i] \quad = \quad \mathcal{O}(1), \text{ for } i = 2, 4.$$

Concerning (6.11) and (6.12), following the steps similar to those in proof of relation (16) in [1], we obtain:

$$Cov(T_2, T_3) = \mathcal{O}\Big(\frac{\chi_x^{1/2}(h_K)}{n\phi_x^2(h_K)}\Big) \quad = \quad \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)$$

and

$$Cov(T_4, T_5) = \mathcal{O}\Big(\frac{\chi_x^{1|2}(h_K)}{n\phi_x^2(h_K)}\Big) \quad = \quad \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right).$$

Thus:

$$\sup_{t\in\mathcal{S}} |\tilde{f}_n(x, y_t) - E[\tilde{f}_n(x, y_t)]| \quad = \quad \mathcal{O}_{a.s.}\left(\sqrt{\frac{\chi_x^{1/2}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{6.18}$$

Finally, the proof of the equality (4.8) of lemma 4.4 can be obtained by considering equations (6.6), (6.7) and (6.18).
For the second part of the lemma (equality (4.9)), it suffices to note that:

$$E[\tilde{m}_n(x)] \quad = \quad 1. \tag{6.19}$$

This leads us to find the result by following the same approach as the first part of the lemma. □

Conclusion

In this work, we proposed a local linear estimator of the conditional density when the data are right-censored and left-truncated in the presence of a functional covariate. We established the almost sure pointwise convergence of this estimator and the almost sure uniform convergence of the mode estimator under the Conditions of weak dependence. Through a numerical study, we automatically selected the two smoothing parameters using the cross-validation method for one and the functional version of the cross-validation ideas for the other in order to prove that our estimators converge at an optimal rate. In the future, we plan to extend our work on the study of the asymptotic normality of the proposed conditional density estimator.

Acknowledgement. We sincerely thank the reviewers for their important contributions that made the article clearer and more readable.

Nous remercions vivement les évaluateurs pour leurs apports importants qui ont permis de rendre l'article plus clair et lisible.

References

1. Abdelkader, B., Fethi, M. & Salah, K. (2020). Strong consistency of local linear estimation of a conditional density function under random censorship. https://doi.org/10.1007/s40065-020-00282-1
2. Agbokou, K. & Gneyou K.E.(2017). On the strong convergence of the hazard rate and its maximum risk point estimators in presence of censorship and functional explanatory covariate. Afrika Statistika, Vol. 12 (3), 2017, pages 1397 - 1416. http://dx.doi.org/10.16929/as/2017.1296.110
3. Agbokou, K., Gneyou, K. E. & Dème, E. H., (2018). Almost sure representations of the condi- tional hazard function and its maximum estimation under right-censoring and left-truncation. Far East J. Theor. Stat., Vol 54, Number 2, 2018, Pages 141-173. http://dx.doi.org/10.17654/TS054020141
4. Agbokou, K.(2023). On the Nature of Elasticity Function: An Investigation and a Kernel Estimation. Volume 2023, Article ID 1346602, 14 pages. https://doi.org/10.1155/2023/1346602
5. Barrientos-Marin, J. (2007). Some Practical Problems of Recent Nonparametric Procedures : Testing, Estimation, and Application. PhD thesis from the Alicante University (Spain). http://hdl.handle.net/10045/9743
6. Barrientos-Marin, J., Ferraty, F. & Vieu, P.(2010). Locally modelled regression and functional data. Journal of Nonparametric Statistics, 22: 5, 617 - 632, First published on: 18 November 2009 (iFirst). https://doi.org/10.1080/10485250903089930
7. Boudada, H. (2022). A nonparametric estimation of the conditional quantile for truncated and functional data, Int. J. Mathematics in Operational Research, Vol. 21, No. 1, pp.127-140. http://dx.doi.org/10.1504/IJMOR.2022.120318
8. Cao, R., Cuevas, A., & Gonzalez M. W. (1994). A comparative study of several smoothing methods in density estimation. Computational Statistics & Data Analysis, 17(2):153-176, 2 1994. ISSN 0167-9473. https://doi.org/10.1016/0167-9473(92)00066-Z
9. Chaib, Y., Barouk, B. & Amiri O. (2021). A nonparametric conditional mode estimate under LTRC model and dependent data. Far East Journal of Mathematical Sciences (FJMS) © 2021 Pushpa Publishing House, Prayagraj, India, Volume 133, Number 1, 2021, Pages 43-80. http://dx.doi.org/10.17654/0972087121003

10. Chaib, Y., Sadki, O. & Boutabia, H. (2013). A nonparametric mode estimate under LTRC model and dependent data, South African Statistical Journal 47 (2013), 91-109. https://hdl.handle.net/10520/EJC143061
11. Dabana, H., Agbokou, K. & Gneyou, K. E. (2024). Kernel estimation of the conditional probability density function and Mode in presence of functional covariate and censorship. Afrika Statistika, Vol. 19 (1), 2024, pages 3771 - 3796. https://dx.doi.org/10.16929/as/2024.3771.327
12. Dabo-Niang, S. & Laksaci, A. (2007). Estimation non paramétrique du mode conditionnel pour variable explicative fonctionnelle. R. Acad. Sci. Paris, Ser. I 344 (2007). C. https://doi.org/10.1016/j.crma.2006.11.022
13. Deheuvels, P. & Einmahl, H. (2000). Functional limit laws for the increments of Kaplan-Meier product limit processes and applications. Ann. Probab. 28, 1301-1335 (2000) https://doi.org/10.1214/aop/1019160336
14. Demongeot, J., Laksaci, A., Madani, F. & Rachdi, M. (2011). A fast functional locally modeled conditional density and mode for functional time-series. In: Recent Advances in Functional Data Analysis and Related Topics. In: Contributions to Statistics, Physica-Verlag, Springer, pp. 85-90. http://dx.doi.org/10.1007/978-3-7908-2736-1_13
15. Ferraty, F. & Vieu, P. (2006). Nonparametric Functional Data Analysis. Theory and practice. Springer Series in Statistics. http://dx.doi.org/10.1007/0-387-36620-2
16. Gneyou, K. E. (2013). A central limit theorem for a nonparametric maximum conditional hazard rate estimator in presence of right censoring. International Journ. Probab. Statist. Vol. 2, No. 3 pp. 110-124. http://doi.org/10.5539/ijsp.v2n3p110
17. Gneyou, K. E. (2014). A strong linear representation for the maximum conditional hazard rate estimator in survival analysis, Journal of Multivariate Analysis 128, 10-18. https://doi.org/10.1016/j.jmva.2014.02.013
18. Hall, P., Marron, J., & Park, B. U. (1992). Smoothed cross-validation. Probability Theory, 92:1-20, 1992. https://doi.org/10.1007/BF01205233
19. He, S. & Yang, G. L. (1998). Estimation of the truncation probability in the random truncation model. Annals of Statistics, 1011-1027. http://dx.doi.org/10.1214/aos/1024691086
20. Khardani, S. & Semmar, S. (2014). Nonparametric conditional density estimation for censored data based on a recursive kernel. Electronic Journal of Statistics 8 (2):2541-56. http://dx.doi.org/10.1214/14-EJS960
21. Kim, C., Oh, M., Yang, S.J. & Choi, H. (2010). A local linear estimation of conditional hazard function in censored data. Journal of the Korean Statistical Society 39(3): 347-355. https://doi.org/10.1016/j.jkss.2010.03.002
22. Laksaci, A., Madani, F. & Rachdi, M. (2013). Kernel conditional density estimation when the regressor is valued in a semi-metric space. Commun. Stat. Theory Methods 42(19), 3544-3570. https://doi.org/10.1080/03610926.2011.633733
23. Liang, H. Y. & Baek, J.I. (2014). Asymptotic normality of conditional density estimation with left-truncated and dependent data. Springer-Verlag Berlin Heidelberg 2014. https://doi.org/10.1007/s00362-014-0635-1
24. Liang, H. Y. & De Una-Alvarez, J. (2010). Asymptotic normality for estimator of conditional mode under left-truncated and dependent observations, Journal of Metrika 72 (2010), 1-19. https://doi.org/10.1007/s00184-009-0237-4
25. Liang, H.Y. & Liu, A. A. (2013). Kernel estimation of conditional density with truncated, censored and dependent data. Journal of Multivariate Analysis 120 (2013) 40-58. http://dx.doi.org/10.1016/j.jmva.2013.05.009
26. Liang, H. Y. & Peng, L. (2010). Asymptotic normality and Berry-Esseen results for conditional density estimator with censored and dependent data. Journal of Multivariate Analysis 101 (5): 1043-54. https://doi.org/10.1016/j.jmva.2010.01.002

27. Ould-Saïd, E. & Tatachak, A. (2007). Asymptotic properties of the kernel estimator of the conditional mode the left-truncated model, C. R. Acad. Sci. Paris Ser. I 344 (2007), 651-656. http://dx.doi.org/10.1016/j.crma.2007.03.023
28. Ould-Saïd, E. & Tatachak, A. (2011). A nonparametric conditional mode estimate under RLT model and strong mixing condition, Int. J. Stat. Econ. 6 (2011), 76-92. http://dx.doi.org/10.17654/0972087121003
29. Parzen, E. (1962). On Estimation of probability Density Function and Mode.The Annals of Mathematical Statistics, 33(3):1065-1076, 9 1962. ISSN 0003-4851. https://doi.org/10.1214/aoms/1177704472
30. Rosenblatt, M. (1956). Remarks on Some Nonparametric Estimates of a Density Function. The Annals of Mathematical Statistics, 27(3), 832–837. http://www.jstor.org/stable/2237390
31. Silverman, B. (1982). On the estimation of a probability density function by the maximum penalized likelihood method. The Annals of Statistics, 10(3):795-810, 1982. https://www.jstor.org/stable/2240905
32. Slaoui, Y. & Khardani, S. (2020). Adaptive recursive kernel conditional density estimators under censoring data. ALEA, Lat. Am. J. Probab. Math. Stat. 17, 389-417. http://dx.doi.org/10.30757/ALEA.v17-16
33. Spierdijk, L. (2008). Nonparametric conditional hazard rate estimation: A local linear approach. Computational Statistics and Data Analysis 52: 2419-2434. https://doi.org/10.1016/j.csda.2007.08.007
34. Woodroofe, M. (1985). Estimating a distribution function with truncated data. The Annals of Statistics, 13 (1), 163-177. https://doi.org/10.1214/aos/1176346584
35. Xianzhou, X. & Meijuan, O. (2019). Non parametric estimation of the conditional density function with right-censored and dependent data, Communications in Statistics - Theory and Methods http://doi.org/10.1080/03610926.2019.1691230

[1] Department of Mathematics, Faculty of sciences, University of Lomé, BP 1515, Togo.
Email address: hezouwedabana8@gmail.com

[2] Department of Mathematics, Faculty of sciences and Technics, University of Kara, BP 239, Kara - Togo.
Email address: ffomestein@gmail.com

[3] Department of Mathematics, Faculty of sciences, University of Lomé, BP 1515, Togo.
Email address: kgneyou@univ-lome.tg