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EVOLUTION TRAIN ALGEBRAS

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Abstract. Through this paper, we show that the criteria for real evolution algebra to be a baric algebra can be extended to any evolution algebra over a commutative field of characteristic $\neq 2$. Then we prove that an evolution algebra $\mathcal{E}$ is a train algebra of rank $r+1$ if and only if the kernel of its weight function is nil of nil-index $r > 1$. We also study special train evolution algebra and characterize idempotents, power-associativity and automorphism in evolution train algebra. Finally we classify up to dimension 5, indecomposable evolution nil-algebra of nil-index 4 that are not power-associative.

1. Introduction

In 2006, J.P. Tian and P. Vojtěchovský introduced the notion of evolution algebras in the literature ([9]). Let $\mathcal{E}$ be a n-dimensional algebra over a commutative field F. We said that $\mathcal{E}$ is an *evolution algebra* if there is a base $B = \{e_1, \dots, e_n\}$ such that

$$e_i e_j = 0, \text{ for } 1 \leq i \neq j \leq n.$$

Such a base is called a *natural base* of $\mathcal{E}$. The multiplication in $\mathcal{E}$ is determined by the products

$$e_i^2 = \sum_{k=1}^{n} a_{ik} e_k \text{ for all } 1 \leq i \leq n \tag{1.1}$$

and $M = (a_{ik})_{1 \leq i,k \leq n}$ is the *structural constants matrix* of $\mathcal{E}$ relative to the natural base B.

Evolution algebras are commutative ([8]) and they are associative if and only if $e_i^2 e_j = 0$ for all $1 \leq i \neq j \leq n$ ([7]).

In ([3]), the authors establish an equivalence between an evolution nil-algebra and nilpotent evolution algebra.

In section 2, we recall some results about evolution nil-algebras and we show that theirs nil-index and index of nilpotency are the same.

In section 3, we characterize baric evolution algebras over a commutative field F of $char(F) \neq 2$ and then we show that there is no evolution train algebra of rank 2 and a baric evolution algebra $(\mathcal{E}, \omega)$ is a train algebra of rank $r+1 > 2$,

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if and only if $\ker \omega$ is nil of nil-index r. Then we study evolution algebras that are special train algebras and characterize idempotents, power-associativity and automorphism in evolution train algebras.

Section 4 is devoted to the classification, up to dimension 5, of indecomposable evolution nil-algebras of nil-index 4 that are not power-associative.

2. Nilpotent evolution algebras

Let $B = \{e_i;\ 1 \le i \le n\}$ be a natural base of finite-dimensional evolution algebra $\mathcal{E}$ over a commutative field F such that the multiplication table in B is given by (1.1). The *principal powers*, of an element $a \in \mathcal{E}$, are defined as follows $a^1 = a$ and $a^{k+1} = a^k a$ $(k \ge 1)$ while that of $\mathcal{E}$ are defined by:

$$\mathcal{E}^1 = \mathcal{E}, \quad \mathcal{E}^{k+1} = \mathcal{E}^k \mathcal{E} \text{ where } k \ge 1.$$

Definition 2.1. We said that algebra $\mathcal{E}$ is:

i) *nilpotent*, if there is a nonzero integer r such that $\mathcal{E}^r = 0$; such a smaller integer is called the *index of nilpotency* of $\mathcal{E}$;

ii) *nil*, if for all $a \in \mathcal{E}$, there is a nonzero integer s such that $a^s = 0$; such a smaller integer is called *index of nilpotency* of a;

iii) a nil-algebra of *bounded index*, if the index of nilpotency of all elements are bounded by some number n; such a smaller n is called the *nil-index* of $\mathcal{E}$.

Theorem 2.2 ([3, Theorem 2.7])**.** *The following statements are equivalent:*

i) The matrix corresponding to $\mathcal{E}$ can be written as

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

ii) $\mathcal{E}$ is nilpotent algebra.

iii) $\mathcal{E}$ is nil-algebra.

The following lemma 2.3 specifies that the index of nilpotency is equal to the nil-index.

Lemma 2.3. *Let $\mathcal{E}$ be a finite-dimensional evolution nil-algebra of nil-index m. Then $\mathcal{E}$ is nilpotent of index of nilpotency m.*

Proof. Let $\mathcal{E}$ be a n-dimensional evolution nil-algebra of nil-index m. Then $m \le n+1$ and $\mathcal{E}$ admits a natural base $B = \{e_i;\ 1 \le i \le n\}$ such that $e_i^2 = \sum_{j=i+1}^{n} a_{ij} e_j$ (Theorem 2.2); moreover, for all $i \in \{1, \dots, n\}$, $e_i^3 = 0$. We have $\mathcal{E}^2 = \mathcal{E}\mathcal{E} = \langle e_i e_j;\ 1 \le i, j \le n \rangle = \langle e_i^2;\ 1 \le i \le n \rangle$ because $e_i e_j = 0$ for $i \ne j$; $\mathcal{E}^3 = \mathcal{E}^2 \mathcal{E} = \langle e_i^2 e_j;\ 1 \le i, j \le n \rangle$ and by induction $\mathcal{E}^m = \mathcal{E}^{m-1}\mathcal{E} = \langle (\cdots (e_{i_1}^2 e_{i_2}) \cdots) e_{i_{m-1}};\ 1 \le i_j \le n \text{ and } 1 \le j \le m-1 \rangle$. Let's show that $\mathcal{E}^m = 0$. We consider the element $x_i = e_{i_1} + e_{i_2} + \cdots + e_{i_{m-1}} = \sum_{k=1}^{m-1} e_{i_k}$ with $1 \le i_1 < i_2 < \ldots < i_{m-1} \le n$. We have $x_i^2 = \sum_{k=1}^{m-1} e_{i_k}^2$, $x_i^3 = \sum_{k,j=1}^{m-1} e_{i_k}^2 e_{i_j}$ and by induction $x_i^m = \sum_{k_1,\dots,k_{m-1}=1}^{m-1} (\cdots (e_{i_{k_1}}^2 e_{i_{k_2}}) \cdots) e_{i_{k_{m-1}}}$.

Since $(\cdots(e_{i_{k_1}}^2 e_{i_{k_2}})\cdots)e_{i_{k_{m-1}}} = a_{i_{k_1}i_{k_2}}\cdots a_{i_{k_{m-2}}i_{k_{m-1}}}e_{i_{m-1}}^2$ and that the structural constants matrix of $\mathcal{E}$ relative to B is upper triangular, it follows that $1 \leq i_{k_1} < i_{k_2} < \cdots < i_{k_{m-1}} \leq n$. As $i_{k_j} \in \{i_1, \ldots, i_{m-1}\}$ for $j \in \{1, \ldots, m-1\}$ and that $(i_1, \ldots, i_{m-1})$ is the unique $(m-1)$-tuple of $\{i_1, \ldots, i_{m-1}\}$ such that $1 \leq i_1 < i_2 < \ldots < i_{m-1} \leq n$, then $\sum_{k_1,\ldots,k_{m-1}=1}^{m-1}(\cdots(e_{i_{k_1}}^2 e_{i_{k_2}})\cdots)e_{i_{k_{m-1}}}$ $= (\cdots(e_{i_1}^2 e_{i_2})\cdots)e_{i_{m-1}}$. We infer that $0 = x_i^m = (\cdots(e_{i_1}^2 e_{i_2})\cdots)e_{i_{m-1}}$. So $\mathcal{E}^m = 0$ and we get the lemma. □

Definition 2.4. Let $\mathcal{E}$ be an evolution algebra:

i) *The annihilator* of $\mathcal{E}$ is defined by $\mathrm{ann}^1(\mathcal{E}) = \mathrm{ann}(\mathcal{E}) = \{x \in \mathcal{E} : x\mathcal{E} = 0\}$.

ii) We also define $\mathrm{ann}^i(\mathcal{E})$ by $\mathrm{ann}^i(\mathcal{E})/\mathrm{ann}^{i-1}(\mathcal{E}) = \mathrm{ann}\left(\mathcal{E}/\mathrm{ann}^{i-1}(\mathcal{E})\right)$, $i \geq 2$.

Let $B = \{e_1, \ldots, e_n\}$ be a natural base of an evolution algebra $\mathcal{E}$ over a commutative field. In ([5, Lemme 2.7]), the authors show that $\mathrm{ann}(\mathcal{E}) = span\{e_i \in B \mid e_i^2 = 0\}$. In ([6]), they show that $\mathrm{ann}^i(\mathcal{E}) = span\{e \in B \mid e^2 \in \mathrm{ann}^{i-1}(\mathcal{E})\}$ and that the base $B = B_1 \cup \cdots \cup B_r$ is a natural base where $B_i = \{e \in B \mid e^2 \in \mathrm{ann}^{i-1}(\mathcal{E}), e \notin \mathrm{ann}^{i-1}(\mathcal{E})\}$. Then, for $\mathcal{U}_i := span\{B_i\}$, $i = 1, \ldots, r$, we have $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_i = \mathrm{ann}^i(\mathcal{E})$ $(i = 1, \ldots, r)$. They prove that $\mathcal{U}_i \oplus \mathcal{U}_1 = \{x \in \mathrm{ann}^i(\mathcal{E}) \mid x\mathrm{ann}^{i-1}(\mathcal{E}) = 0\}$ is an invariant of evolution nil-algebra.

The type of an evolution nil-algebra $\mathcal{E}$ is the sequence $[n_1, n_2, \ldots, n_r]$ where r and n_i are integers defined by $\mathrm{ann}^r(\mathcal{E}) = \mathcal{E}$; $n_i = \dim_F(\mathrm{ann}^i(\mathcal{E})) - \dim_F(\mathrm{ann}^{i-1}(\mathcal{E}))$ and $n_1 + \cdots + n_i = dim_F(\mathrm{ann}^i(\mathcal{E}))$ for all $i \in \{1, \ldots, r\}$.

3. Evolution train algebras

3.1. Baric evolution algebras.

Definition 3.1 ([10, Definition 1.7])**.** We said that an algebra $\mathcal{E}$ over a field F is *baric*, if it admits $\omega : \mathcal{E} \longrightarrow F$ a non trivial algebra homomorphism. The homomorphism ω is called the *weight function* or *weight homomorphism* of $\mathcal{E}$.

Theorem 3.2 ([4, Theorem 3.2])**.** *A n-dimensional evolution algebra $\mathcal{E}$, over the field $\mathbb{R}$, is baric if and only if there is a column $(a_{1i_0}, \ldots, a_{ni_0})^T$ of its strctural constants matrix $M = (a_{ij})_{i,j=1,\ldots n}$, such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$, for $i \neq i_0$. The corresponding weight function is $\omega(x) = a_{i_0i_0}x_{i_0}$ where $x = \sum_{k=1}^n x_i e_i \in \mathcal{E}$.*

This theorem remains true, when we replace the field $\mathbb{R}$ by any commutative field F of $char(F) \neq 2$. In the following, unless otherwise indicated, F designates such a field.

Theorem 3.3. *The evolution algebra $\mathcal{E}$ is baric if and only if there is $i_0 \in \{1, \cdots, n\}$ such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$ for $i \neq i_0$. Moreover, The corresponding weight function ω is defined by: $\omega(e_{i_0}) = a_{i_0i_0}$ and $\omega(e_i) = 0$ for $i \neq i_0$.*

Proof. Suppose that there is a non trivial algebra homomorphism $\omega : \mathcal{E} \longrightarrow F$. Then there is $i_0 \in \{1, \cdots, n\}$ such that $\omega(e_{i_0}) \neq 0$. For $i \neq i_0$, we have $0 = \omega(e_ie_{i_0}) = \omega(e_i)\omega(e_{i_0})$ leads to $\omega(e_i) = 0$ and $0 = \omega(e_i^2) = \sum_{k=1}^n a_{ik}\omega(e_k) = a_{ii_0}\omega(e_{i_o})$ implies $a_{ii_0} = 0$. We have $\omega(e_{i_0}^2) = \sum_{k=1}^n a_{i_0k}\omega(e_k) = a_{i_0i_0}\omega(e_{i_0})$ gives $\omega(e_{i_0})(\omega(e_{i_0}) - a_{i_0i_0}) = 0$. So $\omega(e_{i_0}) = a_{i_0i_0}$ because $\omega(e_{i_0}) \neq 0$.

Conversely, suppose that there is $i_0 \in \{1, \dots, n\}$ such that $a_{i_0 i_0} \neq 0$ and $a_{ii_0} = 0$ for $i \neq i_0$. Then we check that the linear transformation $\omega : \mathcal{E} \longrightarrow F$ defined by $\omega(e_{i_0}) = a_{i_0 i_0}$ and $\omega(e_i) = 0$ for $i \neq i_0$ is a weight function of $\mathcal{E}$. □

Corollary 3.4. *If $(\mathcal{E}, \omega)$ is a baric evolution algebra, then $\mathcal{E}$ admits a natural base $\{u_1, u_2, \dots, u_n\}$ such that $\omega(u_1) = 1$ and $\omega(u_i) = 0$ for $i > 1$. Moreover, $\mathcal{E} = Fu_1 \oplus \ker\omega$ with $u_1 \ker\omega = 0$.*

Proof. Let $(\mathcal{E}, \omega)$ be a baric algebra. Without loss of generality, we can assume that $a_{11} \neq 0$ and $a_{i1} = 0$ for $i > 1$. We set $u_1 = be_1$ with $\omega(u_1) = 1$. We have $b\omega(e_1) = \omega(u_1) = 1$ leads to $b = (\omega(e_1))^{-1}$ so $u_1 = (\omega(e_1))^{-1}e_1$. We have $u_1 e_j = 0$, for $j > 1$ and $u_1^2 = (\omega(e_1))^{-2}e_1^2 = (\omega(e_1))^{-1}e_1 + \sum_{k=2}^{n}(\omega(e_1))^{-2}a_{1k}e_k$. We set $u_k = (\omega(e_1))^{-2}e_k$ for $k > 1$, then the family $\{u_i,\ 1 \leq i \leq n\}$ is a natural base of $\mathcal{E}$ with $\omega(u_1) = 1$ and its multiplication table is defined by

$$u_1^2 = u_1 + \sum_{k=2}^{n} a_{1k}u_k \text{ and } u_j^2 = \sum_{k=2}^{n}(\omega(e_1))^{-2}a_{jk}u_k \text{ for } j > 1. \tag{3.1}$$

We deduce that $\ker\omega =< u_j >_{j\neq 1}$ is an evolution subalgebra of $\mathcal{E}$ (see Definition 3.19) and $\mathcal{E} = Fu_1 \oplus \ker\omega$. □

Remark 3.5. A finite-dimensional baric evolution algebra admits at most n weights functions. The case where there is n weights functions occurs when $e_i^2 = \alpha_i e_i$ with $\alpha_i \neq 0$, $i = 1, \dots, n$.

3.2. **Characterization of evolution train algebras.**

Definition 3.6. We said that a baric algebra $(\mathcal{E}, \omega)$ is a *train algebra of rank r*, if there is scalars $\gamma_i \in F$ and a nonzero integer r such that

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0 \tag{3.2}$$

for all x in $\mathcal{E}$ and r is smaller such integer.

For all $x \in \ker\omega$, we have $x^r = 0$; so $\ker\omega$ is a nil-algebra. Since the weight function is not trivial, by applying it to the identity (3.2), for $\omega(x) \neq 0$, we obtain $1 + \gamma_1 + \cdots + \gamma_{r-1} = 0$. Thus the train algebras of rank 2 are defined by $x^2 - \omega(x)x = 0$.

Proposition 3.7. *There is no evolution train algebra of rank* 2.

Proof. The commutative train algebras of rank 2 are the gametic algebras for simple mendelian inheritance ([10, page 137]). In ([7, Example 2]), the authors show that gametic algebra for simple Mendelian inheritance is not an evolution algebra. We get the proposition. □

Theorem 3.8. *Let $(\mathcal{E}, \omega)$ be a baric evolution algebra. Then $\mathcal{E}$ is a train algebra of rank $r + 1 > 2$ if and only if $\ker(\omega)$ is nil, of nil-index r. The train equation of $\mathcal{E}$ is given by $x^{r+1} - \omega(x)x^r = 0$.*

Proof. Let $(\mathcal{E}, \omega)$ be evolution train algebra with natural base $\{e_1, \dots, e_n\}$ such that $\omega(e_1) = 1$ and $\omega(e_i) = 0$ for $i > 1$; we have $\mathcal{E} = Fe_1 \oplus \ker(\omega)$. Let $x = \alpha e_1 + y$ where $\alpha \in F$ and $y \in \ker(\omega)$. We have $x^2 = \alpha^2 e_1^2 + y^2 = \alpha^2(e_1 + z) + y^2 =$

$\alpha^2 e_1+(\alpha^2 z+y^2)$ where $z=\sum_{k=2}^n a_{1k}e_k$; so $x^2-\omega(x)x=(-\alpha y+\alpha^2 z+y^2)$. We have $x(x^2-\omega(x)x)=(-\alpha y^2+\alpha^2 yz+y^3)$, i.e. $x^3-\omega(x)x^2=-\alpha y^2+\alpha^2 yz+y^3$. Gradually we obtain $x^{k+1}-\omega(x)x^k=-\alpha y^k+\alpha^2\ell_y^{k-1}(z)+y^{k+1}$. Since $\ker(\omega)$ is a nil-algebra, hence nilpotent, there is an integer $k\geq 1$ such that $-\alpha y^k+\alpha^2\ell_y^{k-1}(z)+y^{k+1}=0$. Thus the train equation is of form $x^{s+1}-\omega(x)x^s=0$ with $s>1$ an integer.

Suppose that the nil-index of $N=\ker(\omega)$ is r. Then for all $y,z\in N$, we have $y^r=0$ and $\ell_y^{r-1}(z)=0$ because $(\ker(\omega))^r=0$. So $x^{r+1}-\omega(x)x^r=-\alpha y^r+\alpha^2\ell_y^{r-1}(z)+y^{r+1}=0$ and $\mathcal{E}$ is a train algebra of rank $r+1$. Indeed, if the rank of $\mathcal{E}$ was r, then we would have $x^r-\omega(x)x^{r-1}=0$; either $-\alpha y^{r-1}+\alpha^2\ell_y^{r-2}(z)+y^r=0$. For $\alpha=0$, i.e. $x=y\in\ker(\omega)$, we have $y^r=0$. Thus, $-\alpha y^{r-1}+\alpha^2\ell_y^{r-2}(z)=0$, for all $\alpha\in F$, i.e. $-y^{r-1}+\alpha\ell_y^{r-2}(z)=0$, for all $\alpha\in F^*$. Since $char(F)\neq 2$, we would have $y^{r-1}=0$, for all $y\in\ker(\omega)$. This would contradict the hypothesis since there existed $y_0\in\ker(\omega)$ such that $y_0^{r-1}\neq 0$. Thus, the rank of $\mathcal{E}$ is exactly $r+1$.

In passing, we have shown that if $\mathcal{E}$ is a train algebra of rank r, then for all $y\in\ker(\omega)$, $y^{r-1}=0$, i.e. $N=\ker(\omega)$ is a nil-algebra of nil-index at most $r-1$. In fact, the nil-algebra N is exactly of nil-index $r-1$. □

Lemma 3.9. *Let $(\mathcal{E},\omega)$ be an evolution train algebra. Then, $\mathcal{E}$ admits a natural base $\{u_1,\ldots,u_n\}$ such that $u_1^2=u_1+\sum_{k=2}^n a_{1k}u_k$, $u_j^2=\sum_{k=j+1}^n a_{jk}u_k$ with $2\leq j\leq n$.*

Proof. Since $\mathcal{E}$ is a baric algebra, it admits a natural base $\{e_1,\ldots,e_n\}$ whose multiplication table is of form $e_1^2=e_1+\sum_{k=2}^n a_{1k}e_k$, $e_j^2=\sum_{k=2}^n a_{jk}e_k$ with $2\leq j\leq n$. Since $(\mathcal{E},\omega)$ is a train algebra, it follows that $\ker(\mathcal{E})=<e_2,\ldots,e_n>$ is an evolution nil-algebra and $\ker(\mathcal{E})$ admits a natural base $\{u_2,\ldots,u_n\}$ such that structural constants matrix is of form

$$\begin{pmatrix} 0 & b_{23} & b_{24} & \cdots & b_{2n}\\ 0 & 0 & b_{34} & \cdots & b_{3n}\\ 0 & 0 & 0 & \cdots & b_{4n}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We infer that the family $\{e_1,u_2,u_3,\ldots,u_n\}$ is a natural base of $\mathcal{E}$ and its multiplication table is of form $e_1^2=e_1+\sum_{k=2}^n b_{1k}u_k$, $u_j^2=\sum_{k=j+1}^n b_{jk}u_k$ with $2\leq j\leq n$. We get the proposition. □

3.3. Characterization of special train algebras.

Definition 3.10 ([10, Definition 3.27])**.** Let $(\mathcal{E},\omega)$ be a baric algebra. We said that $\mathcal{E}$ is a *special train algebra* if $N=\ker\omega$ is a nilpotent algebra and its principal powers N^i, $i\in\mathbb{N}^*$, are ideals of $\mathcal{E}$.

Proposition 3.11. *Let $(\mathcal{E},\omega)$ be a finite-dimensional baric algebra. Then $\mathcal{E}$ is a special train algebra if and only if $N=\ker\omega$ is a nil-algebra.*

Proof. Since $\mathcal{E}$ is a finite-dimensional baric algebra, $\mathcal{E}$ admits the following decomposition $\mathcal{E}=Fe_1\oplus N$ where $\omega(e_1)=1$ and $e_1N=0$. Suppose that N

is a nil-algebra. As any finite evolution nil-algebra is nilpotent, N is nilpotent. Hence, $N^{k+1} \subset N^k$. We have $\mathcal{E}N^k = (Fe_1 \oplus N)N^k = N^{k+1} \subset N^k$. So N^k is an ideal of $\mathcal{E}$. We deduce that $\mathcal{E}$ is a special train algebra.

Conversely, suppose that $\mathcal{E}$ is a special train algebra. Then N is a nilpotent algebra. It follows N is a nil-algebra. □

By Theorem 3.8 and Proposition 3.11, we deduce the following theorem

Theorem 3.12. *Any evolution train algebra is a special train algebra.*

3.4. **Idempotents in evolution train algebras.** *A nonzero idempotent* of an algebra $\mathcal{E}$ is an element $e \in \mathcal{E}$ such that $e^2 = e \neq 0$.

Proposition 3.13. *Let $(\mathcal{E}, \omega)$ be a n-dimensional train algebra. Then $\mathcal{E}$ admits one and only one nonzero idempotent.*

Proof. Since $(\mathcal{E}, \omega)$ is a train algebra, it admits a natural base $\{e_1, \ldots, e_n\}$ such that $e_1^2 = e_1 + \sum_{k=2}^n a_{1k}e_k$, $e_j^2 = \sum_{k=j+1}^n a_{jk}e_k$ with $2 \leq j \leq n$. Let $\sigma = \sum_{i=1}^n \sigma_i e_i$ be an element of $\mathcal{E}$. We have $\sigma^2 = \sigma_1^2 e_1 + \sum_{j=2}^n \sigma_1^2 a_{1j}e_j + \sum_{i=2}^n \sum_{j=i+1}^n \sigma_i^2 a_{ij}e_j = \sigma_1^2 e_1 + \sum_{j=2}^n \sigma_1^2 a_{1j}e_j + \sum_{j=2}^n \sum_{i=2}^{j-1} \sigma_i^2 a_{ij}e_j = \sigma_1^2 e_1 + \sum_{j=2}^n (\sum_{i=1}^{j-1} \sigma_i^2 a_{ij})e_j$. So σ is an idempotent if and only if

$$\sigma_1^2 = \sigma_1 \text{ and } \sigma_j = \sigma_1^2 a_{1j} + \sigma_2^2 a_{2j} + \sigma_3^2 a_{3j} + \cdots + \sigma_{j-1}^2 a_{j-1,j} \text{ where } 2 \leq j \leq n. \quad (3.3)$$

We see that if $\sigma_1 = 0$, then $\sigma_2 = \sigma_3 = \cdots = \sigma_n = 0$. So $\sigma_1 = 1$ and (3.3) admits one and only one solution. □

If σ is the unique idempotent of an evolution train algebra $\mathcal{E}$, then the base $\{\sigma, e_2, \ldots, e_n\}$ is not necessarily a natural base of $\mathcal{E}$ because $\sigma e_t = \sigma_t e_t^2$ is not necessarily zero.

Example 3.14. Let $\mathcal{E}$ be a 4-dimensional algebra with a base $\{e, u, v, w\}$ whose nonzero products are defined by $e^2 = e + u$, $u^2 = v$, $v^2 = w$. Then $\mathcal{E}$ is an evolution algebra in a natural base $\{e, u, v, w\}$. Let $\omega : \mathcal{E} \longrightarrow F$ be a linear transformation defined by $\omega(e) = 1$ and $\omega(u) = \omega(v) = \omega(w) = 0$. The Theorem 3.3 tells us that ω is a weight function of $\mathcal{E}$. We set $N = \ker(\omega) =< u, v, w >$ and we have $N^2 =< v, w >$, $N^3 =< w >$, $N^4 = 0$. We infer that $(\mathcal{E}, \omega)$ is a train algebra of rank 5 and its train equation is $x^5 - \omega(x)x^4 = 0$. Moreover, from (3.3) it comes $\sigma = e + u + v + w$ is the unique idempotent of $\mathcal{E}$. Since $\sigma u = u^2 = v \neq 0$, it follows the family $\{\sigma, u, v, w\}$ is not a natural base of $\mathcal{E}$.

Remark 3.15. Let N_1 and N_2 be two finite-dimensional evolution nil-algebras of naturals bases respectively $\{u_1, \ldots, u_{n_1}\}$ and $\{v_1, \ldots, v_{n_2}\}$. We set $\mathcal{E}_1 = Fu_0 \oplus N_1$ and $\mathcal{E}_2 = Fv_0 \oplus N_2$ where $u_0^2 = u_0 + \sum_{k=1}^{n_1} a_{0k}u_k$, $u_0 u_i = 0$ and $v_0^2 = v_0 + \sum_{k=1}^{n_2} a_{0k}v_k$, $v_0 v_j = 0$ with $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. We set respectively ω_1 and ω_2 the weights functions of $\mathcal{E}_1$ and $\mathcal{E}_2$ defined by $\omega_1(u_0) = \omega_2(v_0) = 1$ and $\omega_1(u_i) = \omega_2(v_j) = 0$ with $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. $(\mathcal{E}_1, \omega_1)$ and $(\mathcal{E}_2, \omega_2)$ are evolution train algebras and we set $\sigma_1 = u_0 + \sigma_1'$ and $\sigma_2 = v_0 + \sigma_2'$, the unique idempotents of $(\mathcal{E}_1, \omega_1)$ and $(\mathcal{E}_2, \omega_2)$ respectively. Then $\sigma = e_0 + \sigma_1' + \sigma_2'$ is the unique idempotent of the evolution algebra $\mathcal{E} = Fe_0 \oplus N_1 \oplus N_2$ where $e_0^2 = e_0 + (u_0^2 - u_0) + (v_0^2 - v_0)$, $e_0 u_i = e_0 v_j = 0$ with $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Indeed, $\sigma^2 = e_0^2 + \sigma_1'^2 + \sigma_2'^2 =$

$e_0+(u_0^2+\sigma_1'^2)+(v_0^2+\sigma_2'^2)-(u_0+v_0) = e_0+\sigma_1^2+\sigma_2^2-(u_0+v_0) = e_0+\sigma_1+\sigma_2-(u_0+v_0) = e_0 + \sigma_1' + \sigma_2' = \sigma$.

3.5. Power-associative evolution train algebras.

Definition 3.16. The algebra $\mathcal{E}$ is said:

i) *fourth power-associative* if and only if $x^2x^2 = x^4$ for all $x \in \mathcal{E}$.

ii) *power-associative*, if for all $x \in \mathcal{E}$, $x^ix^j = x^{i+j}$ for all integers $i, j \geq 1$.

Theorem 3.17 ([1])**.** *Let F be a commutative field of char$(F) \neq 2, 3, 5$. Algebra $\mathcal{E}$ is power-associative if and only if $x^2x^2 = x^4$ for all $x \in \mathcal{E}$.*

Proposition 3.18. *Let $\mathcal{E} = Fe_1 \oplus \ker\omega$ be a finite-dimensional evolution train algebra over a commutative field of char$(F) \neq 2, 3, 5$, where $e_1^2 = e_1 + z$ with $z \in \ker\omega$ and $e_1 \ker\omega = 0$. Then $\mathcal{E}$ is a power-associative algebra if and only the following statements are satisfied:*

i) $\ker\omega$ *is a power-associative evolution algebra;*

ii) $z \in ann(\ker\omega)$.

Proof. We set $x = \alpha e_1 + y$ with $\alpha \in F$ and $y \in \ker\omega$. We have the following equalities $x^2 = \alpha^2 e_1^2 + y^2 = \alpha^2(e_1 + z) + y^2$; $x^3 = \alpha^3 e_1^2 + \alpha^2 yz + y^3 = \alpha^3(e_1 + z) + \alpha^2 yz + y^3$; $x^4 = \alpha^4 e_1^2 + \alpha^3 yz + \alpha^2 y(yz) + y^4$ and $x^2x^2 = \alpha^4 e_1^2 + \alpha^4 z^2 + y^2y^2 + 2\alpha^2 y^2 z$.

Suppose that $\mathcal{E}$ is a power-associative algebra. Then the equality $x^2x^2 = x^4$ leads to

$$\alpha^4 z^2 - \alpha^3 yz + \alpha^2(2y^2z - y(yz)) + y^2y^2 - y^4 = 0. \tag{3.4}$$

By replacing α by $-\alpha$ in (3.4), we get

$$\alpha^4 z^2 + \alpha^3 yz + \alpha^2(2y^2z - y(yz)) + y^2y^2 - y^4 = 0. \tag{3.5}$$

The difference of (3.4) and (3.5) gives $2\alpha^3 yz = 0$. We deduce that $yz = 0$, for all $y \in \ker\omega$, i.e. $z\ker\omega = 0$. We get *ii*). The equality (3.4) becomes $y^2y^2 - y^4 = 0$ for all $y \in \ker\omega$. The Theorem 3.17 tells us that $\ker\omega$ is a power-associative evolution algebra and we obtain *i*).

Suppose that *i*) and *ii*) are satisfied. Then $x^4 = \alpha^4 e_1^2 + y^4$ and $x^2x^2 = \alpha^4 e_1^2 + y^2y^2 = \alpha^4 e_1^2 + y^4 = x^2x^2$. We deduce that $\mathcal{E}$ is a power-associative evolution algebra. □

Definition 3.19 ([2, Definition 2.4])**.** *An evolution sub-algebra* of an evolution algebra $\mathcal{E}$ is a sub-algebra $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}'$ is an evolution algebra, i.e. $\mathcal{E}'$ have a natural base. We said that $\mathcal{E}'$ have *extension property* if there is a natural base B' of $\mathcal{E}'$ which can be extended to natural base B of $\mathcal{E}$.

Remark 3.20. Let $\mathcal{E} = Fe_1 \oplus \ker\omega$ be a finite-dimensional power-associative evolution train algebra, where $e_1^2 = e_1 + z$ with $z \in \ker\omega$, $z \in \text{ann}(\ker\omega)$ and $e_1 \ker\omega = 0$. Then $e = e_1^2$ is the unique idempotent of $\mathcal{E}$ and the sub-algebra generated by e have the extension property [7, Lemme 5]. This result is also verified by direct calculation. Indeed, $e^2 = e_1^2 e_1^2 = e_1^2 + z^2 = e_1^2$ and for all $y \in \ker\omega$, we have $e_1^2 y = zy = 0$. Then, there is an orthogonal family $\{u_2, \ldots, u_n\}$ such that $\ker\omega =< u_2, \ldots, u_n >$ and the family $\{e, u_2, \ldots, u_n\}$ is a natural base of $\mathcal{E}$. Thus, the Peirce decomposition of $\mathcal{E}$ is given by $\mathcal{E} = \mathcal{E}_1(e) \oplus \mathcal{E}_0(e)$ where

$\mathcal{E}_1(e) = Fe$ and $\mathcal{E}_0(e) = \{x \in \mathcal{E}, xe = 0\}$. We see that $\mathcal{E}_0(e) = \ker\omega$; hence $\mathcal{E} = Fe \oplus \ker\omega$ is a direct sum of algebras.

Now, we give an example of evolution train algebra that is not power-associative and admits an idempotent e such that the sub-algebra generated by e is not provided with extension property and whose kernel is power-associative.

Example 3.21. Let $\mathcal{E}$ be a 5-dimensional algebra with a base $\{e_1, x, u, v, w\}$ whose nonzero products are defined by $e_1^2 = e_1 + u + v$, $x^2 = u + v$, $u^2 = w$, $v^2 = -w$. Then $\mathcal{E}$ is an evolution algebra in a natural base $\{e_1, x, u, v, w\}$. Let $\varphi : \mathcal{E} \longrightarrow F$ be a linear transformation defined by $\varphi(e_1) = 1$ and $\varphi(x) = \varphi(u) = \varphi(v) = \varphi(w) = 0$. The linear transformation φ is a weight function of $\mathcal{E}$ and $(\mathcal{E}, \varphi)$ is an evolution train algebra. Moreover $e_1^2 e_1^2 = e_1^2 + u^2 + v^2 = e_1^2$ and $\ker\omega =< x, u, v, w >$ is power-associative evolution algebra that is not associative. However, $\mathcal{E}$ is not power-associative because $(u + v)u = u^2 = w \neq 0$. Let's look for the vectors $e_2 = a_2 x + a_3 u + a_4 v$ and $e_3 = b_2 x + b_3 u + b_4 v$ of $\ker\omega$ such that the family $\{e_1^2, x, e_2, e_3, w\}$ is a natural base of $\mathcal{E}$. We have $0 = e_2 x = a_2 x^2 = a_2(u + v)$ leads to $a_2 = 0$; $0 = e_3 x = b_2 x^2 = b_2(u + v)$ implies $b_2 = 0$; $0 = e_1^2 e_2 = (a_3 - a_4)w$ gives $a_3 = a_4$ and $0 = e_1^2 e_3 = (b_3 - b_4)w$ gives $b_3 = b_4$. Thus $e_2 = a_3(u + v)$, $e_3 = b_3(u + v)$ and the family $\{e_1^2, x, e_2, e_3, w\}$ is not a base of $\mathcal{E}$. So the sub-algebra generated by e_1^2 is not provided with the extension property.

3.6. **Automorphism in evolution train algebras.** Let $\mathcal{E} = Fe_1 \oplus \ker\omega$ be a finite-dimensional evolution train algebra such that $\omega(e_1) = 1$ and $e_1 \ker\omega = 0$. Then $e_1^2 = e_1 + x$ with $x \in \ker\omega$ and let $\sigma \in \mathrm{Aut}(\mathcal{E})$ be an automorphism of $\mathcal{E}$. Since the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\omega} & F \\ {\scriptstyle\sigma}\downarrow & \nearrow_{\omega} & \\ \mathcal{E} & & \end{array}$$

is commutative, $\omega \circ \sigma = \omega$. We have $\omega \circ \sigma(e_1) = \omega(e_1) = 1$, hence $\sigma(e_1) = e_1 + y$ with $y \in \ker\omega$. For all $z \in \ker\omega$, we have $0 = \omega(z) = \omega \circ \sigma(z)$ leads to $\sigma(\ker\omega) \subseteq \ker\omega$. The ideal $\ker\omega$ being stable under σ, we have $\sigma(\ker\omega) = \ker\omega$ and $\sigma_{|\ker\omega} \in \mathrm{Aut}(\ker\omega)$, where $\sigma_{|\ker\omega}$ is the restriction of σ to $\ker\omega$. Now $0 = \sigma(e_1 z) = \sigma(e_1)\sigma(z) = (e_1 + y)\sigma(z) = y\sigma(z)$, so $0 = y\ker\omega$, i.e. $y \in \mathrm{ann}(\ker\omega)$. We have $\sigma(e_1^2) = \sigma(e_1 + x) = \sigma(e_1) + \sigma(x) = e_1 + y + \sigma(x)$ and $\sigma(e_1)^2 = e_1^2 + y^2 = e_1^2 = e_1 + x$. Thus, $\sigma(e_1^2) = \sigma(e_1)^2$ gives $e_1 + y + \sigma(x) = e_1 + x$. Hence $y = x - \sigma(x) \in \mathrm{ann}(\ker\omega)$ and $\sigma(e_1) = e_1 + x - \sigma(x)$. We infer that $\mathrm{Aut}(\mathcal{E}) = \{\sigma \mid \sigma(e_1) = e_1 + x - \sigma(x),\ x - \sigma(x) \in \mathrm{ann}(\ker\omega),\ \sigma_{|\ker\omega} \in \mathrm{Aut}(\ker\omega)\}$. Thus, the knowledge of $\sigma_{|\ker\omega}$ makes it possible to determine $\sigma(e_1)$ and consequently σ.

Example 3.22. We consider an evolution train algebra $\mathcal{E} = Fe_1 \oplus \ker\omega$ where $e_1 \ker\omega = 0$, $e_1^2 = e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4$ and $\ker\omega =< e_2, e_3, e_4 >$ is an evolution nil-algebra of nil-index 4 defined by $e_2^2 = e_3$, $e_3^2 = e_4$, $e_4^2 = 0$. Let $\sigma \in \mathrm{Aut}(\mathcal{E})$ and $\varphi = \sigma_{|\ker\omega} \in \mathrm{Aut}(\ker\omega)$. We set $x = a_{12}e_2 + a_{13}e_3 + a_{14}e_4$ and $\varphi(e_2) = \sigma_{22}e_2 + \sigma_{23}e_3 + \sigma_{24}e_4$. We have $\varphi(e_3) = \varphi(e_2^2) = \varphi(e_2)^2 = \sigma_{22}^2 e_3 + \sigma_{23}^2 e_4$, $\varphi(e_4) = \varphi(e_3^2) = \varphi(e_3)^2 = \sigma_{22}^4 e_4$ and $0 = \varphi(e_2 e_3) = \varphi(e_2)\varphi(e_3) = \sigma_{22}^2\sigma_{23}e_4$ leads to $\sigma_{23} = 0$ because the linear transformation φ is an automorphism of

$\ker\omega$. We deduce that $\varphi(e_2) = \sigma_{22}e_2 + \sigma_{24}e_4$, $\varphi(e_3) = \sigma_{22}^2 e_3$, $\varphi(e_4) = \sigma_{22}^4 e_4$ and $x - \varphi(x) = a_{12}(1-\sigma_{22})e_2 + a_{13}(1-\sigma_{22}^2)e_3 + (a_{14} - a_{12}\sigma_{24} - a_{14}\sigma_{22}^4)e_4$. Moreover $x - \varphi(x) \in \operatorname{ann}(\ker\omega) = Fe_4$ if and only if $a_{12}(1-\sigma_{22}) = a_{13}(1-\sigma_{22}^2) = 0$. For

i) $\sigma_{22} = 1$, $\sigma(e_1) = e_1 - a_{12}\sigma_{24}e_4$, $\sigma(e_2) = e_2 + \sigma_{24}e_4$, $\sigma(e_3) = e_3$, $\sigma(e_4) = e_4$;

ii) $\sigma_{22} \neq 1$, then $a_{12} = 0$ and $\sigma(e_1) = e_1 + a_{14}(1-\sigma_{22}^4)e_4$, $\sigma(e_2) = \sigma_{22}e_2 + \sigma_{24}e_4$, $\sigma(e_3) = \sigma_{22}^2 e_3$, $\sigma(e_4) = \sigma_{22}^4 e_4$ with $a_{13}(1+\sigma_{22}) = 0$.

In ([7]), the authors give the classification of evolution nil-algebras of nil-index 2, 3 and 4. In the particular case of nil-index 4, they give the classification only for the power-associative algebras. Here, we give the classification of evolution nil-algebras of nil-index 4 up to diemension 5 and that are not power-associative.

4. Nil-algebra of nil-index 4 that are not power-associative

Theorem 4.1 ([7, Corollary 1]). *Let $B = \{e_1, \dots, e_n\}$ be a natural base of an evolution nil-algebra $\mathcal{E}$. Then $\mathcal{E}$ is fourth power-associative if and only if*

i) $e_i^2 e_j^2 = 0$ *for all* $1 \le i \le j \le n$;

ii) $(e_i^2 e_j)e_k = 0$ *for all* $1 \le i, j, k \le n$.

Definition 4.2. The algebra $\mathcal{E}$ is *decomposable* if there is the nonzero ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{E} = \mathcal{I} \oplus \mathcal{J}$. Otherwise, it is *indecomposable*.

Lemma 4.3 ([6, Corollary 2.6]). *Let $\mathcal{E}$ be a finite-dimensional evolution algebra such that $\dim_F(ann(\mathcal{E})) \ge \frac{1}{2}\dim_F(\mathcal{E}) \ge 1$. Then $\mathcal{E}$ is decomposable.*

In the following, N is an indecomposable evolution nil-algebra of nil-index 4, up to dimension 5, that is not power-associative. Let $B = \{e_1, \dots, e_n\}$ be a natural base of N. As $0 = N^4$, then $(e_i^2 e_j)e_k = 0$ for all $1 \le i, j, k \le n$. Thus, there are $i_1, i_2 \in \{1, \dots, n\}$ such that $e_{i_1}^2 e_{i_2}^2 \neq 0$ otherwise N would be power-associative. The type of N is $[n_1, n_2, n_3]$ where n_1, n_2, n_3 are nonzero integers such that $n_1 + n_2 + n_3 = \dim(N)$. Necessarily $\dim(N) \ge 3$.

In ([7, § 4.3]), the authors denote by $N_{i,j}$ the j-th indecomposable power-associative evolution nil-algebra of dimension i. We will continue with this notation. In the sens that if N_{i,j_0} is the last indecomposable power-associative evolution nil-algebra of dimension i, N_{i,j_0+k} would be k-th indecomposable nil-algebra of dimension i that is not power-associative.

With regard to Theorem 3.17 and Theorem 4.1, let F be a commutative field of $char(F) \neq 2, 3, 5$.

4.1. Case of 3-dimensional.

Theorem 4.4. *Let N be 3-dimensional indecomposable evolution nil-algebra of nil-index 4 that is not power-associative. Then N is isomorphic to the evolution evolution algebra $N_{3,4} : e_1^2 = e_2, e_2^2 = e_3$ and $e_3^2 = 0$ of type $[1, 1, 1]$.*

Proof. $\dim(N) = 3$, leads to $\dim(\operatorname{ann}(N)) < 1.5$; so $\dim(\operatorname{ann}(N)) = 1$ and the type of N is $[1, 1, 1]$. Let $B = \{e_1, e_2, e_3\}$ be a natural base of N such that $\operatorname{ann}(N) = Fe_3$ and $\operatorname{ann}^2(N) =< e_2, e_3 >$. The multiplication table in B is of form $e_1^2 = a_{12}e_2 + a_{13}e_3$, $e_2^2 = a_{23}e_3$, $e_3^2 = 0$, with $a_{12}, a_{23} \in F^*$. We set $u_2 = a_{12}e_2 + a_{13}e_3$ and $u_3 = u_2^2$; the family $\{e_1, u_2, u_3\}$ is a natural base of N and N is isomorphic to $N_{3,4} : e_1^2 = u_2$, $u_2^2 = u_3$, $u_3^2 = 0$. ∎

4.2. **Case of 4-dimensional.**

Theorem 4.5. *Let N be 4-dimensional indecomposable evolution nil-algebra of nil-index 4 that is not power-associative. Then N is isomorphic to one and only one evolution algebras in table 1.*

Table 1

N	*Multiplication*	*Type*
$N_{4,7}(\alpha)$	$e_1^2 = e_3$, $e_2^2 = \alpha e_3$, $e_3^2 = e_4$, $e_4^2 = 0$ *with* $\alpha \in F^*$	$[1, 1, 2]$
$N_{4,8}(\alpha, \beta)$	$e_1^2 = e_3$, $e_2^2 = \alpha e_3 + \beta e_4$, $e_3^2 = e_4$, $e_4^2 = 0$ *with* $\alpha, \beta \in F^*$	
$N_{4,9}(\alpha)$	$e_1^2 = e_2$, $e_2^2 = e_4$, $e_3^2 = \alpha e_4$, $e_4^2 = 0$ *with* $\alpha \in F^*$	$[1, 2, 1]$

Proof. $\dim(N) = 4$, leads to $\dim(\mathrm{ann}(N)) < 2$; so $\dim(\mathrm{ann}(N)) = 1$ and the possible types of N are $[1, 1, 2]$ and $[1, 2, 1]$. Let $B = \{e_1, e_2, e_3, e_4\}$ be a natural base of N such that $\mathrm{ann}(N) = Fe_4$.

- N is of type $[1, 1, 2]$. We take $\mathrm{ann}^2(N) =< e_3, e_4 >$ and we have $\mathcal{U}_1 = Fe_4$, $\mathcal{U}_2 = Fe_3$, $\mathcal{U}_3 = Fe_1 \oplus Fe_2$. The multiplication table in B is of form $e_1^2 = a_{13}e_3 + a_{14}e_4$, $e_2^2 = a_{23}e_3 + a_{24}e_4$, $e_3^2 = a_{34}e_4$, $e_4^2 = 0$ with $a_{13}, a_{23}, a_{34} \in F^*$. We set $u_3 = a_{13}e_3 + a_{14}e_4$, $u_4 = u_3^2$; the family $\{e_1, e_2, u_3, u_4\}$ is a natural base of N and its multiplication table is of form $e_1^2 = u_3$, $e_2^2 = \alpha u_3 + \beta u_4$, $u_3^2 = u_4$, $u_4^2 = 0$, with $\alpha \in F^*$ because $a_{13}, a_{23} \in F^*$. We have $(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 =< u_3, \beta u_4 >$.
  If $\dim_F(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 1$, then $\beta = 0$ and N is isomorphic to $N_{4,7}(\alpha) : e_1^2 = u_3$, $e_2^2 = \alpha u_3$, $u_3^2 = u_4$, $u_4^2 = 0$, with $\alpha \in F^*$.
  If $\dim_F(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 2$, then $\beta \neq 0$ and N is isomorphic to $N_{4,8}(\alpha, \beta) : e_1^2 = u_3$, $e_2^2 = \alpha u_3 + \beta u_4$, $u_3^2 = u_4$, $u_4^2 = 0$, with $\alpha, \beta \in F^*$.
- N is of type $[1, 2, 1]$. We take $\mathrm{ann}^2(N) =< e_2, e_3, e_4 >$. The multiplication table in B is of form $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4$, $e_2^2 = a_{24}e_4$, $e_3^2 = a_{34}e_4$, $e_4^2 = 0$ with $a_{24}, a_{34} \in F^*$ and $(a_{12}, a_{13}) \neq (0, 0)$. Moreover $a_{12}^2 a_{24} + a_{13}^2 a_{34} \neq 0$ because $e_1^2 e_1^2 \neq 0$, otherwise N would be power-associative. We set $u_2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4$ and $u_4 = u_2^2$. Let's us determine a vector $u_3 = ae_2 + be_3$ such that the family $\{e_1, u_2, u_3, u_4\}$ is a natural base of N. The orthogonality of u_2 and u_3 gives $a_{12}a_{24}a + a_{13}a_{34}b = 0$. We distinguish two cases
  i) $a_{12} = 0$, then we can take $u_3 = e_2$;
  ii) $a_{12} \neq 0$, then $a = -(a_{12}a_{24})^{-1}(a_{13}a_{34})b$.
  We can take $u_3 = -(a_{12}a_{24})^{-1}(a_{13}a_{34})e_2 + e_3$.
  In each cases, the multiplication table in $\{e_1, u_2, u_3, u_4\}$ is of form $e_1^2 = u_2$, $u_2^2 = u_4$, $u_3^2 = \alpha u_4$, $u_4^2 = 0$ with $\alpha \in F^*$. We deduce that N is isomorphic to $N_{4,9}(\alpha)$.

□

4.3. **Case of 5-dimensional.** If $\dim(N) = 5$ then $\dim(\mathrm{ann}(N)) < 2.5$; so $\dim(\mathrm{ann}(N)) = 1$ or 2. The possible types of N are $[1, 1, 3]$, $[1, 2, 2]$, $[1, 3, 1]$, $[2, 1, 2]$ and $[2, 2, 1]$.

Theorem 4.6. *Let N be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[1,1,3]$ that is not power-associative. Then N is isomorphic to one and only one evolution algebras in table 2.*

Table 2

N	*Multiplication*
$N_{5,13}(\alpha,\beta)$	$e_1^2=e_4,\ e_2^2=\alpha e_4,\ e_3^2=\beta e_4,\ e_4^2=e_5,\ e_5^2=0$ *with* $\alpha,\beta\in F^*$.
$N_{5,14}(\alpha,\beta,\gamma)$	$e_1^2=e_4,\ e_2^2=\alpha e_4+\beta e_5,\ e_3^2=\gamma(\alpha e_4+\beta e_5),\ e_4^2=e_5,\ e_5^2=0$ *with* $\alpha,\beta,\gamma\in F^*$.
$N_{5,15}(\alpha,\beta,\gamma,\delta)$	$e_1^2=e_4,\ e_2^2=\alpha e_4+\beta e_5,\ e_3^2=\gamma e_4+\delta e_5,\ e_4^2=e_5,\ e_5^2=0$ *with* $\alpha,\beta,\gamma,\delta\in F^*$ *and* $\alpha\delta-\beta\gamma\neq 0$

Proof. N is of type $[1,1,3]$. Let $B=\{e_1,\dots,e_5\}$ be a natural base of N such that $\mathrm{ann}(N)=Fe_5$ and $\mathrm{ann}^2(N)=<e_4,e_5>$. We have $\mathcal{U}_1=Fe_5$, $\mathcal{U}_2=Fe_4$ and $\mathcal{U}_3=Fe_1\oplus Fe_2\oplus Fe_3$; the multiplication table in B is of form $e_1^2=a_{14}e_4+a_{15}e_5$, $e_2^2=a_{24}e_4+a_{25}e_5$, $e_3^2=a_{34}e_4+a_{35}e_5$, $e_4^2=a_{45}e_5$, $e_5^2=0$, with $a_{14},a_{24},a_{34},a_{45}\in F^*$. Since $e_i^2e_i^2=a_{i4}^2a_{45}e_5\neq 0$, for $1\le i\le 3$, we set $u_4=e_1^2$ and $u_5=e_1^2e_1^2$; the family $\{e_1,e_2,e_3,u_4,u_5\}$ is a natural base of N and its multiplication table is of form $e_1^2=u_4$, $e_2^2=\alpha_4u_4+\alpha_5u_5$, $e_3^2=\beta_4u_4+\beta_5u_5$, $u_4^2=u_5$, $u_5^2=0$, with $\alpha_4,\beta_4\in F^*$ because $a_{14}a_{24}a_{34}\neq 0$. We have $(\mathcal{U}_3\oplus\mathcal{U}_1)^2=<u_4,\alpha_5u_5,\beta_5u_5>$.

If $\dim(\mathcal{U}_3\oplus\mathcal{U}_1)^2=1$, then $\alpha_5=\beta_5=0$ and N is isomorphic to $N_{5,13}(\alpha_4,\beta_4)$: $e_1^2=u_4$, $e_2^2=\alpha_4u_4$, $e_3^2=\beta_4u_4$, $u_4^2=u_5$, $u_5^2=0$, with $\alpha_4,\beta_4\in F^*$.

If $\dim(\mathcal{U}_3\oplus\mathcal{U}_1)^2=2$, then $(\alpha_5,\beta_5)\neq(0,0)$. We distinguish four cases:

i) $det(e_2^2,e_3^2)=\alpha_4\beta_5-\alpha_5\beta_4=0$, then $\beta_5=\alpha_4^{-1}\alpha_5\beta_4$ and necessarily $\beta_5\alpha_5\neq 0$. So $e_1^2=u_4$, $e_2^2=\alpha_4u_4+\alpha_5u_5$, $e_3^2=\beta_4u_4+\alpha_4^{-1}\alpha_5\beta_4u_5=\beta_4\alpha_4^{-1}(\alpha_4u_4+\alpha_5u_5)$, $u_4^2=u_5$, $u_5^2=0$, with $\alpha_4,\alpha_5,\beta_4\in F^*$. We deduce that N is isomorphic to $N_{5,14}(\alpha_4,\alpha_5,\beta_4\alpha_4^{-1})$.

ii) $det(e_2^2,e_3^2)\neq 0$ and $\beta_5=0$. We set $v_4=e_2^2$ and $v_5=v_4^2$. We have $e_1^2=\alpha_4^{-1}v_4-\alpha_4^{-2}\alpha_5v_5$, $e_2^2=v_4$, $e_3^2=\beta_4(\alpha_4^{-1}v_4-\alpha_4^{-2}\alpha_5v_5)$, $v_4^2=v_5$ and $v_5^2=0$. We obtain $N_{5,14}(\alpha_4^{-1},-\alpha_4^{-2}\alpha_5,\beta_4)$.

iii) $det(e_2^2,e_3^2)\neq 0$ and $\alpha_5=0$. By permuting the vectors e_2 and e_3, we get case *ii)*.

iv) $det(e_2^2,e_3^2)\neq 0$ and $\alpha_5\beta_5\neq 0$. Then $e_1^2=u_4$, $e_2^2=\alpha_4u_4+\alpha_5u_5$, $e_3^2=\beta_4u_4+\beta_5u_5$, $u_4^2=u_5$, $u_5^2=0$ with $\alpha_4,\alpha_5,\beta_4,\beta_5\in F^*$ and $\alpha_4\beta_5-\alpha_5\beta_4\neq 0$. So N is isomorphic to $N_{5,15}(\alpha_4,\alpha_5,\beta_4,\beta_5)$.

□

Theorem 4.7. *Let N be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[1,2,2]$ that is not power-associative. Then N is isomorphic to one and only one evolution algebras in table 3.*

Table 3

N	*Multiplication*
$N_{5,16}(\alpha,\beta)$	$e_1^2=e_3,\ e_2^2=\alpha e_3,\ e_3^2=e_5, e_4^2=\beta e_5,\ e_5^2=0$ *with* $\alpha,\beta\in F^*$.
$N_{5,17}(\alpha,\beta,\gamma)$	$e_1^2=e_3,\ e_2^2=\alpha e_3+\beta e_5,\ e_3^2=e_5, e_4^2=\gamma e_5,\ e_5^2=0$ *with* $\alpha,\beta,\gamma\in F^*$.

N	*Multiplication*
$N_{5,18}(\alpha,\beta)$	$e_1^2=e_3,\ e_2^2=\alpha e_3+e_4,\ e_3^2=e_5, e_4^2=\beta e_5,\ e_5^2=0$ *with* $\alpha\in F$ *and* $\beta\in F^*$.
$N_{5,19}(\alpha)$	$e_1^2=e_3+e_4,\ e_2^2=\alpha(e_3-e_4),\ e_3^2=e_5, e_4^2=-e_5,\ e_5^2=0$ *with* $\alpha\in F^*$

Proof. N is of type $[1,2,2]$. Let $B=\{e_1,\dots,e_5\}$ be a natural base of N such that $\text{ann}(N)=<e_5>$ and $\text{ann}^2(N)=<e_3,e_4,e_5>$. We have $\mathcal{U}_1=Fe_5$, $\mathcal{U}_2=Fe_3\oplus Fe_4$, $\mathcal{U}_3=Fe_1\oplus Fe_2$ and a multiplication table in B is of form $e_1^2=a_{13}e_3+a_{14}e_4+a_{15}e_5=a+a_{15}e_5$, $e_2^2=a_{23}e_3+a_{24}e_4+a_{25}e_5=b+a_{25}e_5$, $e_3^2=a_{35}e_5$, $e_4^2=a_{45}e_5$ and $e_5^2=0$ with $a,b\neq 0$ and $a_{35},a_{45}\in F^*$. Then, either $a^2=b^2=0$, i.e. $e_1^2e_1^2=e_2^2e_2^2=0$, either $a^2\neq 0$ or $b^2\neq 0$.

- By symmetry, we suppose that $a^2\neq 0$. We set $u_3=a+a_{15}e_5$ and $u_5=a^2$. Let's us determine the vector $u_4=c_3e_3+c_4e_4\in\mathcal{U}_2$ such that the family $\{e_1,e_2,u_3,u_4,u_5\}$ is a natural base of N. The orthogonality of the vectors u_3 and u_4 implies $c_3a_{13}a_{35}+c_4a_{14}a_{45}=0$. We distinguish three cases:
  - *i)* $a_{13}=0$, then $c_4=0$ and we can take $u_4=e_3$;
  - *ii)* $a_{14}=0$, then $c_3=0$ and we can take $u_4=e_4$;
  - *iii)* $a_{13}a_{14}\neq 0$, then $c_4=-c_3(a_{13}a_{35})(a_{14}a_{45})^{-1}$ and we can take $u_4=e_3-(a_{13}a_{35})(a_{14}a_{45})^{-1}e_4$.

  In each cases, the multiplication table in the natural base $\{e_1,e_2,u_3,u_4,u_5\}$ is of form $e_1^2=u_3$, $e_2^2=\alpha u_3+\beta u_4+\gamma u_5$, $u_3^2=u_5$, $u_4^2=\delta u_5$, $u_5^2=0$ with $(\alpha,\beta)\neq(0,0)$, $\delta\in F^*$ and we have $N^2=<u_3,\beta u_4,u_5>$.

  If $\dim(N^2)=2$, then $\beta=0$ and $(\mathcal{U}_3\oplus\mathcal{U}_1)^2=<u_3,\gamma u_5>$.
  - *a)* $\dim(\mathcal{U}_3\oplus U_1)^2=1$, gives $\gamma=0$ and N is isomorphic to $N_{5,16}(\alpha,\delta): e_1^2=u_3$, $e_2^2=\alpha u_3$, $u_3^2=u_5$, $u_4^2=\delta u_5$, $u_5^2=0$ with $\alpha,\delta\in F^*$.
  - *b)* $\dim(\mathcal{U}_3\oplus U_1)^2=2$, leads to $\gamma\neq 0$ and N is isomorphic to $N_{5,17}(\alpha,\gamma,\delta)$: $e_1^2=u_3$, $e_2^2=\alpha u_3+\gamma u_5$, $u_3^2=u_5$, $u_4^2=\delta u_5$, $u_5^2=0$ with $\alpha,\gamma,\delta\in F^*$.

  If $\dim(N^2)=3$, then $\beta\neq 0$ and we set $u_4'=\beta u_4+\gamma u_5$. The family $\{e_1,e_2,u_3,u_4',u_5\}$ is a natural base of N and N is isomorphic to $N_{5,18}(\alpha,\delta')$: $e_1^2=u_3$, $e_2^2=\alpha u_3+u_4'$, $u_3^2=u_5$, $(u_4')^2=\beta^2\delta u_5=\delta' u_5$, $u_5^2=0$, with $\alpha\in F$ and $\delta'\in F^*$.
- Suppose that $a^2=b^2=0$. We have $a^2=0$ implies $a_{13},a_{14}\in F^*$, otherwise $a=0$. Similarly $a_{23},a_{24}\in F^*$, otherwise $b=0$. We set $u_3=a_{13}e_3+a_{15}e_5$, $u_4=a_{14}e_4$ and $u_5=u_3^2$; as $0=e_1^2e_1^2=(u_3+u_4)^2=u_3^2+u_4^2$, then $u_4^2=-u_3^2=-u_5$ and the family $\{e_1,e_2,u_3,u_4,u_5\}$ is a natural base of N. The multiplication table in this base is of form $e_1^2=u_3+u_4$, $e_2^2=\alpha u_3+\beta u_4+\gamma u_5$, $u_3^2=u_5$, $u_4^2=-u_5$, $u_5^2=0$ with $(\alpha,\beta)\neq(0,0)$. We have $0=e_2^2e_2^2=(\alpha^2-\beta^2)u_5$ leads to $\alpha=\beta$ or $\alpha=-\beta$. If $\alpha=\beta$, then $e_1^2e_2^2=(\alpha-\beta)u_5=0$ and N would be power-associative. We infer that $\alpha=-\beta\neq 0$ and we set $v_3=u_3+\frac{1}{2}\alpha^{-1}\gamma u_5$ and $v_4=u_4-\frac{1}{2}\alpha^{-1}\gamma u_5$. Then N is isomorphic to $N_{5,19}(\alpha)$: $e_1^2=v_3+v_4$, $e_2^2=\alpha(v_3-v_4)$, $v_3^2=u_5$, $v_4^2=-u_5$, $u_5^2=0$ with $\alpha\in F^*$.

□

Theorem 4.8. *Let N be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[1,3,1]$ that is not power-associative. Then N is isomorphic to $N_{5,20}(\alpha,\beta): e_1^2=e_2, e_2^2=e_5, e_3^2=\alpha e_5, e_4^2=\beta e_5, e_5^2=0$ with $\alpha,\beta\in F^*$.*

Proof. N is of type $[1,3,1]$. Let $B=\{e_1,\dots,e_5\}$ be a natural base of N such that $\mathrm{ann}(N)=<e_5>$ and $\mathrm{ann}^2(N)=<e_2,e_3,e_4,e_5>$. We have $\mathcal{U}_1=Fe_5$, $\mathcal{U}_2=Fe_2\oplus Fe_3\oplus Fe_4$ and $\mathcal{U}_3=Fe_1$. The multiplication table in B is of form $e_1^2=a_{12}e_2+a_{13}e_3+a_{14}e_4+a_{15}e_5=a+a_{15}e_5$, $e_2^2=a_{25}e_5$, $e_3^2=a_{35}e_5$, $e_4^2=a_{45}e_5$, $e_5^2=0$ with $a\neq 0$ and $a_{25},a_{35},a_{45}\in F^*$. We have $0\neq e_1^2e_1^2=(a_{12}^2a_{25}+a_{13}^2a_{35}+a_{14}^2a_{45})e_1^2$ otherwise N would be power-associative. In the following, we assume that $a_{12}\neq 0$, otherwise we reorder the vectors e_2, e_3 and e_4 of natural base. We set $u_2=e_1^2$, $u_5=e_1^2e_1^2$ and let's look for the vectors $u_3=b_2e_2+b_3e_3+b_4e_4$, $u_4=c_2e_2+c_3e_3+c_4e_4\in\mathcal{U}_2$ such that the family $\{e_1,u_2,u_3,u_4,,u_5\}$ is a natural base of N. The orthogonality of u_2 with the vectors u_3 and u_4 leads to $b_2=-(b_3a_{13}a_{35}+b_4a_{14}a_{45})(a_{12}a_{25})^{-1}$ and $c_2=-(c_3a_{13}a_{35}+c_4a_{14}a_{45})(a_{12}a_{25})^{-1}$. We deduce that

$$\begin{aligned}u_3 &= -(b_3a_{13}a_{35}+b_4a_{14}a_{45})(a_{12}a_{25})^{-1}e_2+b_3e_3+b_4e_4,\\ u_4 &= -(c_3a_{13}a_{35}+c_4a_{14}a_{45})(a_{12}a_{25})^{-1}e_2+c_3e_3+c_4e_4.\end{aligned}$$

The orthogonality of u_3 and u_4 gives

$$\begin{aligned}&b_3a_{35}(c_3(a_{12}^2a_{25}+a_{13}^2a_{35})+c_4a_{13}a_{14}a_{45})+\\ &b_4a_{45}(c_4(a_{12}^2a_{25}+a_{14}^2a_{45})+c_3a_{13}a_{14}a_{35})=0.\end{aligned}\tag{4.1}$$

Moreover the family $\{e_1,u_2,u_3,u_4,u_5\}$ is linearly independent if and only if the family $\{u_3,u_4\}$ is linearly independent. The family $\{u_3,u_4\}$ is linearly independent if and only if $rang(\{u_3,u_4\})=2$, i.e. $b_3c_4-b_4c_3\neq 0$. We distinguish the following cases:

i) $a_{12}^2a_{25}+a_{13}^2a_{35}\neq 0$, we can take $b_4=0$, then $b_3\neq 0$ and $c_3=-(a_{13}a_{14}a_{45})(a_{12}^2a_{25}+a_{13}^2a_{35})^{-1}c_4$. In this case

$$\begin{aligned}u_3 &= -(a_{13}a_{35})(a_{12}a_{25})^{-1}e_2+e_3,\\ u_4 &= -a_{12}a_{14}a_{45}(a_{12}^2a_{25}+a_{13}^2a_{35})^{-1}e_2-(a_{13}a_{14}a_{45})(a_{12}^2a_{25}+a_{13}^2a_{35})^{-1}e_3+e_4\ ;\end{aligned}$$

ii) $a_{12}^2a_{25}+a_{14}^2a_{45}\neq 0$, we can take $b_3=0$, then $b_4\neq 0$ and $c_4=-(a_{13}a_{14}a_{35})(a_{12}^2a_{25}+a_{14}^2a_{45})^{-1}c_3$. In this case,

$$\begin{aligned}u_3 &= -(a_{14}a_{45})(a_{12}a_{25})^{-1}e_2+e_4,\\ u_4 &= -a_{12}a_{13}a_{35}(a_{12}^2a_{25}+a_{14}^2a_{45})^{-1}e_2+e_3-(a_{13}a_{14}a_{35})(a_{12}^2a_{25}+a_{14}^2a_{45})^{-1}e_4\ ;\end{aligned}$$

iii) $a_{12}^2a_{25}=-a_{13}^2a_{35}=-a_{14}^2a_{45}$, then (4.1) becomes $b_3c_4+b_4c_3=0$. We can take $b_4=b_3$ and $c_4=-c_3$. In this cases,

$$\begin{aligned}u_3 &= -(a_{13}a_{35}+a_{14}a_{45})(a_{12}a_{25})^{-1}e_2+e_3+e_4,\\ u_4 &= -(a_{13}a_{35}-a_{14}a_{45})(a_{12}a_{25})^{-1}e_2+e_3-e_4.\end{aligned}$$

In each cases, N is isomorphic to $N_{5,20}(\alpha,\beta)$: $e_1^2=u_2$, $u_2^2=u_5$, $u_3^2=\alpha u_5$, $u_4^2=\beta u_5$, $u_5^2=0$ with $\alpha,\beta\in F^*$. □

Theorem 4.9. *Let N be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[2,1,2]$ that is not power-associative. Then N is isomorphic to $N_{5,21}(\alpha)$: $e_1^2=e_3$, $e_2^2=\alpha e_3+e_5$, $e_3^2=e_4$, $e_4^2=e_5^2=0$ with $\alpha\in F^*$.*

Proof. N is of type $[2,1,2]$. Let $B=\{e_1,\dots,e_5\}$ be a natural base of N such that $\mathrm{ann}(N)=<e_4,e_5>$ and $\mathrm{ann}^2(N)=<e_3,e_4,e_5>$. We have $\mathcal{U}_1=Fe_4\oplus Fe_5$, $\mathcal{U}_2=Fe_3$, $\mathcal{U}_3=Fe_1\oplus Fe_2$ and the multiplication table in B is of form $e_1^2=$

$a_{13}e_3 + a_{14}e_4 + a_{15}e_5$, $e_2^2 = a_{23}e_3 + a_{24}e_4 + a_{25}e_5$, $e_3^2 = a_{34}e_4 + a_{35}e_5$, $e_4^2 = e_5^2 = 0$ with $a_{13}, a_{23} \in F^*$ and $(a_{34}, a_{35}) \neq (0,0)$. We set $u_3 = e_1^2$, $u_4 = e_1^2 e_1^2$ and let $u_5 \in \operatorname{ann}(N)$ such that $\operatorname{ann}(N) = Fu_4 \oplus Fu_5$, then the family $\{e_1, e_2, u_3, u_4, u_5\}$ is a natural base of N and its multiplication table is of form $e_1^2 = u_3$, $e_2^2 = \alpha u_3 + \beta u_4 + \gamma u_5$, $u_3^2 = u_4$, $u_4^2 = u_5^2 = 0$ with $\alpha \in F^*$. If $\gamma = 0$, then $N =< e_1, e_2, u_3, u_4 > \oplus < u_5 >$ is decomposable. So $\gamma \neq 0$ and we set $u_5' = \beta u_4 + \gamma u_5$. The family $\{e_1, e_2, u_3, u_4, u_5'\}$ is a natural base of N and N is isomorphic to $N_{5,21}(\alpha) : e_1^2 = u_3$, $e_2^2 = \alpha u_3 + u_5'$, $u_3^2 = u_4$, $u_4^2 = (u_5')^2 = 0$ with $\alpha \in F^*$. ☐

Theorem 4.10. *Let N be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[2,2,1]$ that is not power-associative. Then N is isomorphic to $N_{5,22} : e_1^2 = u_2 + u_3$, $u_2^2 = u_4$, $u_3^2 = u_5$, $u_4^2 = u_5^2 = 0$.*

Proof. N is of type $[2,2,1]$. Let $B = \{e_1, \ldots, e_5\}$ be a natural base of N such that $\operatorname{ann}(N) =< e_4, e_5 >$ and $\operatorname{ann}^2(N) =< e_2, e_3, e_4, e_5 >$. We have $\mathcal{U}_1 = Fe_4 \oplus Fe_5$, $\mathcal{U}_2 = Fe_2 \oplus Fe_3$, $\mathcal{U}_3 = Fe_1$ and the multiplication table in B is of form $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 + a_{15}e_5$, $e_2^2 = a_{24}e_4 + a_{25}e_5$, $e_3^2 = a_{34}e_4 + a_{35}e_5$, $e_4^2 = e_5^2 = 0$ with $(a_{12}, a_{13}) \neq (0,0)$, $(a_{24}, a_{25}) \neq (0,0)$ and $(a_{34}, a_{35}) \neq (0,0)$. We have $0 \neq e_1^2 e_1^2 = a_{12}^2 e_2^2 + a_{13}^2 e_3^2$ otherwise N would be power-associative. We assume that $a_{12} \neq 0$ otherwise, we permute the vectors e_2 and e_3 of the natural base. Moreover the family $\{e_2^2, e_3^2\}$ is a linearly independent otherwise $N =< e_1, a_{12}e_2 + a_{14}e_4 + a_{15}e_5, e_3, e_2^2 > \oplus < u_5 >$ would be decomposable with $\operatorname{ann}(N) = Fe_2^2 \oplus Fu_5$: impossible. In the same way $a_{13} \neq 0$, otherwise $N =< e_1, a_{12}e_2 + a_{14}e_4 + a_{15}e_5, e_2^2 > \oplus < e_3, e_3^2 >$ would be decomposable. We set $u_2 = a_{12}e_2 + a_{14}e_4$, $u_3 = a_{13}e_3 + a_{15}e_5$, $u_4 = u_2^2$ and $u_5 = u_3^2$. So the family $\{e_1, u_2, u_3, u_4, u_5\}$ is a natural base of N and N is isomorphic to $N_{5,22} : e_1^2 = u_2 + u_3$, $u_2^2 = u_4$, $u_3^2 = u_5$, $u_4^2 = u_5^2 = 0$. ☐

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