EVOLUTION TRAIN ALGEBRAS

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Abstract. Through this paper, we show that the criteria for real evolution algebra to be a baric algebra can be extended to any evolution algebra over a commutative field of characteristic \( \neq 2 \). Then we prove that an evolution algebra \( \mathcal{E} \) is a train algebra of rank \( r + 1 \) if and only if the kernel of its weight function is nil of nil-index \( r > 1 \). We also study special train evolution algebra and characterize idempotents, power-associativity and automorphism in evolution train algebra. Finally we classify up to dimension 5, indecomposable evolution nil-algebra of nil-index 4 that are not power-associative.

1. Introduction

In 2006, J.P. Tian and P. Vojtěchovský introduced the notion of evolution algebras in the literature ([9]). Let \( \mathcal{E} \) be a \( n \)-dimensional algebra over a commutative field \( F \). We said that \( \mathcal{E} \) is an evolution algebra if there is a base \( B = \{ e_1, \ldots, e_n \} \) such that
\[
e_i e_j = 0, \text{ for } 1 \leq i \neq j \leq n.
\]
Such a base is called a natural base of \( \mathcal{E} \). The multiplication in \( \mathcal{E} \) is determined by the products
\[
e_i^2 = \sum_{k=1}^{n} a_{ik} e_k \text{ for all } 1 \leq i \leq n
\]
and \( M = (a_{ik})_{1 \leq i, k \leq n} \) is the structural constants matrix of \( \mathcal{E} \) relative to the natural base \( B \).

Evolution algebras are commutative ([8]) and they are associative if and only if \( e_i^2 e_j = 0 \) for all \( 1 \leq i \neq j \leq n \) ([7]).

In ([3]), the authors establish an equivalence between an evolution nil-algebra and nilpotent evolution algebra.

In section 2, we recall some results about evolution nil-algebras and we show that theirs nil-index and index of nilpotency are the same.

In section 3, we characterize baric evolution algebras over a commutative field \( F \) of \( \text{char}(F) \neq 2 \) and then we show that there is no evolution train algebra of rank 2 and a baric evolution algebra \( (\mathcal{E}, \omega) \) is a train algebra of rank \( r + 1 > 2 \),
We have an evolution train algebra. Then we study evolution algebras that are
special train algebras and characterize idempotents, power-associativity and
automorphism in evolution train algebras.

Section 4 is devoted to the classification, up to dimension 5, of indecomposable
evolution nil-algebras of nil-index 4 that are not power-associative.

2. Nilpotent evolution algebras

Let \( B = \{e_i; 1 \leq i \leq n\} \) be a natural base of finite-dimensional evolution
algebra \( \mathcal{E} \) over a commutative field \( F \) such that the multiplication table in \( B \) is
given by (1.1). The principal powers, of an element \( a \in \mathcal{E} \), are defined as follows
\( a^1 = a \) and \( a^{k+1} = a^k a \) \( (k \geq 1) \) while that of \( \mathcal{E} \) are defined by:
\[
\mathcal{E}^1 = \mathcal{E}, \quad \mathcal{E}^{k+1} = \mathcal{E}^k \mathcal{E} \quad \text{where} \quad k \geq 1.
\]

**Definition 2.1.** We said that algebra \( \mathcal{E} \) is:

\( i) \) nilpotent, if there is a nonzero integer \( r \) such that \( \mathcal{E}^r = 0 \); such a smaller
integer is called the index of nilpotency of \( \mathcal{E} \);

\( ii) \) nil, if for all \( a \in \mathcal{E} \), there is a nonzero integer \( s \) such that \( a^s = 0 \); such a
smaller integer is called index of nilpotency of \( a \);

\( iii) \) a nil-algebra of bounded index, if the index of nilpotency of all elements are
bounded by some number \( n \); such a smaller \( n \) is called the nil-index of \( \mathcal{E} \).

**Theorem 2.2** ([3, Theorem 2.7]). The following statements are equivalent:

\( i) \) The matrix corresponding to \( \mathcal{E} \) can be written as

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & 0 & a_{23} & \cdots & a_{2n} \\
0 & 0 & 0 & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

\( ii) \) \( \mathcal{E} \) is nilpotent algebra.

\( iii) \) \( \mathcal{E} \) is nil-algebra.

The following lemma 2.3 specifies that the index of nilpotency is equal to the
nil-index.

**Lemma 2.3.** Let \( \mathcal{E} \) be a finite-dimensional evolution nil-algebra of nil-index \( m \).
Then \( \mathcal{E} \) is nilpotent of index of nilpotency \( m \).

**Proof.** Let \( \mathcal{E} \) be a \( n \)-dimensional evolution nil-algebra of nil-index \( m \). Then
\( m \leq n + 1 \) and \( \mathcal{E} \) admits a natural base \( B = \{e_i; 1 \leq i \leq n\} \) such that
\( e_i^2 = \sum_{j=i+1}^{n} a_{ij} e_j \) (Theorem 2.2); moreover, for all \( i \in \{1, \ldots, n\} \), \( e_i^3 = 0 \).
We have \( \mathcal{E}^2 = \mathcal{E} \mathcal{E} = \langle e_i e_j; 1 \leq i, j \leq n \rangle = \langle e_i^2; 1 \leq i \leq n \rangle \) because
e\( e_i e_j = 0 \) for \( i \neq j \); \( \mathcal{E}^3 = \mathcal{E}^2 \mathcal{E} = \langle e_i^2 e_j; 1 \leq i, j \leq n \rangle \) and by induction
\( \mathcal{E}^m = \mathcal{E}^{m-1} \mathcal{E} = \langle (e_i^2 e_j) \cdots e_{i_{m-1}}; 1 \leq i_j \leq n \text{ and } 1 \leq j \leq m-1 \rangle \). Let's
show that \( \mathcal{E}^m = 0 \). We consider the element \( x_i = e_{i_1} + e_{i_2} + \cdots + e_{i_{m-1}} =
\sum_{k=1}^{m-1} e_{i_k} \) with \( 1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n \). We have \( x_i^2 = \sum_{k=1}^{m-1} e_{i_k}^2 \),
x\( i_3 = \sum_{k,j=1}^{m-1} e_{i_k}^2 e_{i_j} \) and by induction \( x_i^m = \sum_{k_1,\ldots,k_{m-1}=1}^{m-1} (\cdots (e_{i_{k_1}} e_{i_{k_2}}) \cdots e_{i_{k_{m-1}}}) \).
Since \((\cdots (e_{i_1}^2 e_{i_2}) \cdots) e_{i_{m-1}} = a_{i_1i_2} \cdots a_{i_{m-2}i_{m-1}} e_{i_{m-1}}^2\) and that the structural constants matrix of \(E\) relative to \(B\) is upper triangular, it follows that 

\[1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n.\] 

As \(i_{k_j} \in \{i_1, \ldots, i_{m-1}\}\) for \(j \in \{1, \ldots, m-1\}\) and that \((i_1, \ldots, i_{m-1})\) is the unique \((m-1)\)-tuple of \(\{i_1, \ldots, i_{m-1}\}\) such that 

\[1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n,\] 

then 

\[\sum_{k_1, \ldots, k_{m-1}=1}^{m-1} (\cdots (e_{i_1}^2 e_{i_2}) \cdots) e_{i_{m-1}} = (\cdots (e_{i_1}^2 e_{i_2}) \cdots) e_{i_{m-1}}.\] 

We infer that \(0 = x_i = (\cdots (e_{i_1}^2 e_{i_2}) \cdots) e_{i_{m-1}}.\) So \(E^m = 0\) and we get the lemma.

\[\square\]

**Definition 2.4.** Let \(E\) be an evolution algebra:

i) The annihilator of \(E\) is defined by \(\text{ann}(E) = \{x \in E : xE = 0\}\).

ii) We also define \(\text{ann}^i(E)\) by \(\text{ann}^1(E)/\text{ann}^{i-1}(E) = \text{ann}(E/\text{ann}^{i-1}(E)), i \geq 2\).

Let \(B = \{e_1, \ldots, e_n\}\) be a natural base of an evolution algebra \(E\) over a commutative field. In \([5, \text{Lemme 2.7}]\), the authors show that \(\text{ann}(E) = \text{span}\{e_i \in B | e_i^2 = 0\}\). In \([6]\), they show that \(\text{ann}(E) = \text{span}\{e \in B | e_i^2 \in \text{ann}^{i-1}(E)\}\) and that the base \(B = B_1 \cup \cdots \cup B_r\) is a natural base where \(B_i = \{e \in B | e_i^2 \in \text{ann}^{i-1}(E), e \notin \text{ann}^{i-1}(E)\}\). Then, for \(U_i := \text{span}(B_i), i = 1, \ldots, r,\) we have \(U_i \oplus \cdots \oplus U_i = \text{ann}(E) (i = 1, \ldots, r)\). They prove that \(U_i \oplus U_i = \{x \in \text{ann}(E) | x \text{ann}^{i-1}(E) = 0\}\) is an invariant of evolution nil-algebra.

The type of an evolution nil-algebra \(E\) is the sequence \([n_1, n_2, \ldots, n_r]\) where \(r\) and \(n_i\) are integers defined by \(\text{ann}(E) = \hat{E}; n_i = \dim_F(\text{ann}(E)) - \dim_F(\text{ann}^{i-1}(E))\) and \(n_1 + \cdots + n_i = \dim_F(\text{ann}(E))\) for all \(i \in \{1, \ldots, r\}\).

3. Evolution train algebras


**Definition 3.1** ([10, Definition 1.7]). We said that an algebra \(E\) over a field \(F\) is baric, if it admits \(\omega : E \to F\) a non trivial algebra homomorphism. The homomorphism \(\omega\) is called the weight function or weight homomorphism of \(E\).

**Theorem 3.2** ([4, Theorem 3.2]). A \(n\)-dimensional evolution algebra \(E\), over the field \(\mathbb{R}\), is baric if and only if there is a column \((a_{i_10}, \ldots, a_{n0})^T\) of its strctural constants matrix \(M = (a_{ij}), i,j=1,\ldots,n,\), such that \(a_{i_00} \neq 0\) and \(a_{ii0} = 0, i \neq i_0\). The corresponding weight function is \(\omega(x) = a_{i_00}x_{i_0}\) where \(x = \sum_{k=1}^n x_k e_k \in E\).

This theorem remains true, when we replace the field \(\mathbb{R}\) by any commutative field \(F\) of \(\text{char}(F) \neq 2\). In the following, unless otherwise indicated, \(F\) designates such a field.

**Theorem 3.3.** The evolution algebra \(E\) is baric if and only if there is \(i_0 \in \{1, \cdots, n\}\) such that \(a_{i_0i_0} \neq 0\) and \(a_{ii0} = 0, i \neq i_0\). Moreover, The corresponding weight function \(\omega\) is defined by: \(\omega(e_{i_0}) = a_{i_0i_0}\) and \(\omega(e_i) = 0\) for \(i \neq i_0\).

**Proof.** Suppose that there is a non trivial algebra homomorphism \(\omega : E \to F\). Then there is \(i_0 \in \{1, \cdots, n\}\) such that \(\omega(e_{i_0}) \neq 0\). For \(i \neq i_0\), we have \(0 = \omega(e_i e_{i_0}) = \omega(e_i) \omega(e_{i_0})\) leads to \(\omega(e_i) = 0\) and \(0 = \omega(e_{i_0}^2) = \sum_{k=1}^n a_{i_0k} \omega(e_k) = a_{i_0i_0} \omega(e_{i_0})\) implies \(a_{i_0i_0} = 0\). We have \(\omega(e_{i_0}^2) = \sum_{k=1}^n a_{i_0k} \omega(e_k) = a_{i_0i_0} \omega(e_{i_0})\) gives \(\omega(e_{i_0}) (\omega(e_{i_0}) - a_{i_0i_0}) = 0\). So \(\omega(e_{i_0}) = a_{i_0i_0}\) because \(\omega(e_{i_0}) \neq 0\).
Conversely, suppose that there is \( i_0 \in \{1, \ldots, n\} \) such that \( a_{i_0 i_0} \neq 0 \) and \( a_{i_0 i} = 0 \) for \( i \neq i_0 \). Then we check that the linear transformation \( \omega : \mathcal{E} \rightarrow F \) defined by \( \omega(e_{i_0}) = a_{i_0 i_0} \) and \( \omega(e_i) = 0 \) for \( i \neq i_0 \) is a weight function of \( \mathcal{E} \).

**Corollary 3.4.** If \((\mathcal{E}, \omega)\) is a baric evolution algebra, then \( \mathcal{E} \) admits a natural base \( \{u_1, u_2, \ldots, u_n\} \) such that \( \omega(u_1) = 1 \) and \( \omega(u_i) = 0 \) for \( i > 1 \). Moreover, \( \mathcal{E} = Fu_1 \oplus \ker \omega \) with \( u_i \ker \omega = 0 \).

**Proof.** Let \((\mathcal{E}, \omega)\) be a baric algebra. Without loss of generality, we can assume that \( a_{11} \neq 0 \) and \( a_{i1} = 0 \) for \( i > 1 \). We set \( u_1 = be_1 \) with \( \omega(u_1) = 1 \). We have \( b \omega(e_1) = \omega(u_1) = 1 \) leads to \( b = (\omega(e_1))^{-1} \) so \( u_1 = (\omega(e_1))^{-1} e_1 \). We have \( u_1 e_j = 0 \), for \( j > 1 \) and \( u_1^2 = (\omega(e_1))^{-2} e_1^2 = (\omega(e_1))^{-1} e_1 + \sum_{k=2}^{n}(\omega(e_1))^{-2} a_{1k} e_k \).

We set \( u_k = (\omega(e_1))^{-2} e_k \) for \( k > 1 \), then the family \( \{u_i, 1 \leq i \leq n\} \) is a natural base of \( \mathcal{E} \) with \( \omega(u_1) = 1 \) and its multiplication table is defined by

\[
u_1^2 = u_1 + \sum_{k=2}^{n} a_{1k} u_k \text{ and } u_j^2 = \sum_{k=2}^{n}(\alpha_i e_i) a_{jk} u_k \text{ for } j > 1.
\]

We deduce that \( \ker \omega \) is a subalgebra of \( \mathcal{E} \) (see Definition 3.19) and \( \mathcal{E} = Fu_1 \oplus \ker \omega \).

**Remark 3.5.** A finite-dimensional baric evolution algebra admits at most \( n \) weights functions. The case where there is \( n \) weights functions occurs when \( e_i^2 = \alpha_i e_i \) with \( \alpha_i \neq 0, \ i = 1, \ldots, n \).

### 3.2. Characterization of evolution train algebras.

**Definition 3.6.** We said that a baric algebra \((\mathcal{E}, \omega)\) is a train algebra of rank \( r \), if there is scalars \( \gamma_i \in F \) and a nonzero integer \( r \) such that

\[
x^r + \gamma_1 \omega(x)x^{r-1} + \cdots + \gamma_{r-1} \omega(x)^{r-1} x = 0
\]

for all \( x \in \mathcal{E} \) and \( r \) is smaller such integer.

For all \( x \in \ker \omega \), we have \( x^r = 0 \); so \( \ker \omega \) is a nil-algebra. Since the weight function is not trivial, by applying it to the identity (3.2), for \( \omega(x) \neq 0 \), we obtain \( 1 + \gamma_1 + \cdots + \gamma_{r-1} = 0 \). Thus the train algebras of rank 2 are defined by

\[
x^2 = \omega(x)x = 0.
\]

**Proposition 3.7.** There is no evolution train algebra of rank 2.

**Proof.** The commutative train algebras of rank 2 are the gametic algebras for simple mendelian inheritance ([10, page 137]). In ([7, Example 2]), the authors show that gametic algebra for simple Mendelian inheritance is not an evolution algebra. We get the proposition.

**Theorem 3.8.** Let \((\mathcal{E}, \omega)\) be a baric evolution algebra. Then \( \mathcal{E} \) is a train algebra of rank \( r + 1 = 2 \) if and only if \( \ker(\omega) \) is nil, of nil-index \( r \). The train equation of \( \mathcal{E} \) is given by \( x^{r+1} - \omega(x)x^r = 0 \).

**Proof.** Let \((\mathcal{E}, \omega)\) be evolution train algebra with natural base \( \{e_1, \ldots, e_n\} \) such that \( \omega(e_1) = 1 \) and \( \omega(e_i) = 0 \) for \( i > 1 \); we have \( \mathcal{E} = Fe_1 \oplus \ker(\omega) \). Let \( x = \alpha e_1 + y \) where \( \alpha \in F \) and \( y \in \ker(\omega) \). We have \( x^2 = \alpha^2 e_1^2 + y^2 = \alpha^2 (e_1 + z) + y^2 = \alpha^2 z + y^2 \).
\(\alpha^2e_1 + (\alpha^2z + y^2)\) where \(z = \sum_{k=2}^n a_{1k}e_k\), so \(x^2 - \omega(x)x = (-\alpha y + \alpha^2 z + y^2)\). We have \(x(x^2 - \omega(x)x) = (-\alpha y + \alpha^2 z + y^2)^2\), i.e. \(x^3 - \omega(x)x^2 = -\alpha y^2 + \alpha^2 y + y^3\). Gradually we obtain \(x^{k+1} - \omega(x)x^k = -\alpha y^k + \alpha^2 \ell_{y}^{k-1}(z) + y^{k+1}\). Since \(\ker(\omega)\) is a nil-algebra, hence nilpotent, there is an integer \(k \geq 1\) such that \(-\alpha y^k + \alpha^2 \ell_{y}^{k-1}(z) + y^{k+1} = 0\). Thus the train equation is of form \(x^{k+1} - \omega(x)x^s = 0\) with \(s > 1\) an integer.

Suppose that the nil-index of \(N = \ker(\omega)\) is \(r\). Then for all \(y, z \in N\), we have \(y^r = 0\) and \(\ell_{y}^{r-1}(z) = 0\) because \((\ker(\omega))^r = 0\). So \(x^{r+1} - \omega(x)x^r = -\alpha y^r + \alpha^2 \ell_{y}^{r-1}(z) + y^{r+1} = 0\) and \(E\) is a train algebra of rank \(r+1\). Indeed, if the rank of \(E\) was \(r\), then we would have \(x^{r} - \omega(x)x^{r-1} = 0\); either \(-\alpha y^{r-1} + \alpha^2 \ell_{y}^{r-2}(z) + y^{r} = 0\). For \(\alpha = 0\), i.e. \(x = y \in \ker(\omega)\), we have \(y^r = 0\). Thus, \(-\alpha y^{r-1} + \alpha^2 \ell_{y}^{r-2}(z) = 0\), for all \(\alpha \in F\), i.e. \(-y^{r-1} + \alpha \ell_{y}^{r-2}(z) = 0\), for all \(\alpha \in F^*\). Since \(\text{char}(F) \neq 2\), we would have \(y^{r-1} = 0\), for all \(y \in \ker(\omega)\). This would contradict the hypothesis since there existed \(y_0 \in \ker(\omega)\) such that \(y_0^{r-1} \neq 0\). Thus, the rank of \(E\) is exactly \(r+1\).

In passing, we have shown that if \(E\) is a train algebra of rank \(r\), then for all \(y \in \ker(\omega)\), \(y^{r-1} = 0\), i.e. \(N = \ker(\omega)\) is a nil-algebra of nil-index at most \(r-1\). In fact, the nil-algebra \(N\) is exactly of nil-index \(r-1\).

**Lemma 3.9.** Let \((E, \omega)\) be an evolution train algebra. Then, \(E\) admits a natural base \(\{u_1, \ldots, u_n\}\) such that \(u_1^2 = u_1 + \sum_{k=2}^n a_{1k}u_k\), \(u_2^2 = \sum_{k=j+1}^n a_{jk}u_k\) with \(2 \leq j \leq n\).

**Proof.** Since \(E\) is a baric algebra, it admits a natural base \(\{e_1, \ldots, e_n\}\) whose multiplication table is of form \(e_1^2 = e_1 + \sum_{k=2}^n a_{1k}e_k\), \(e_2^2 = \sum_{k=2}^n a_{jk}e_k\) with \(2 \leq j \leq n\). Since \((E, \omega)\) is a train algebra, it follows that \(\ker(E) = \langle e_2, \ldots, e_n \rangle\) is an evolution nil-algebra and \(\ker(E)\) admits a natural base \(\{u_2, \ldots, u_n\}\) such that structural constants matrix is of form

\[
\begin{pmatrix}
0 & b_{23} & b_{24} & \cdots & b_{2n} \\
0 & 0 & b_{34} & \cdots & b_{3n} \\
0 & 0 & 0 & \cdots & b_{4n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

We infer that the family \(\{e_1, u_2, u_3, \ldots, u_n\}\) is a natural base of \(E\) and its multiplication table is of form \(e_1^2 = e_1 + \sum_{k=2}^n b_{1k}u_k\), \(u_2^2 = \sum_{k=j+1}^n b_{jk}u_k\) with \(2 \leq j \leq n\). We get the proposition. \(\square\)

### 3.3. Characterization of special train algebras

**Definition 3.10 ([10, Definition 3.27]).** Let \((E, \omega)\) be a baric algebra. We said that \(E\) is a special train algebra if \(N = \ker(\omega)\) is a nilpotent algebra and its principal powers \(N^i, i \in \mathbb{N}^*,\) are ideals of \(E\).

**Proposition 3.11.** Let \((E, \omega)\) be a finite-dimensional baric algebra. Then \(E\) is a special train algebra if and only if \(N = \ker(\omega)\) is a nil-algebra.

**Proof.** Since \(E\) is a finite-dimensional baric algebra, \(E\) admits the following decomposition \(E = Fe_1 \oplus N\) where \(\omega(e_1) = 1\) and \(e_1N = 0\). Suppose that \(N\)
Hence, \( N^k \subset N^{k+1} \). We have \( \mathcal{E}N^k = (F_e \oplus N)N^k = N^{k+1} \subset N^k \). So \( N^k \) is an ideal of \( \mathcal{E} \). We deduce that \( \mathcal{E} \) is a special train algebra.

Conversely, suppose that \( \mathcal{E} \) is a special train algebra. Then \( N \) is a nilpotent algebra. It follows \( N \) is a nil-algebra. \( \square \)

By Theorem 3.8 and Proposition 3.11, we deduce the following theorem

**Theorem 3.12.** Any evolution train algebra is a special train algebra.

3.4. Idempotents in evolution train algebras. A nonzero idempotent of an algebra \( \mathcal{E} \) is an element \( e \in \mathcal{E} \) such that \( e^2 = e \neq 0 \).

**Proposition 3.13.** Let \(( \mathcal{E}, \omega )\) be a \( n \)-dimensional train algebra. Then \( \mathcal{E} \) admits one and only one nonzero idempotent.

**Proof.** Since \(( \mathcal{E}, \omega )\) is a train algebra, it admits a natural base \( \{ e_1, \ldots, e_n \} \) such that \( e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k}e_k \), \( e_2^2 = \sum_{k=j+1}^{n} a_{jk}e_k \) with \( 2 \leq j \leq n \). Let \( \sigma = \sum_{i=1}^{n} \sigma_i e_i \) be an element of \( \mathcal{E} \). We have \( \sigma^2 = \sigma_1^2 e_1 + \sum_{j=2}^{n} \sigma_1^j a_{ij}e_j + \sum_{j=i+1}^{n} \sum_{i=1}^{n} \sigma_2^j a_{ij}e_j = \sigma_1^2 e_1 + \sum_{j=2}^{n} \sigma_1^j a_{ij}e_j + \sum_{j=i+1}^{n} \sum_{i=1}^{n} \sigma_2^j a_{ij}e_j \). So \( \sigma \) is an idempotent if and only if

\[
\sigma_1^2 = \sigma_1 \quad \text{and} \quad \sigma_j = \sigma_1^2 a_{ij} + \sigma_2^2 a_{ij} + \sigma_3^2 a_{ij} + \cdots + \sigma_{j-1}^2 a_{ij} \quad \text{where} \quad 2 \leq j \leq n. \quad (3.3)
\]

We see that if \( \sigma_1 = 0 \), then \( \sigma_2 = \cdots = \sigma_n = 0 \). So \( \sigma_1 = 1 \) and (3.3) admits one and only one solution. \( \square \)

If \( \sigma \) is the unique idempotent of an evolution train algebra \( \mathcal{E} \), then the base \( \{ \sigma, e_2, \ldots, e_n \} \) is not necessarily a natural base of \( \mathcal{E} \) because \( \sigma e_i = \sigma e_i^2 \) is not necessarily zero.

**Example 3.14.** Let \( \mathcal{E} \) be a 4-dimensional algebra with a base \( \{ e, u, v, w \} \) whose nonzero products are defined by \( e^2 = e + u, u^2 = v, v^2 = w \). Then \( \mathcal{E} \) is an evolution algebra in a natural base \( \{ e, u, v, w \} \). Let \( \omega : \mathcal{E} \rightarrow F \) be a linear transformation defined by \( \omega(e) = 1 \) and \( \omega(u) = \omega(v) = \omega(w) = 0 \). The Theorem 3.3 tells us that \( \omega \) is a weight function of \( \mathcal{E} \). We set \( N = \ker(\omega) = \langle u, v, w \rangle \) and we have \( N^2 = \langle v, w \rangle, N^3 = \langle w \rangle, N^4 = 0 \). We infer that \( (\mathcal{E}, \omega) \) is an algebra of rank 5 and its train equation is \( x^5 - \omega(x)x^4 = 0 \). Moreover, from (3.3) it comes \( \sigma = e + u + v + w \) is the unique idempotent of \( \mathcal{E} \). Since \( \sigma u = u^2 = v \neq 0 \), it follows the family \( \{ \sigma, u, v, w \} \) is not a natural base of \( \mathcal{E} \).

**Remark 3.15.** Let \( N_1 \) and \( N_2 \) be two finite-dimensional evolution nil-algebras of naturals bases respectively \( \{ u_1, \ldots, u_{n_1} \} \) and \( \{ v_1, \ldots, v_{n_2} \} \). We set \( \mathcal{E}_1 = F_{u_0} \oplus N_1 \) and \( \mathcal{E}_2 = F_{v_0} \oplus N_2 \) where \( u_0^2 = u_0 + \sum_{k=1}^{n_1} a_{0k}u_k \), \( u_0u_i = 0 \) and \( v_0^2 = v_0 + \sum_{k=1}^{n_2} a_{0k}v_k \), \( v_0v_j = 0 \) with \( i = 1, \ldots, n_1 \) and \( j = 1, \ldots, n_2 \). We set respectively \( \omega_1 \) and \( \omega_2 \) the weights functions of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) defined by \( \omega_1(u_0) = \omega_2(v_0) = 1 \) and \( \omega_1(u_i) = \omega_2(v_j) = 0 \) with \( i = 1, \ldots, n_1 \) and \( j = 1, \ldots, n_2 \). \( (\mathcal{E}_1, \omega_1) \) and \( (\mathcal{E}_2, \omega_2) \) are evolution train algebras and we set \( \sigma_1 = u_0 + \sigma_1' \) and \( \sigma_2 = v_0 + \sigma_2' \), the unique idempotents of \( (\mathcal{E}_1, \omega_1) \) and \( (\mathcal{E}_2, \omega_2) \) respectively. Then \( \sigma = e_0 + \sigma_1' + \sigma_2' \) is the unique idempotent of the evolution algebra \( \mathcal{E} = F_{e_0} \oplus N_1 \oplus N_2 \) where \( e_0^2 = e_0 + (u_0^2 - u_0) + (v_0^2 - v_0) \), \( e_0u_i = e_0v_j = 0 \) with \( i = 1, \ldots, n_1 \) and \( j = 1, \ldots, n_2 \). Indeed, \( \sigma^2 = e_0^2 + \sigma_1'^2 + \sigma_2'^2 = \ldots \)
\[ e_0 + (u_0^2 + \sigma_1^2) + (v_0^2 + \sigma_2^2) - (u_0 + v_0) = e_0 + \sigma_1^2 + \sigma_2^2 - (u_0 + v_0) = e_0 + \sigma_1' + \sigma_2' = \sigma. \]

3.5. Power-associative evolution train algebras.

**Definition 3.16.** The algebra \( E \) is said:

i) fourth power-associative if and only if \( x^2x^2 = x^4 \) for all \( x \in E \).

ii) power-associative, if for all \( x \in E \), \( x^ix^j = x^{i+j} \) for all integers \( i, j \geq 1 \).

**Theorem 3.17** ([1]). Let \( F \) be a commutative field of \( \text{char}(F) \neq 2, 3, 5 \). Algebra \( E \) is power-associative if and only if \( x^2x^2 = x^4 \) for all \( x \in E \).

**Proposition 3.18.** Let \( E = F e_1 \oplus \ker \omega \) be a finite-dimensional evolution train algebra over a commutative field of \( \text{char}(F) \neq 2, 3, 5 \), where \( e_1^2 = e_1 + z \) with \( z \in \ker \omega \) and \( e_1 \ker \omega = 0 \). Then \( E \) is a power-associative algebra if and only the following statements are satisfied:

i) \( \ker \omega \) is a power-associative evolution algebra;

ii) \( z \in \text{ann}(\ker \omega) \).

**Proof.** We set \( x = \alpha e_1 + y \) with \( \alpha \in F \) and \( y \in \ker \omega \). We have the following equalities \( x^2 = \alpha^2 e_1^2 + y^2 = \alpha^2(e_1 + z) + y^2; x^3 = \alpha^3 e_1^2 + \alpha^2 yz + y^3 = \alpha^3(e_1 + z) + \alpha^2 yz + y^3; x^4 = \alpha^4 e_1^2 + \alpha^3 yz + \alpha^2 y^2 + y^4 \) and \( x^2x^2 = \alpha^4 e_1^2 + \alpha^3 yz + \alpha^2 y^2 + 2\alpha^2 y^2 z \).

Suppose that \( E \) is a power-associative algebra. Then the equality \( x^2x^2 = x^4 \) leads to
\[ \alpha^4 z^2 - \alpha^3 yz + \alpha^2 (2y^2 z - y(yz)) + y^2 y^2 - y^4 = 0. \] (3.4)

By replacing \( \alpha \) by \(-\alpha\) in (3.4), we get
\[ \alpha^4 z^2 + \alpha^3 yz + \alpha^2 (2y^2 z - y(yz)) + y^2 y^2 - y^4 = 0. \] (3.5)
The difference of (3.4) and (3.5) gives \( 2\alpha^3 yz = 0 \). We deduce that \( yz = 0 \), for all \( y \in \ker \omega \), i.e. \( z \ker \omega = 0 \). We get ii). The equality (3.4) becomes \( y^2 y^2 - y^4 = 0 \) for all \( y \in ker \omega \). The Theorem 3.17 tells us that \( \ker \omega \) is a power-associative evolution algebra and we obtain i).

Suppose that i) and ii) are satisfied. Then \( x^4 = \alpha^4 e_1^2 + y^4 \) and \( x^2x^2 = \alpha^4 e_1^2 + y^2 y^2 = \alpha^4 e_1^2 + y^4 = x^2 x^2 \). We deduce that \( E \) is a power-associative evolution algebra. \( \square \)

**Definition 3.19** ([2, Definition 2.4]). An evolution sub-algebra of an evolution algebra \( E \) is a sub-algebra \( E' \subset E \) such that \( E' \) is an evolution algebra, i.e. \( E' \) have a natural base. We said that \( E' \) have extension property if there is a natural base \( B' \) of \( E' \) which can be extended to natural base \( B \) of \( E \).

**Remark 3.20.** Let \( E = F e_1 \oplus \ker \omega \) be a finite-dimensional power-associative evolution train algebra, where \( e_1^2 = e_1 + z \) with \( z \in \ker \omega \), \( z \in \text{ann}(\ker \omega) \) and \( e_1 \ker \omega = 0 \). Then \( e = e_1^2 \) is the unique idempotent of \( E \) and the sub-algebra generated by \( e \) have the extension property [7, Lemme 5]. This result is also verified by direct calculation. Indeed, \( e^2 = e_1^2 e_1^2 = e_1^2 + z^2 = e_1^2 \) and for all \( y \in ker \omega \), we have \( e_1^2 y = zy = 0 \). Then, there is an orthogonal family \( \{u_2, \ldots, u_n\} \) such that \( ker \omega = < u_2, \ldots, u_n > \) and the family \( \{e, u_2, \ldots, u_n\} \) is a natural base of \( E \). Thus, the Peirce decomposition of \( E \) is given by \( E = E_1(e) \oplus E_0(e) \) where
\( \mathcal{E}(e) = Fe \) and \( \mathcal{E}_0(e) = \{ x \in \mathcal{E}, xe = 0 \} \). We see that \( \mathcal{E}_0(e) = \ker \omega \); hence \( \mathcal{E} = Fe \oplus \ker \omega \) is a direct sum of algebras.

Now, we give an example of evolution train algebra that is not power-associative and admits an idempotent \( e \) such that the sub-algebra generated by \( e \) is not provided with extension property and whose kernel is power-associative.

**Example 3.21.** Let \( \mathcal{E} \) be a 5-dimensional algebra with a base \( \{ e_1, x, u, v, w \} \) whose nonzero products are defined by \( e_1^2 = e_1 + u + v, x^2 = u + v, u^2 = w, v^2 = -w \). Then \( \mathcal{E} \) is an evolution algebra in a natural base \( \{ e_1, x, u, v, w \} \).

Let \( \varphi : \mathcal{E} \rightarrow F \) be a linear transformation defined by \( \varphi(e_1) = 1 \) and \( \varphi(x) = \varphi(u) = \varphi(v) = \varphi(w) = 0 \). The linear transformation \( \varphi \) is a weight function of \( \mathcal{E} \) and \( (\mathcal{E}, \varphi) \) is an evolution train algebra. Moreover \( e_1^2e_1^2 = e_1^2 + u^2 + v^2 = e_1^2 \) and \( \ker \omega = \langle x, u, v, w \rangle \) is a natural base of \( \mathcal{E} \). We have \( 0 = e_2x = a_2x^2 = a_2(u + v) \) leads to \( a_2 = 0 \); \( 0 = e_3x = b_2x^2 = b_2(u + v) \) implies \( b_2 = 0 \); \( 0 = e_4^2e_2 = (a_3 - a_4)w \) gives \( a_3 = a_4 \) and \( 0 = e_4^3e_3 = (b_3 - b_4)w \) gives \( b_3 = b_4 \).

Thus \( e_2 = a_3(u + v), e_3 = b_3(u + v) \) and the family \( \{ e_1^2, x, e_2, e_3, w \} \) is not a base of \( \mathcal{E} \). So the sub-algebra generated by \( e_1^2 \) is not provided with the extension property.

**3.6. Automorphism in evolution train algebras.** Let \( \mathcal{E} = Fe_1 \oplus \ker \omega \) be a finite-dimensional evolution train algebra such that \( \omega(e_1) = 1 \) and \( e_1 \ker \omega = 0 \). Then \( e_1^2 = e_1 + x \) with \( x \in \ker \omega \) and let \( \sigma \in \text{Aut}(\mathcal{E}) \) be an automorphism of \( \mathcal{E} \). Since the following diagram

\[
\begin{array}{c}
\mathcal{E} \\
\sigma \downarrow \omega \\
\mathcal{E}
\end{array}
\]

is commutative, \( \omega \circ \sigma = \omega \). We have \( \omega \circ \sigma(e_1) = \omega(e_1) = 1 \), hence \( \sigma(e_1) = e_1 + y \) with \( y \in \ker \omega \). For all \( z \in \ker \omega \), we have \( 0 = \omega(z) = \omega \circ \sigma(z) \) leads to \( \sigma(\ker \omega) \subseteq \ker \omega \). The ideal \( \ker \omega \) being stable under \( \sigma \), we have \( \sigma(\ker \omega) = \ker \omega \) and \( \sigma|_{\ker \omega} \in \text{Aut}(\ker \omega) \), where \( \sigma|_{\ker \omega} \) is the restriction of \( \sigma \) to \( \ker \omega \). Now \( 0 = e_1z \) such that \( \sigma(e_1z) = \sigma(e_1)\sigma(z) = (e_1 + y)\sigma(z) = y\sigma(z) \), so \( 0 = y \ker \omega \), i.e. \( y \in \text{ann}(\ker \omega) \). We have \( \sigma(e_1^2) = \sigma(e_1 + x) = \sigma(e_1) + \sigma(x) = e_1 + y + \sigma(x) \) and \( \sigma(e_1) = e_1 + y - e_1 = e_1 + x \). Thus, \( \sigma(e_1^2) = e_1^2 + y^2 = e_1^2 = e_1 + x \).

**Example 3.22.** We consider an evolution train algebra \( \mathcal{E} = Fe_1 \oplus \ker \omega \) where \( e_1 \ker \omega = 0, e_1^2 = e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4 \) and \( \ker \omega = \langle e_2, e_3, e_4 \rangle \) is an evolution nil-algebra of nil-index 4 defined by \( e_2^2 = e_3, e_3^2 = e_4, e_4^2 = 0 \). Let \( \sigma \in \text{Aut}(\mathcal{E}) \) and \( \varphi = \sigma|_{\ker \omega} \in \text{Aut}(\ker \omega) \). We set \( x = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 \) and \( \varphi(e_2) = \sigma_{22}e_2 + \sigma_{23}e_3 + \sigma_{24}e_4 \). We have \( \varphi(e_3) = \varphi(e_3)^2 = \varphi(e_2)^2 = \sigma_{22}^2e_3 + \sigma_{23}^2e_4 \), \( \varphi(e_4) = \varphi(e_4)^2 = \varphi(e_3)^2 = \sigma_{24}^2e_4 \) and \( 0 = \varphi(e_2^2) = \varphi(e_2)e_4 \). Thus, \( \sigma_{23} = 0 \) because the linear transformation \( \varphi \) is an automorphism of
ker \omega. We deduce that \( \varphi(e_2) = \sigma_{22}e_2 + \sigma_{24}e_4 \), \( \varphi(e_3) = \sigma_{22}^2e_3 \), \( \varphi(e_4) = \sigma_{22}^4e_4 \) and
\[
x - \varphi(x) = a_{12}(1 - \sigma_{22})e_2 + a_{13}(1 - \sigma_{22}^2)e_3 + (a_{14} - a_{12}\sigma_{24} - a_{14}\sigma_{22}^2)e_4.
\]
Moreover \( x - \varphi(x) \in \text{ann}(\ker \omega) = F e_4 \) if and only if \( a_{12}(1 - \sigma_{22}) = a_{13}(1 - \sigma_{22}^2) = 0 \). For
\( i \) \( \sigma_{22} = 1 \), \( \sigma(e_1) = e_1 - a_{12}\sigma_{24}e_4 \), \( \sigma(e_2) = e_2 + \sigma_{24}e_4 \), \( \sigma(e_3) = e_3 \), \( \sigma(e_4) = e_4 \); \( ii \) \( \sigma_{22} \neq 1 \), then \( a_{12} = 0 \) and \( \sigma(e_1) = e_1 + a_{14}(1 - \sigma_{22}^2)e_4 \), \( \sigma(e_2) = \sigma_{22}e_2 + \sigma_{24}e_4 \), \( \sigma(e_3) = \sigma_{22}^2e_3 \), \( \sigma(e_4) = \sigma_{22}^4e_4 \) with \( a_{13}(1 + \sigma_{22}) = 0 \).

In ([7]), the authors give the classification of evolution nil-algebras of nil-index 2, 3 and 4. In the particular case of nil-index 4, they give the classification only for the power-associative algebras. Here, we give the classification of evolution nil-algebras of nil-index 4 up to dimension 5 and that are not power-associative.

4. Nil-algebra of nil-index 4 that are not power-associative

**Theorem 4.1** ([7, Corollary 1]). Let \( B = \{e_1, \ldots, e_n\} \) be a natural base of an evolution nil-algebra \( E \). Then \( E \) is fourth power-associative if and only if
\( i \) \( e_i^2 e_j^2 = 0 \) for all \( 1 \leq i \leq j \leq n \);
\( ii \) \( (e_i^2 e_j)e_k = 0 \) for all \( 1 \leq i,j,k \leq n \).

**Definition 4.2.** The algebra \( E \) is decomposable if there is the nonzero ideals \( \mathcal{I} \) and \( \mathcal{J} \) such that \( E = \mathcal{I} \oplus \mathcal{J} \). Otherwise, it is indecomposable.

**Lemma 4.3** ([6, Corollary 2.6]). Let \( E \) be a finite-dimensional evolution algebra such that \( \dim_F(\text{ann}(E)) \geq \frac{1}{2} \dim_F(E) \geq 1 \). Then \( E \) is decomposable.

In the following, \( N \) is an indecomposable evolution nil-algebra of nil-index 4, up to dimension 5, that is not power-associative. Let \( B = \{e_1, \ldots, e_n\} \) be a natural base of \( N \). As \( 0 = N^4 \), then \( (e_i^2 e_j)e_k = 0 \) for all \( 1 \leq i,j,k \leq n \). Thus, there are \( i_1, i_2 \in \{1, \ldots, n\} \) such that \( e_{i_1}^2 e_{i_2}^2 \neq 0 \) otherwise \( N \) would be power-associative. The type of \( N \) is \( [n_1, n_2, n_3] \) where \( n_1, n_2, n_3 \) are nonzero integers such that \( n_1 + n_2 + n_3 = \dim(N) \). Necessarily \( \dim(N) \geq 3 \).

In ([7, § 4.3]), the authors denote by \( N_{i,j} \) the \( j \)-th indecomposable power-associative evolution nil-algebra of dimension \( i \). We will continue with this notation. In the sense that if \( N_{i,j_0} \) is the last indecomposable power-associative evolution nil-algebra of dimension \( i \), \( N_{i,j_0+k} \) would be \( k \)-th indecomposable nil-algebra of dimension \( i \) that is not power-associative.

With regard to Theorem 3.17 and Theorem 4.1, let \( F \) be a commutative field of \( \text{char}(F) \neq 2, 3, 5 \).

4.1. Case of 3-dimensional.

**Theorem 4.4.** Let \( N \) be 3-dimensional indecomposable evolution nil-algebra of nil-index 4 that is not power-associative. Then \( N \) is isomorphic to the evolution algebra \( N_{3,4} : e_1^2 = e_2, e_3^2 = e_3 \) and \( e_4^2 = 0 \) of type \( [1, 1, 1] \).

**Proof.** \( \dim(N) = 3 \), leads to \( \dim(\text{ann}(N)) < 1.5 \); so \( \dim(\text{ann}(N)) = 1 \) and the type of \( N \) is \( [1, 1, 1] \). Let \( B = \{e_1, e_2, e_3\} \) be a natural base of \( N \) such that \( \text{ann}(N) = Fe_3 \) and \( \text{ann}^2(N) = \langle e_2, e_3 \rangle \). The multiplication table in \( B \) is of form \( e_1^2 = a_{12} e_2 + a_{13} e_3 \), \( e_2^2 = a_{23} e_3 \), \( e_3^2 = 0 \), with \( a_{12}, a_{23} \in F^* \). We set \( u_2 = a_{12} e_2 + a_{13} e_3 \) and \( u_3 = u_2^2 \); the family \( \{e_1, u_2, u_3\} \) is a natural base of \( N \) and \( N \) is isomorphic to \( N_{3,4} : e_1^2 = u_2, u_2^2 = u_3, u_3^2 = 0 \). \( \square \)
4.2. Case of 4-dimensional.

**Theorem 4.5.** Let $N$ be 4-dimensional indecomposable evolution nil-algebra of nil-index 4 that is not power-associative. Then $N$ is isomorphic to one and only one evolution algebras in table 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Multiplication</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{4,7}(\alpha)$</td>
<td>$e^2 = e_3$, $e_2 = \alpha e_3$, $e_3 = e_4$, $e_4 = 0$ with $\alpha \in F^*$</td>
<td>$[1, 1, 2]$</td>
</tr>
<tr>
<td>$N_{4,8}(\alpha, \beta)$</td>
<td>$e^2 = e_3$, $e_2 = \alpha e_3 + \beta e_4$, $e_3 = e_4$, $e_4 = 0$ with $\alpha, \beta \in F^*$</td>
<td></td>
</tr>
<tr>
<td>$N_{4,9}(\alpha)$</td>
<td>$e_1^2 = e_2$, $e_2^2 = e_4$, $e_3 = \alpha e_4$, $e_4 = 0$ with $\alpha \in F^*$</td>
<td>$[1, 2, 1]$</td>
</tr>
</tbody>
</table>

*Proof.* $\dim(N) = 4$, leads to $\dim(\text{ann}(N)) < 2$; so $\dim(\text{ann}(N)) = 1$ and the possible types of $N$ are $[1, 1, 2]$ and $[1, 2, 1]$. Let $B = \{e_1, e_2, e_3, e_4\}$ be a natural base of $N$ such that $\text{ann}(N) = Fe_4$.

- $N$ is of type $[1, 1, 2]$. We take $\text{ann}^2(N) = \langle e_3, e_4 \rangle$ and we have $U_1 = Fe_4$, $U_2 = Fe_3$, $U_3 = Fe_1 \oplus Fe_2$. The multiplication table in $B$ is of form $e^2 = a_{13}e_3 + a_{14}e_4$, $e_1^2 = a_{23}e_3 + a_{24}e_4$, $e_2^2 = a_{34}e_4$, $e_4 = 0$ with $a_{13}, a_{23}, a_{34} \in F^*$. We set $u_3 = a_{13}e_3 + a_{14}e_4$, $u_4 = u_3^2$; the family $\{e_1, e_2, u_3, u_4\}$ is a natural base of $N$ and its multiplication table is of form $e^2 = u_3$, $e_2 = \alpha u_3$, $u_4 = u_3$, $u_4^2 = 0$, with $\alpha \in F^*$ because $a_{13}, a_{23} \in F^*$. We have $(U_3 \oplus U_1)^2 = \langle u_3, \beta u_4 \rangle$.

If $\dim_F(U_3 \oplus U_1)^2 = 1$, then $\beta = 0$ and $N$ is isomorphic to $N_{4,7}(\alpha) : e_1^2 = u_3$, $e_2 = \alpha u_3$, $u_3 = u_4$, $u_4^2 = 0$, with $\alpha \in F^*$.

If $\dim_F(U_3 \oplus U_1)^2 = 2$, then $\beta \neq 0$ and $N$ is isomorphic to $N_{4,8}(\alpha, \beta) : e_1^2 = u_3$, $e_2 = \alpha u_3 + \beta u_4$, $u_3 = u_4$, $u_4^2 = 0$, with $\alpha, \beta \in F^*$.

- $N$ is of type $[1, 2, 1]$. We take $\text{ann}^2(N) = \langle e_2, e_3, e_4 \rangle$. The multiplication table in $B$ is of form $e^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4$, $e_1^2 = a_{24}e_4$, $e_2^2 = a_{34}e_4$, $e_3 = 0$ with $a_{24}, a_{34} \in F^*$ and $(a_{12}, a_{13}) \neq (0, 0)$. Moreover $a_{12}^2a_{24} + a_{13}a_{34} \neq 0$ because $e_1^2e_1^2 \neq 0$, otherwise $N$ would be power-associative. We set $u_2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4$ and $u_4 = u_2^2$. Let’s us determine a vector $u_3 = ae_2 + be_3$ such that the family $\{e_1, u_2, u_3, u_4\}$ is a natural base of $N$. The orthogonality of $a_2$ and $u_3$ gives $a_{12}a_{24}a + a_{13}a_{34}b = 0$. We distinguish two cases

i) $a_{12} = 0$, then we can take $u_3 = e_2$;

ii) $a_{12} \neq 0$, then $a = -(a_{12}a_{24})^{-1}(a_{13}a_{34})b$.

We can take $u_3 = -(a_{12}a_{24})^{-1}(a_{13}a_{34})e_2 + e_3$.

In each cases, the multiplication table in $\{e_1, u_2, u_3, u_4\}$ is of form $e^2 = u_2$, $u_2^2 = u_4$, $u_3^2 = \alpha u_4$, $u_4^2 = 0$ with $\alpha \in F^*$. We deduce that $N$ is isomorphic to $N_{4,9}(\alpha)$.

□

4.3. Case of 5-dimensional. If $\dim(N) = 5$ then $\dim(\text{ann}(N)) < 2.5$; so $\dim(\text{ann}(N)) = 1$ or 2. The possible types of $N$ are $[1, 1, 3]$, $[1, 2, 2]$, $[1, 3, 1]$, $[2, 1, 2]$ and $[2, 2, 1]$. 
Theorem 4.6. Let $N$ be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[1,1,3]$ that is not power-associative. Then $N$ is isomorphic to one and only one evolution algebras in table 2.

| Table 2 |
|------------------|------------------|
| $N$               | Multiplication   |
| $N_{5,13}(\alpha, \beta)$ | $e_1^2 = e_4$, $e_2^2 = \alpha e_4$, $e_3^2 = e_5$, $e_4^2 = e_5$, $e_5^2 = 0$ with $\alpha, \beta \in F^*$. |
| $N_{5,14}(\alpha, \beta, \gamma)$ | $e_1^2 = e_4$, $e_2^2 = \alpha e_4 + \beta e_5$, $e_3^2 = \gamma (\alpha e_4 + \beta e_5)$, $e_4^2 = e_5$, $e_5^2 = 0$ with $\alpha, \beta, \gamma \in F^*$. |
| $N_{5,15}(\alpha, \beta, \gamma, \delta)$ | $e_1^2 = e_4$, $e_2^2 = \alpha e_4 + \beta e_5$, $e_3^2 = \gamma e_4 + \delta e_5$, $e_4^2 = e_5$, $e_5^2 = 0$ with $\alpha, \beta, \gamma, \delta \in F^*$ and $\alpha \delta - \beta \gamma \neq 0$. |

Proof. $N$ is of type $[1,1,3]$. Let $B = \{e_1, \ldots, e_5\}$ be a natural base of $N$ such that $\text{ann}(N) = Fe_5$ and $\text{ann}^2(N) = \langle e_4, e_5 \rangle$. We have $U_1 = Fe_5$, $U_2 = Fe_4$ and $U_3 = Fe_1 \oplus Fe_2 \oplus Fe_3$; the multiplication table in $B$ is of form $e_1^2 = a_{14} e_4 + a_{15} e_5$, $e_2^2 = a_{24} e_4 + a_{25} e_5$, $e_3^2 = a_{34} e_4 + a_{35} e_5$, $e_4^2 = a_{45} e_5$, $e_5^2 = 0$, with $a_{14}, a_{24}, a_{34}, a_{45} \in F^*$. Since $e_1^2 e_i^2 = a_{i4} a_{45} e_5 \neq 0$, for $1 \leq i \leq 3$, we set $u_4 = e_1^2$ and $u_5 = e_3^2 e_1^2$; the family $\{e_1, e_2, e_3, u_4, u_5\}$ is a natural base of $N$ and its multiplication table is of form $e_1^2 = u_4$, $e_2^2 = \alpha_4 u_4 + \alpha_5 u_5$, $e_3^2 = \beta_4 u_4 + \beta_5 u_5$, $u_4^2 = u_5$, $u_5^2 = 0$, with $\alpha_4, \beta_4 \in F^*$ because $a_{i4} a_{24} a_{34} \neq 0$. We have $\langle U_3 \oplus U_4 \rangle = \langle u_4, \alpha_5 u_5, \beta_5 u_5 \rangle$.

If $\dim(U_3 \oplus U_4)^2 = 1$, then $\alpha_5 = \beta_5 = 0$ and $N$ is isomorphic to $N_{5,13}(\alpha_4, \beta_4)$: $e_1^2 = u_4$, $e_2^2 = \alpha_4 u_4$, $e_3^2 = \beta_4 u_4$, $u_4^2 = u_5$, $u_5^2 = 0$, with $\alpha_4, \beta_4 \in F^*$.

If $\dim(U_3 \oplus U_4)^2 = 2$, then $(\alpha_5, \beta_5) \neq (0,0)$. We distinguish the four cases:

i) $\det(e_2^2, e_3^2) = \alpha_4 \beta_5 - \alpha_5 \beta_4 = 0$, then $\beta_5 = \alpha_4^{-1} \alpha_5 \beta_4$ and necessarily $\beta_5 \alpha_5 \neq 0$.

So $e_1^2 = u_4$, $e_2^2 = \alpha_4 u_4 + \alpha_5 u_5$, $e_3^2 = \beta_4 u_4 + \beta_5 u_5 = \beta_4 \alpha_4^{-1} (\alpha_4 u_4 + \alpha_5 u_5)$, $u_4^2 = u_5$, $u_5^2 = 0$, with $\alpha_4, \alpha_5, \beta_4 \in F^*$. We deduce that $N$ is isomorphic to $N_{5,14}(\alpha_4, \alpha_5, \beta_4 \alpha_4^{-1})$.

ii) $\det(e_2^2, e_3^2) \neq 0$ and $\beta_5 = 0$. We set $v_4 = e_2^2$ and $v_5 = e_3^2$. We have $e_1^2 = \alpha_4^{-1} v_4 - \alpha_4^{-2} \alpha_5 v_5$, $e_2^2 = v_4$, $e_3^2 = \beta_4 (\alpha_4^{-1} v_4 - \alpha_4^{-2} \alpha_5 v_5)$, $v_4^2 = v_5$ and $v_5^2 = 0$. We obtain $N_{5,14}(\alpha_4^{-1}, -\alpha_4^{-2} \alpha_5, \beta_4)$.

iii) $\det(e_2^2, e_3^2) \neq 0$ and $\alpha_5 = 0$. By permuting the vectors $e_2$ and $e_3$, we get case ii).

iv) $\det(e_2^2, e_3^2) \neq 0$ and $\alpha_5 \beta_5 \neq 0$. Then $e_1^2 = u_4$, $e_2^2 = \alpha_4 u_4 + \alpha_5 u_5$, $e_3^2 = \beta_4 u_4 + \beta_5 u_5$, $u_4^2 = u_5$, $u_5^2 = 0$ with $\alpha_4, \alpha_5, \beta_4 \in F^*$ and $\alpha_4 \beta_5 - \alpha_5 \beta_4 \neq 0$.

So $N$ is isomorphic to $N_{5,15}(\alpha_4, \alpha_5, \beta_4, \beta_5)$.

$\square$

Theorem 4.7. Let $N$ be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[1,2,2]$ that is not power-associative. Then $N$ is isomorphic to one and only one evolution algebras in table 3.

| Table 3 |
|------------------|------------------|
| $N$               | Multiplication   |
| $N_{5,16}(\alpha, \beta)$ | $e_1^2 = e_3$, $e_2^2 = \alpha e_3$, $e_3^2 = e_5$, $e_4^2 = e_5$, $e_5^2 = 0$ with $\alpha, \beta \in F^*$. |
| $N_{5,17}(\alpha, \beta, \gamma)$ | $e_1^2 = e_3$, $e_2^2 = \alpha e_3 + \beta e_5$, $e_3^2 = e_5$, $e_4^2 = \gamma e_5$, $e_5^2 = 0$ with $\alpha, \beta, \gamma \in F^*$. |
Proving N is of type \([1, 2, 2]\). Let \(B = \{e_1, \ldots, e_5\}\) be a natural base of \(N\) such that \(\text{ann}(N) = \langle e_5 \rangle \) and \(\text{ann}^2(N) = \langle e_3, e_4, e_5 \rangle\). We have \(U_1 = F e_5, U_2 = F e_3 \oplus F e_4, U_3 = F e_1 \oplus F e_2\) and a multiplication table in \(B\) of form \(e_1^2 = a_{13}e_3 + a_{14}e_4 + a_{15}e_5 = a + a_{15}e_5, e_2^2 = a_{23}e_3 + a_{24}e_4 + a_{25}e_5 = b + a_{25}e_5, e_3^2 = a_{35}e_5, e_4^2 = a_{45}e_5\) and \(e_5^2 = 0\) with \(a, b \neq 0\) and \(a_{35}, a_{45} \in F^*\). Then, either \(a^2 = b^2 = 0\), i.e. \(e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_5^2 = 0\), or \(a^2 \neq 0\) or \(b^2 \neq 0\).

- By symmetry, we suppose that \(a^2 \neq 0\). We set \(u_3 = a + a_{15}e_5\) and \(u_5 = a^2\).

Let's us determine the vector \(u_4 = c_3e_3 + c_4e_4 \in U_2\) such that the family \(\{e_1, e_2, u_3, u_4, u_5\}\) is a natural base of \(N\). The orthogonality of the vectors \(u_3\) and \(u_4\) implies \(c_3a_{13}a_{35} + c_4a_{14}a_{45} = 0\). We distinguish three cases:

i) \(a_{13} = 0\), then \(c_4 = 0\) and we can take \(u_4 = e_3\);

ii) \(a_{14} = 0\), then \(c_3 = 0\) and we can take \(u_4 = e_4\);

iii) \(a_{13}a_{14} \neq 0\), then \(c_4 = -\frac{c_3a_{13}a_{35}}{(a_{13}a_{35})(a_{14}a_{45})^{-1}}\) and we can take \(u_4 = e_3 - \frac{c_3a_{13}a_{35}}{(a_{13}a_{35})(a_{14}a_{45})^{-1}}e_4\).

In each case, the multiplication table in the natural base \(\{e_1, e_2, u_3, u_4, u_5\}\) is of form \(e_1^2 = u_3, e_2^2 = \alpha u_3 + \beta u_4 + \gamma u_5, u_3^2 = u_5, u_4^2 = \delta u_5, u_5^2 = 0\) with \(\alpha, \beta \neq (0, 0), \delta \in F^*\) and we have \(N^2 = \langle u_3, \beta u_4, u_5 \rangle\).

If \(\dim(N^2) = 2\), then \(\beta = 0\) and \((U_3 \oplus U_1)^2 = \langle u_3, \gamma u_5 \rangle\).

a) \(\dim((U_3 \oplus U_1)^2) = 1\), gives \(\gamma = 0\) and \(N\) is isomorphic to \(N_{5,16}(\alpha, \delta) : e_1^2 = u_3, e_2^2 = \alpha u_3, u_3^2 = u_5, u_4^2 = \delta u_5, u_5^2 = 0\) with \(\alpha, \delta \in F^*\).

b) \(\dim((U_3 \oplus U_1)^2) = 2\), leads to \(\gamma \neq 0\) and \(N\) is isomorphic to \(N_{5,17}(\alpha, \gamma, \delta) : e_1^2 = u_3, e_2^2 = \alpha u_3 + \gamma u_5, u_3^2 = u_5, u_4^2 = \delta u_5, u_5^2 = 0\) with \(\alpha, \gamma, \delta \in F^*\).

If \(\dim(N^2) = 3\), then \(\beta \neq 0\) and we set \(u_4' = \beta u_4 + \gamma u_5\). The family \(\{e_1, e_2, u_3, u_4', u_5\}\) is a natural base of \(N\) and \(N\) is isomorphic to \(N_{5,18}(\alpha, \delta') : e_1^2 = u_3, e_2^2 = \alpha u_3 + \gamma u_5', u_3^2 = u_5, (u_4')^2 = \beta^2 \delta u_5 = \delta' u_5, u_5^2 = 0\) with \(\alpha \in F\) and \(\delta' \in F^*\).

- Suppose that \(a^2 = b^2 = 0\). We have \(a^2 = 0\) implies \(a_{13}, a_{14} \in F^*\), otherwise \(a = 0\). Similarly \(a_{23}, a_{24} \in F^*\), otherwise \(b = 0\). We set \(u_3 = a_{13}e_3 + a_{15}e_5, u_4 = a_{14}e_4\) and \(u_5 = u_3^2\); as \(0 = e_1^2 e_2^2 = (u_3 + u_4)^2 = u_3^2 + u_4^2\), then \(u_4^2 = -u_3^2 = -u_5\) and the family \(\{e_1, e_2, u_3, u_4, u_5\}\) is a natural base of \(N\). The multiplication table in this base is of form \(e_1^2 = u_3 + u_4, e_2^2 = \alpha u_3 + \beta u_4 + \gamma u_5, u_3^2 = u_5, u_4^2 = -u_5, u_5^2 = 0\) with \((\alpha, \beta) \neq (0, 0)\). We have \(0 = e_2^2 e_3^2 = (\alpha^2 - \beta^2) u_5\) leads to \(\alpha = \beta\) or \(\alpha = -\beta\). If \(\alpha = \beta\), then \(e_1^2 e_2^2 = (\alpha - \beta) u_5 = 0\) and \(N\) would be power-associative. We infer that \(\alpha = -\beta \neq 0\) and we set \(v_3 = u_3 + \frac{1}{2} \alpha^{-1} \gamma u_5\) and \(v_4 = u_4 - \frac{1}{2} \alpha^{-1} \gamma u_5\). Then \(N\) is isomorphic to \(N_{5,19}(\alpha) : e_1^2 = v_3 + v_4, e_2^2 = \alpha(v_3 - v_4), v_3^2 = u_5, v_4^2 = -u_5, u_5^2 = 0\) with \(\alpha \in F^*\).

\(\square\)

**Theorem 4.8.** Let \(N\) be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type \([1, 3, 1]\) that is not power-associative. Then \(N\) is isomorphic to \(N_{5,20}(\alpha, \beta) : e_1^2 = e_2, e_2^2 = e_5, e_3^2 = \alpha e_5, e_4^2 = \beta e_5, e_5^2 = 0\) with \(\alpha, \beta \in F^*\).
Proof. $N$ is of type $[1, 3, 1]$. Let $B = \{e_1, \ldots, e_5\}$ be a natural base of $N$ such that $\text{ann}(N) = \langle e_5 \rangle$ and $\text{ann}^2(N) = \langle e_2, e_5 \rangle$. We have $\mathcal{U}_1 = Fe_5$, $\mathcal{U}_2 = Fe_2 \oplus Fe_3 \oplus Fe_4$ and $\mathcal{U}_3 = Fe_1$. The multiplication table in $B$ is of form $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 + a_{15}e_5 = a + a_{15}e_5$, $e_2^2 = a_{25}e_5$, $e_3^2 = a_{35}e_5$, $e_4^2 = a_{45}e_5$, $e_5^2 = 0$ with $a \neq 0$ and $a_{25}, a_{35}, a_{45} \in F^*$. We have $0 \neq e_1^2e_1^2 = (a_{12}^2a_{25} + a_{13}^2a_{35} + a_{14}^2a_{45})e_1^2$ otherwise $N$ would be power-associative. In the following, we assume that $a_{12} \neq 0$, otherwise we reorder the vectors $e_2$, $e_3$ and $e_4$ of natural base. We set $u_2 = e_1^2$, $u_5 = e_1^2e_1^2$ and let's look for the vectors $u_3 = b_2e_2 + b_3e_3 + b_4e_4$, $u_4 = c_2e_2 + c_3e_3 + c_4e_4 \in \mathcal{U}_2$ such that the family $\{e_1, u_2, u_3, u_4, u_5\}$ is a natural base of $N$. The orthogonality of $u_2$ with the vectors $u_3$ and $u_4$ leads to $b_2 = - (b_3a_{13}a_{35} + b_4a_{14}a_{45})(a_{12}a_{25})^{-1}$ and $c_2 = -(c_3a_{13}a_{35} + c_4a_{14}a_{45})(a_{12}a_{25})^{-1}$. We deduce that

\[ u_3 = -(b_3a_{13}a_{35} + b_4a_{14}a_{45})(a_{12}a_{25})^{-1}e_2 + b_3e_3 + b_4e_4, \]
\[ u_4 = -(c_3a_{13}a_{35} + c_4a_{14}a_{45})(a_{12}a_{25})^{-1}e_2 + c_3e_3 + c_4e_4. \]

The orthogonality of $u_3$ and $u_4$ gives

\[ b_3a_{12}a_{25}(c_3(a_{12}^2a_{25} + a_{13}^2a_{35}) + c_4a_{13}a_{14}a_{45}) + b_4a_{12}a_{25}(c_3(a_{12}^2a_{25} + a_{14}^2a_{45}) + c_4a_{14}a_{13}a_{35}) = 0. \] (4.1)

Moreover the family $\{e_1, u_2, u_3, u_4, u_5\}$ is linearly independent if and only if the family $\{u_3, u_4\}$ is linearly independent. The family $\{u_3, u_4\}$ is linearly independent if and only if $\text{rang}(\{u_3, u_4\}) = 2$, i.e., $b_3c_4 - b_4c_3 \neq 0$. We distinguish the following cases:

\[ i) \quad a_{12}^2a_{25} + a_{13}^2a_{35} \neq 0, \text{ we can take } b_4 = 0, \text{ then } b_3 \neq 0 \text{ and } c_3 = -(a_{13}a_{14}a_{45})(a_{12}^2a_{25} + a_{13}^2a_{35})^{-1}c_4. \text{ In this case, } \]
\[ u_3 = -(a_{13}a_{35})(a_{12}a_{25})^{-1}e_2 + e_3, \]
\[ u_4 = -a_{12}a_{13}a_{45}(a_{12}a_{25} + a_{13}^2a_{35})^{-1}e_2 - (a_{13}a_{14}a_{45})(a_{12}a_{25} + a_{13}^2a_{35})^{-1}e_3 + e_4; \]

\[ ii) \quad a_{12}^2a_{25} + a_{14}^2a_{45} \neq 0, \text{ we can take } b_3 = 0, \text{ then } b_4 \neq 0 \text{ and } c_4 = -(a_{13}a_{14}a_{35})(a_{12}a_{25} + a_{14}^2a_{45})^{-1}c_3. \text{ In this case, } \]
\[ u_3 = -(a_{14}a_{45})(a_{12}a_{25})^{-1}e_2 + e_4, \]
\[ u_4 = -a_{12}a_{13}a_{45}(a_{12}a_{25} + a_{14}^2a_{45})^{-1}e_2 + e_3 - (a_{13}a_{14}a_{35})(a_{12}a_{25} + a_{14}^2a_{45})^{-1}e_4; \]

\[ iii) \quad a_{12}^2a_{25} = -a_{13}^2a_{35} = -a_{14}a_{45}, \text{ then } (4.1) \text{ becomes } b_3c_4 + b_4c_3 = 0. \text{ We can take } b_4 = b_3 \text{ and } c_4 = -c_3. \text{ In this cases, } \]
\[ u_3 = -(a_{13}a_{35} + a_{14}a_{45})(a_{12}a_{25})^{-1}e_2 + e_3 + e_4, \]
\[ u_4 = -(a_{13}a_{35} - a_{14}a_{45})(a_{12}a_{25})^{-1}e_2 + e_3 - e_4. \]

In each cases, $N$ is isomorphic to $N_{5,20}(\alpha, \beta) : e_1^2 = u_2, e_2^2 = u_5, e_3 = \alpha u_5, u_4^2 = \beta u_5, u_5^2 = 0$ with $\alpha, \beta \in F^*$. □

**Theorem 4.9.** Let $N$ be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type $[2, 1, 2]$ that is not power-associative. Then $N$ is isomorphic to $N_{5,21}(\alpha) : e_1^2 = e_3, e_2^2 = \alpha e_3 + e_5, e_3^2 = e_4, e_4^2 = e_5 = 0$ with $\alpha \in F^*$.

**Proof.** $N$ is of type $[2, 1, 2]$. Let $B = \{e_1, \ldots, e_5\}$ be a natural base of $N$ such that $\text{ann}(N) = \langle e_5 \rangle$ and $\text{ann}^2(N) = \langle e_3, e_5 \rangle$. We have $\mathcal{U}_1 = Fe_5 \oplus Fe_5$, $\mathcal{U}_2 = Fe_3$, $\mathcal{U}_3 = Fe_1 \oplus Fe_2$ and the multiplication table in $B$ is of form $e_1^2 =
The family \( \{ \text{nil-index} N \} \) is isomorphic to Power-associative evolution algebras.

The family \( \{ e_1, e_2, u_3, u_4, u_5 \} \) is a natural base of \( N \) and \( N \) is isomorphic to \( N_{5,21}(\alpha) : e_1^2 = u_3, e_2^2 = \alpha u_3 + u_5, u_3^2 = u_4, u_4^2 = (u_5')^2 = 0 \) with \( \alpha \in F^* \).

**Theorem 4.10.** Let \( N \) be 5-dimensional indecomposable evolution nil-algebra of nil-index 4 and of type \([2, 2, 1]\) that is not power-associative. Then \( N \) is isomorphic to \( N_{5,22} : e_1^2 = u_2 + u_3, u_2^2 = u_4, u_3^2 = u_5, u_4^2 = u_5^2 = 0 \).

**Proof.** \( N \) is of type \([2, 2, 1]\). Let \( B = \{ e_1, \ldots, e_5 \} \) be a natural base of \( N \) such that \( \text{ann}(N) = \langle e_4, e_5 \rangle \) and \( \text{ann}^2(N) = \langle e_2, e_3, e_4, e_5 \rangle \). We have \( U_1 = F e_4 \oplus F e_5 \), \( U_2 = F e_2 \oplus F e_3 \), \( U_3 = F e_1 \) and the multiplication table in \( B \) is of form \( e_1^2 = a_{12} e_2 + a_{13} e_3 + a_{14} e_4 + a_{15} e_5, e_2^2 = a_{24} e_4 + a_{25} e_5, e_3^2 = a_{34} e_4 + a_{35} e_5, e_4^2 = e_5^2 = 0 \) with \( (a_{12}, a_{13}) \neq (0, 0) \). We have \( 0 \neq e_1^2 e_1 = a_{12} e_2 + a_{13} e_3^2 \) otherwise \( N \) would be power-associative. We assume that \( a_{12} \neq 0 \) otherwise, we permute the vectors \( e_2 \) and \( e_3 \) of the natural base. Moreover the family \( \{ e_2^2, e_3^2 \} \) is a linearly independent otherwise \( N = \langle e_1, a_{12} e_2 + a_{14} e_4 + a_{15} e_5, e_3, e_2^2 \rangle \) would be decomposable with \( \text{ann}(N) = Fe_2^2 \oplus Fu_5 \): impossible. In the same way \( a_{13} \neq 0 \) otherwise \( N = \langle e_1, a_{12} e_2 + a_{14} e_4 + a_{15} e_5, e_2^2 \rangle \) would be decomposable. We set \( u_2 = a_{12} e_2 + a_{14} e_4, u_3 = a_{13} e_3 + a_{15} e_5 \), \( u_4 = u_2^2 \) and \( u_5 = u_2^2 \). So the family \( \{ e_1, u_2, u_3, u_4, u_5 \} \) is a natural base of \( N \) and \( N \) is isomorphic to \( N_{5,22} : e_1^2 = u_2 + u_3, u_2^2 = u_4, u_3^2 = u_5, u_4^2 = u_5^2 = 0 \).  

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**References**


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