Gulf Journal of Mathematics
Vol 8, Issue 1 (2020) 32-36

WHEN EVERY FINITELY GENERATED PRIME PROPER IDEAL IS REGULAR

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Abstract. In this paper we introduce and investigate a class of those rings in which every finitely generated prime proper ideal is regular. We establish the transfer of this notion to the trivial ring extension, direct product and homomorphic image, and then generate new and original families of rings satisfying this property.

1. Introduction and preliminaries

All rings considered in this paper are commutative with identity elements and all modules are unital. We use "local" to refer to (not necessarily Noetherian) ring with a unique maximal ideal.

In this paper, we are interested in those rings in which every finitely generated prime proper ideal is regular and which we called a $FGPR$-rings. Clearly, any integral domain is an $FGPR$-ring.

Let A be a ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A (see [1, Theorem 3.2]). Suitable background on commutative trivial ring extensions is [1, 3, 13, 16, 17].

The purpose of this paper is to give some simple methods in order to construct $FGPR$-rings. For this, we investigate the transfer of $FGPR$-property to the trivial ring extension, direct product and homomorphic image.

Date: Received: Feb 2, 2020; Accepted: Mar 17, 2020.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.
Key words and phrases. finitely generated prime ideal, trivial ring extension, direct product, homomorphic image.

2. Main results

A ring R is called an $FGPR$-ring if every finitely generated prime proper ideal is regular. Any integral domain is an FPR-ring.

In this section, we study the possible transfer of the $FGPR$-property to various trivial extension contexts, homomorphic images and to direct products. First, we examine the context of trivial ring extensions of a domain A by a K-vector space E, where $K := qf(A)$.

Theorem 2.1. *Let A be a domain, $K = qf(A)$, E be a K-vector space, and $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is an $FGPR$-ring.*
2. *R is not an integral domain.*

Proof. 1) Let $P := \sum_{i=1}^{n} R(b_i, e_i)$ be a nonzero finitely generated proper prime ideal of R, where $(b_i, e_i) \in P$ for each $i = 1, \ldots, n$. Then P has the form $P := Q \propto E$ by [3, Theorem 25.1], where Q is a prime ideal of A. Since $0 \propto E$ is not a finitely generated ideal of R (since E and K are not finitely generated A-modules), then $Q \neq 0$. Therefore, P is a regular ideal (since each (b_i, e_i) is a nonzero-divisor for every $b_i \neq 0$), as desired.

2) Straightforward. □

The following is an example of $FGPR$-ring with zero-divisors.

Example 2.2. *Let $R := \mathbb{Z} \propto \mathbb{Q}$ be the trivial ring extension of $\mathbb{Z}$ by $\mathbb{Q}$. Then :*

1. *R is an $FGPR$-ring by Theorem 2.1(1).*
2. *R is not an integral domain.*

Next, we explore a different context; namely, the trivial ring extension of a local domain (A, M) by an A-module E such that $ME = 0$.

Theorem 2.3. *Let (A, M) be a local domain, $E(\neq 0)$ an A-module with $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *Assume that A is an integral domain. Then, R is an $FGPR$-ring if and only if $dim_{(A/M)}(E) = \infty$.*
2. *Assume that A is a ring with zero-divisors. Then, R is an $FGPR$-ring if and only if one of the following conditions holds:*
   (a) *There is no finitely generated prime proper ideal of A.*
   (b) *$dim_{(A/M)}(E) = \infty$.*

Proof. 1) Assume that A is an integral domain. Our aim is to show that R is an $FGPR$-ring if and only if $dim_{(A/M)}(E) = \infty$.

Assume that E is an (A/M)-vector space of infinite rank. Our aim is to show that there exists no proper finitely generated prime ideal of R, and so R is an $FGPR$-ring. Deny. Let $Q := P \propto E = \sum_{i=1}^{n} R(a_i, e_i)$ be a finitely generated prime ideal of R, where $(a_i, e_i) \in Q$ for each $i = 1, \ldots, n$. Hence, $E = \sum_{i=1}^{n} Ae_i$ since $a_iE = 0$ (since $a_i \in M$) for each $i = 1, \ldots, n$, a contradiction since E is an (A/M)-vector space of infinite rank. Therefore, R is an $FGPR$-ring, as desired.

Conversely, assume that R is an $FGPR$-ring, that is that there exists no proper finitely generated prime ideal of R since R is a total ring. So, if E is an (A/M)-vector space of finite rank, then $Q := 0 \propto E$ is a proper finitely generated prime ideal of R and so it is regular, a desired contradiction since R is a total ring.

2) Now, assume that A is a ring with zero-divisors.

Assume that R is an $FGPR$-ring and $dim_{(A/M)}(E) < \infty$. Our aim is to show that there is no finitely generated prime proper ideal of A. Deny. Let $P := \sum_{i=1}^{n} Aa_i$ be a finitely generated prime proper ideal of A. Then $Q := P \propto E = \sum_{i=1}^{n} R(a_i, e_i)$ is a finitely generated prime proper ideal of R for some $e_i \in E$ (since $dim_{(A/M)}(E) < \infty$) and so it is regular, a desired contradiction since R is a total ring.

Conversely, assume that there is no finitely generated prime proper ideal of A or $dim_{(A/M)}(E) = \infty$. Our aim is to show that R is an $FGPR$-ring, that is there is no finitely generated prime proper ideal of A (since R is a total ring). Deny. Let $Q := P \propto E = \sum_{i=1}^{n} R(a_i, e_i)$ be a finitely generated prime ideal of R, where $(a_i, e_i) \in Q$ for each $i = 1, \ldots, n$. Hence, $E = \sum_{i=1}^{n} Ae_i$ since $a_iE = 0$ (since $a_i \in M$) for each $i = 1, \ldots, n$, and so $dim_{(A/M)}(E) < \infty$.
On the other hand, $P = \sum_{i=1}^{n} Aa_i$ is a finitely generated prime ideal of A. Also, $P \neq 0$ since $R/(0 \propto E) \cong A$ is not an integral domain. Therefore, P is a finitely generated prime proper ideal of A, a desired contradiction.
Hence, R is an $FGPR$-ring and this completes the proof of the Theorem. □

The following are new examples of $FGPR$-rings with zero-divisors.

Example 2.4. *Let $R := K[[X]] \propto K^{\infty}$ be the trivial ring extension of $K[[X]]$ by K^{∞}, where $K := K[[X]]/XK[[X]]$, K is a field, and X is an indeterminates over K. Then :*

(1) *R is an $FGPR$-ring by Theorem 2.2(1).*
(2) *R is not an integral domain.*

Example 2.5. *Let $A := K \propto K^{\infty}$ be the trivial ring extension of K by K^{∞}, where K is a field and $M := 0 \propto K^{\infty}$. Set $R := A \propto E$, where $E := A/M$. Then :*

(1) *R is an $FGPR$-ring.*
(2) *R is not an integral domain.*

Proof. 1) The ring $A := K \propto K^{\infty}$ is a local ring with zero-divisors and with unique prime ideal $M := 0 \propto K^{\infty}$ which is not not finitely generated. Hence, there is no finitely generated prime proper ideal of A. Also, $dim_{(A/M)}(E) = 1$. Therefore, by Theorem 2.2(2), R is an $FGPR$-ring.

2) Straightforward. □

We end this paper by showing that the direct product of rings is a never an $FGPR$-rings.

Proposition 2.6. *Let R be a finite direct product of some family of rings $(R_i)_{i=1,\dots,n}$, where $n \geq 2$. Then R is a never an $FGPR$-ring.*

Proof. It suffices to check the proof for $n = 2$ and let $(R_i)_{i=1,2}$ be a two rings. The following two exact sequences of $(R_1 \times R_2)$:

$$R_1 \times 0 \to R_1 \times R_2 \to 0 \times R_2 \to 0$$

$$0 \to R_2 \times 0 \to R_1 \times R_2 \to R_1 \times 0 \to 0$$

means that $R_1 \times 0$ and $0 \times R_2$ are two finitely generated prime proper ideals of $R_1 \times R_2$. But, $R_1 \times 0$ (and also $0 \times R_2$) is not regular since $(R_1 \times 0)(0 \times R_2) = 0$, as desired. □

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