Gulf Journal of Mathematics
Vol 20, Issue 1 (2025) 360-380

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i.2993

EIGENVALUE LOCATION AND CONDITION NUMBER BOUNDS OF EVEN GRADE POLYNOMIAL EIGENVALUE PROBLEMS

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Abstract. We consider even grade polynomial eigenvalue problems (PEP) to study their eigenvalue location and condition number bounds. We converted the even grade PEP into a Quadratic eigenvalue problem (QEP) of high dimension via quadratification techniques. We study an eigenvalue localization method that applies block Geršgorin sets to certain linearization classes of quadratic matrix polynomials originating from well-established vector spaces. We presented a few results on the bounding ratios of condition number of certain linearization classes with respect to that of original PEP and QEP induced due to quadratification. Relevant computational works are performed using a model numerical example to show the effectiveness of the results in terms of calculating eigenvalue and bounds of condition number ratios.

1. Introduction

Matrix polynomial $P(\lambda) := \sum_{j=0}^{k} \lambda^j P_j$ of degree k has received attention by the researchers [6, 10] over the recent and past times, where $P_j \in \mathbb{C}^{n\times n}$. The PEP associated with a given matrix polynomial $P(\lambda)$ of degree k, is to find a complex scalar λ and the vector $0 \neq u \in \mathbb{C}^n$ such that $P(\lambda)u = 0$. Then, the pair (λ, u) is called eigenpair of PEP associated with matrix polynomial $P(\lambda)$. A PEP is called regular and singular according as $\det(P(\lambda)) \neq 0$ and $\det(P(\lambda)) \equiv 0$, respectively. Literature for spectral theory and numerical solution of PEP, we may refer [13]. The survey works [23] contains an extensive works on the overview of PEP. Applications of matrix polynomials prompts quick advancements in the related theories [7, 8, 9], computational techniques [21], as well as the softwares [1, 17]. The classical approach to investigate the PEP is by linearization techniques, which covert a given PEP into a generalized eigenvalue problem of large dimension. But before doing so, it is critical to comprehend how the linearization process influences correctness and stability of the computed solution. The pioneer paper [22] opens up the new direction on the study of vector space of linearizations for matrix polynomials [11]. Eigenvalue problem with quadratic matrix polynomial grained attention by the scholars [3, 4, 5, 14] as it appears

Date: Received: May 8, 2025; Accepted: May 29, 2025.

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2020 *Mathematics Subject Classification.* Primary 15A18; Secondary 15A22, 65F15, 65F35.

Key words and phrases. Condition number, Linearization, Matrix polynomial, Polynomial eigenvalue problem, Quadratification.

in many scientific and industrial problems. The literature on linearizations for quadratic matrix polynomials are found in [18, 19]. Depending upon the different types of linearization concerned, the approximations of eigenvalues and the study about their locations in the complex plane has its significance in the context of present research [2, 20]. Moreover, in the existing literature, QEP are well developed both numerically and theoretically [18, 25, 27]. It can be proved that PEP of even grade can be converted into a QEP, which opens up new direction of future research on the numerical and theoretical study of PEP. In the current paper, we study quadratification techniques of even grade PEP. We also study two popular linearization classes for regular QEP induced due to quadratification of even grade PEP and impact of their linearizations classes on the computation of eigenpairs.

The rest of the article is organized as follows: In Section 2 we present some notation and preliminary definitions, which will help to obtain main findings of the article. In Section 3, we discuss the quadratification techniques of even grade PEP and its related linearization classes. Section 4 contains few results on upper bounds of condition number ratios of $\mathbb{P}(\lambda)$ with respect to their linearization classes. Finally, in Section 5, a concluding remark is given on the whole research work.

2. Preliminaries

The following notations and basic definitions will be used throughout the paper. In the whole article, $i := 1 : k$ represents the values of i from $i = 1, \ldots, k$. The set of all $n \times m$ matrices is denoted by $\mathbb{C}^{n\times m}$, and I_n denote the identity matrix of order n. A^{-1} and A^T represents the inverse and transpose of any matrix A respectively. $\sigma_{min}(A)$ and $\lambda_{min}(A)$ denotes the smallest singular value and smallest eigenvalue in magnitude of any matrix A. The conjugate transpose of a vector x is denoted by x^* and $\otimes$ denotes the usual Kronecker product. The notation $||.||_p, 1 \le p \le \infty$ is used to denote the induced operator norm on $\mathbb{C}^{n\times m}$.

Definition 2.1. [28] A polynomial eigenvalue problem of degree 2 of the form $\mathbb{Q}(\lambda)x := (\lambda^2\mathbb{Q}_2 + \lambda\mathbb{Q}_1 + \mathbb{Q}_0)x = 0, \quad 0 \neq x \in \mathbb{C}^n, \mathbb{Q}_i \in \mathbb{C}^{n\times n}$ is called QEP, where the problem is to find the scalars $\lambda \in \mathbb{C}$ for some $0 \neq x \in \mathbb{C}^n$.

Definition 2.2. [22] $L(\lambda)$ is said to be a linearization of $\mathbb{P}(\lambda)$ if there exist $kn \times kn$ unimodular matrix polynomials $U(\lambda)$, $V(\lambda)$ such that $U(\lambda)L(\lambda)V(\lambda) = diag(I_{(k-1)n}, P(\lambda))$ for $\lambda \in \mathbb{C}$.

Definition 2.3. [22] A reversal of $\mathbb{P}(\lambda)$, denoted as rev $\mathbb{P}(\lambda)$, is defined as the polynomial rev $\mathbb{P}(\lambda) := \lambda^k\mathbb{P}(\frac{1}{\lambda})$.

Definition 2.4. [22] If rev $L(\lambda) := B + \lambda A$ is a linearization of rev $\mathbb{P}(\lambda)$, then $L(\lambda)$ is said to be a strong linearization of $\mathbb{P}(\lambda)$, where rev $\mathbb{P}(\lambda) := \lambda^k\mathbb{P}(\frac{1}{\lambda})$.

Theorem 2.5. [12] *All eigenvalues of a square matrix A lie within certain disks in the complex plane whose center are equal to the diagonal entries and radius* *are the sum of absolute values of non-diagonal entries given by*

$$|\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|, i := 1 : k.$$

This theorem is termed as Geršgorin Disc Theorem.

For any $k \times k$ matrix A with complex entries, we define the set $L = \{1, 2, ..., l\}$ with a partition $0 = l_0 < l_1 < ... < l_t = l$, so that A can be partitioned as

$$A = \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}$$

where, each block submatrix $A_{m,n}$ is complex with size $(l_m - l_{m-1}) \times (l_n - l_{n-1})$. Then the block Gersgorin set [12] containing all eigenvalues λ of A is given by

$$G(A) = \bigcup_{i=1}^{k} \left\{ \lambda \in \mathbb{C} : \|(A_{ii} - \lambda I)^{-1}\|_p^{-1} \leq \sum_{n=1, i\neq n}^{k} \|A_{i,n}\|_p \right\} \tag{2.1}$$

3. Even grade PEP

We consider the following PEP with $\mathbb{P}(\lambda)$ as defined in (3.1)

$$\mathbb{P}(\lambda) := \sum_{j=0}^{2m} \lambda^j P_j \text{ where } P_j \in \mathbb{C}^{n\times n} \tag{3.1}$$

for $0 \leq j \leq 2m$ and m is a positive integer and it is termed as even grade PEP. It can be converted in to a QEP using quadratification techniques. For that, denote $B_1(\lambda) = \lambda^2 P_2 + \lambda P_1 + P_0; B_2(\lambda) = \lambda^2 P_4 + \lambda P_3; B_3(\lambda) = \lambda^2 P_6 + \lambda P_5; \dots ; B_m(\lambda) = \lambda^2 P_{2m} + \lambda P_{2m-1}$. Define

$$\Lambda(\lambda) = \begin{bmatrix} \lambda^{2j-2} & \lambda^{2j-4} & \dots & \lambda^2 & 1 \end{bmatrix}^T ; j := 2 : m$$

Then, the first companion form can be written as

$$\mathbb{Q}(\lambda) := \begin{bmatrix} B_m(\lambda) & B_{m-1}(\lambda) & \dots & B_2(\lambda) & B_1(\lambda) \\ -I_n & \lambda^2 I_n & \dots & O & O \\ O & -I_n & \lambda^2 I_n & \dots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \dots & -I_n & \lambda^2 I_n \end{bmatrix} = \lambda^2 \mathbb{Q}_2 + \lambda \mathbb{Q}_1 + \mathbb{Q}_0 \tag{3.2}$$

where,

$$\mathbb{Q}_2 = \begin{bmatrix} P_{2m} & P_{2m-2} & \dots & P_2 \\ O & I_n & \dots & O \\ O & O & \dots & O \\ \vdots & \ddots & \ddots & \vdots \\ O & O & \dots & I_n \end{bmatrix}_{mn\times mn}$$

$$\mathbb{Q}_1 = \begin{bmatrix} P_{2m-1} & P_{2m-3} & \dots & P_1 \\ O & O & \dots & O \\ \vdots & \ddots & \ddots & \vdots \\ O & O & \dots & O \end{bmatrix}_{mn\times mn} ; \ \mathbb{Q}_0 = \begin{bmatrix} O & O & \dots & P_0 \\ -I_n & O & \dots & O \\ \vdots & \ddots & \ddots & \vdots \\ O & O & -I_n & O \end{bmatrix}_{mn\times mn}$$

The second companion form of (3.2) can be written as $\hat{\mathbb{Q}}(\lambda) = \lambda^2\mathbb{Q}_2^T + \lambda\mathbb{Q}_1^T + \mathbb{Q}_0^T$. It can be proved that both the quadratification $\mathbb{Q}(\lambda)$ and $\hat{\mathbb{Q}}(\lambda)$ are strong quadratifications. The matrix polynomial $\mathbb{Q}(\lambda)$ generates a QEP, where the problem is to find the pair (λ, Z) such that $\mathbb{Q}(\lambda)Z = 0$ for $\Lambda(\lambda) \otimes x = Z \in \mathbb{C}^{mn}$. From the structure of $\mathbb{Q}_2$, it is seen that if P_{2m} is nonsingular, then $\mathbb{Q}_2$ will also be nonsingular. Therefore, the QEP will be regular if P_{2m} is nonsingular. It is proved in [28], (Theorem 1) that the block Gersgorin set $G(A)$ defined in (2.1) contains the spectrum of regular QEP.

3.1. **Linearization of first companion form $\mathbb{Q}(\lambda)$.** Linearization is the classical process to solve eigenvalue problems associated with matrix polynomials. In this process PEP converted into a GEP of high dimension, where we search the spectral parameters for which the linearized matrix pencil is not of full rank. It is worth noting, that only strong linearizations are reliable to work, which preserve the complete eigenstructure of singular matrix polynomial [24]. For the QEP of the form (3.2), consider the two common linearization $\mathbb{C}_1(\lambda)$ and $\mathbb{C}_2(\lambda)$ given by

$$\mathbb{C}_1(\lambda) = \lambda \begin{bmatrix} \mathbb{Q}_2 & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} \mathbb{Q}_1 & \mathbb{Q}_0 \\ -I_n & 0 \end{bmatrix} \tag{3.3}$$

$$\mathbb{C}_2(\lambda) = \lambda \begin{bmatrix} \mathbb{Q}_2 & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} \mathbb{Q}_1 & -I_n \\ \mathbb{Q}_0 & 0 \end{bmatrix} \tag{3.4}$$

If $\mathbb{Q}_2$ and $\mathbb{Q}_0$ are nonsingular, then the following two symmetric linearization for QEP was given in [16].

$$\hat{\mathbb{C}}_1(\lambda) = \lambda \begin{bmatrix} \mathbb{Q}_2 & 0 \\ 0 & -\mathbb{Q}_0 \end{bmatrix} + \begin{bmatrix} \mathbb{Q}_1 & \mathbb{Q}_0 \\ \mathbb{Q}_0 & 0 \end{bmatrix} \tag{3.5}$$

$$\hat{\mathbb{C}}_2(\lambda) = \lambda \begin{bmatrix} 0 & \mathbb{Q}_2 \\ \mathbb{Q}_2 & \mathbb{Q}_1 \end{bmatrix} + \begin{bmatrix} -\mathbb{Q}_2 & 0 \\ 0 & \mathbb{Q}_0 \end{bmatrix} \tag{3.6}$$

The matrix polynomial $\mathbb{Q}(\lambda)$ and its linearization have the same finite eigenvalues [13]. The linearized GEP are generally solved using QZ method for matrices of smaller size or Krylov subspace method for the problem consisting of sparse matrices. Again, the first and second companion form namely, $\mathbb{C}_1(\lambda)$ and $\mathbb{C}_2(\lambda)$ are strong linearization irrespective of whether QEP is regular or singular. Three vector spaces of potential linearizations of matrix polynomial of arbitrary grade were introduced in [22], namely the right and left ansatz vector spaces denoted by $\mathbb{L}_1(\mathbb{Q})$ and $\mathbb{L}_2(\mathbb{Q})$ respectively and the double ansatz space $\mathbb{DL}(\mathbb{Q}) = \mathbb{L}_1(\mathbb{Q}) \cap \mathbb{L}_2(\mathbb{Q})$. For QEP defined in (3.2), they can be written as

$$\mathbb{L}_1(\mathbb{Q}) = \mathbb{L}(\lambda) = \lambda X + Y : \mathbb{L}(\lambda).(\Lambda \otimes I_n) = v \otimes P(\lambda), v \in \mathbb{C}^2\} \tag{3.7}$$

$$\mathbb{L}_2(\mathbb{Q}) = \mathbb{L}(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n).\mathbb{L}(\lambda) = w^T \otimes P(\lambda), w \in \mathbb{C}^2\} \tag{3.8}$$

where, $\mathbb{L}(\lambda)$ is any linearized classes defined in (3.3)- (3.6), $\Lambda = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ and $X, Y \in \mathbb{C}^{2mn\times 2mn}$. The equations (3.7) and (3.8) are respectively called right and left ansatz equations and the corresponding vectors v and w are called left and right ansatz vectors. The recovery of eigenvectors of regular problem from linearized matrix pencil has been well studied in [22], which is further extended to singular case to discuss the recovery of minimal bases in [24].

3.2. **Even grade PEP induces a regular QEP.** Denote $\mathbb{P}_n$ be the set of all $n \times n$ quadratic matrix polynomials and $\mathbb{Q}(\lambda)$ be any matrix polynomial in $\mathbb{P}_n$ such that

$$\mathbb{Q}(\lambda) = \lambda^2\mathbb{Q}_2 + \lambda\mathbb{Q}_1 + \mathbb{Q}_0 \in \mathbb{C}^{n\times n}, \mathbb{Q}_2 \neq 0 \tag{3.9}$$

The following theorem is required for locating eigenvalues of the QEP.

Theorem 3.1. [28] *Let $\mathbb{Q}(\lambda) \in \mathbb{P}_n$ be defined in (3.9). Then, the general form of a strong linearization $\mathbb{L}(\lambda)$ in $\mathbb{L}_1(\mathbb{Q})$ and $\mathbb{L}_2(\mathbb{Q})$ with right ansatz vector $v = \alpha e_1, \alpha \in \mathbb{C} \setminus 0$ are respectively,*

$$\mathbb{L}(\lambda) = \lambda \left[\begin{array}{c|c} \alpha\mathbb{Q}_2 & -W_1 \\ \hline 0 & W_2 \end{array}\right] + \left[\begin{array}{c|c} W_1 + \alpha\mathbb{Q}_1 & \alpha\mathbb{Q}_0 \\ \hline W_2 & 0 \end{array}\right] \tag{3.10}$$

$$\mathbb{L}(\lambda) = \lambda \left[\begin{array}{c|c} \alpha\mathbb{Q}_2 & 0 \\ \hline -W_1 & -W_2 \end{array}\right] + \left[\begin{array}{c|c} W_1 + \alpha\mathbb{Q}_1 & W_2 \\ \hline \alpha\mathbb{Q}_0 & 0 \end{array}\right] \tag{3.11}$$

It is to be noted that if $\alpha = 1, W_1 = 0$ and $W_2 = -I_n$ in equation (3.10), then it corresponds to first linearization class $\mathbb{C}_1(\lambda)$. Similarly, if $\alpha = 1, W_1 = 0$ and $W_2 = -I_n$ in equation (3.11), then it corresponds to second linearization class $\mathbb{C}_2(\lambda)$. Motivated by the works of [28], we study few sets that localize eigenvalue of regular QEP represented by (3.2) which is being induced due to quadrification. The block Gersgorin sets arising from $\mathbb{C}_1(\lambda)$ belonging to the vector spaces $\mathbb{L}_1(\mathbb{Q})$, $\mathbb{L}_2(\mathbb{Q})$ and $\mathbb{DL} = \mathbb{L}_1(\mathbb{Q}) \cap \mathbb{L}_2(\mathbb{Q})$ are strong linearization of $\mathbb{Q}(\lambda)$. Generalizing the theorem 6 and 7 presented in [28], the following two theorems provide the block Gersgorin sets associated with the strong linearization in $\mathbb{L}_1(\mathbb{Q})$ and $\mathbb{L}_2(\mathbb{Q})$ with ansatz vectors αe_1 and αe_2 respectively.

Theorem 3.2. [28] *Let, $\mathbb{Q}(\lambda) = \lambda^2\mathbb{Q}_2 + \lambda\mathbb{Q}_1 + \mathbb{Q}_0$ with non-zero matrix $\mathbb{Q}_0$. Then the block Gersgorin set containing all eigenvalues of $\mathbb{Q}(\lambda)$ is $G(Q) = S_1 \cup S_2$ where, S_1 and S_2 are the sets that takes different form based on the linearizations considered. The following four forms of S_1 and S_2 are generally takes into consideration in the study.*

(1) *Let $L(\lambda)$ be first companion form of linearization. Then,*

$$S_1 = \left\{ \lambda \in \mathbb{C}^\infty : \left\| (\lambda\mathbb{Q}_2 + \mathbb{Q}_1)^{-1} \right\|_p^{-1} \leq \|\mathbb{Q}_0\|_p \right\} \tag{3.12}$$

$$S_2 = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \tag{3.13}$$

(2) *Let $L(\lambda)$ be second companion form of linearization. Then,*

$$S_1 = \left\{ \lambda \in \mathbb{C}^\infty : \left\| (\lambda\mathbb{Q}_2 + \mathbb{Q}_1)^{-1} \right\|_p^{-1} \leq 1 \right\} \tag{3.14}$$

$$S_2 = \{\lambda \in \mathbb{C} : |\lambda| \leq \|\mathbb{Q}_0\|_p\} \tag{3.15}$$

(3) *Let,* $L(\lambda) \in \mathbb{L}_1(\mathbb{Q})$ *with right ansatz vector* $v = \beta e_1, \beta \in \mathbb{C}\backslash\{0\}$ *as given in (3.10). Then,*

$$S_1 = \left\{\lambda \in \mathbb{C}^{\infty} : \left\|(\beta(\lambda \mathbb{Q}_2 + \mathbb{Q}_1) + U_1)^{-1}\right\|_p^{-1} \leq \|\beta \mathbb{Q}_0 - \lambda U_1\|_p\right\} \tag{3.16}$$

$$S_2 = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \|U_2\|_p \|U_2^{-1}\|_p\right\}. \tag{3.17}$$

(4) *Let,* $L(\lambda) \in \mathbb{L}_2(\mathbb{Q})$ *with left ansatz vector* $v = \beta e_1, \beta \in \mathbb{C}\backslash\{0\}$ *as given in (3.11). Then,*

$$S_1 = \left\{\lambda \in \mathbb{C}^{\infty} : \left\|(\beta(\lambda \mathbb{Q}_2 + \mathbb{Q}_1) + U_1)^{-1}\right\|_p^{-1} \leq \|U_2\|_p\right\} \tag{3.18}$$

$$S_2 = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \|U_2^{-1}\|_p \|\beta \mathbb{Q}_0 - \lambda U_1\|_p\right\} \tag{3.19}$$

where $U_2 \in \mathbb{C}^{n,n}$ *is non-singular and* $U_1 \in \mathbb{C}^{n,n}$ *is arbitrary.*

(5) *Let,* $L(\lambda) \in \mathbb{L}_1(\mathbb{Q})$ *with corresponding ansatz vector* $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$ *which is not a multiple of* e_1 *or* e_2 *then*

$$S_1 = \left\{\lambda \in \mathbb{C}^{\infty} : \left\|(\lambda \mathbb{Q}_2 + \mathbb{Q}_1 + K_1)^{-1}\right\|_p^{-1} \leq \|\mathbb{Q}_0 - \beta K_1\|_p\right\} \tag{3.20}$$

$$S_2 = \left\{\lambda \in \mathbb{C} : \|(\mathbb{Q}_0 - \beta K_2)^{-1}\|_p \leq \|\lambda \mathbb{Q}_2 + \mathbb{Q}_1 + K_2\|_p\right\} \tag{3.21}$$

where $U_1, U_2 \in \mathbb{C}^{n,n}$ *and* $K_1 = \dfrac{U_1}{v_1}, K_2 = \dfrac{U_2}{v_2}$ *such that* $K_1 - K_2$ *is non-singular.*

From theorem 3.2, it is clear that if the matrix pencil $\lambda \mathbb{Q}_2 + \mathbb{Q}_1$ is singular, then the Block Gersgorin set becomes $\mathbb{C}^{\infty}$ corresponding to first and second companion forms.

Example 1. Consider the PEP defined in (3.22) of grade 10.

$$\mathbb{P}(\lambda) = P_{10}\lambda^{10} + P_9\lambda^9 + P_8\lambda^8 + P_7\lambda^7 + P_6\lambda^6 + P_5\lambda^5 + P_4\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 \tag{3.22}$$

where,

$$P_{10} = \begin{bmatrix} 2.5475 & 1.1784 & -1.3256 \\ 1.4237 & -0.1754 & 0.2254 \\ 1.1123 & 2.4736 & -0.1258 \end{bmatrix}, P_9 = \begin{bmatrix} 2.3342 & 1.4412 & 2.4733 \\ 0.2684 & 0.2513 & 0.2222 \\ 0.2147 & 2.1124 & -1.2214 \end{bmatrix}$$

$$P_8 = \begin{bmatrix} 2.3564 & 1.1427 & 1.1111 \\ 2.1236 & 2.1254 & -2.2254 \\ 1.7852 & 1.3480 & -0.4475 \end{bmatrix}, P_7 = \begin{bmatrix} 1.2236 & 2.2235 & 2.2258 \\ 1.2654 & 2.2256 & 0.2587 \\ 0.2564 & 1.2256 & -2.4875 \end{bmatrix}$$

$$P_6 = \begin{bmatrix} 0.1236 & 2.1147 & 0.1245 \\ 0.2654 & 1.3324 & 2.4474 \\ 1.3564 & -1.3254 & -2.3354 \end{bmatrix}, P_5 = \begin{bmatrix} 1.2247 & 0.2548 & 0.4122 \\ 0.1258 & -2.2365 & 1.4477 \\ 1.3987 & 2.2654 & 1.3365 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 1.2347 & 2.3347 & 0.0412 \\ 0.3362 & -1.3214 & 1.2256 \\ 0.2222 & 0.5555 & 1.7423 \end{bmatrix}, P_3 = \begin{bmatrix} 0.2213 & 0.3211 & 2.3368 \\ -2.5412 & 2.8888 & 2.7777 \\ 6.1235 & 2.3369 & 2.5698 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 2.4432 & 0.6684 & 2.7896 \\ -0.2236 & 1.3256 & 1.3325 \\ -2.2213 & 1.3254 & 1.3325 \end{bmatrix}, P_1 = \begin{bmatrix} 2.3365 & 1.3323 & 2.3222 \\ 1.3362 & 2.3333 & 0.2365 \\ 3.2256 & -0.1654 & 2.3365 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 2.3365 & -0.2222 & -1.2456 \\ -1.2654 & 3.2258 & 1.3698 \\ -2.2548 & -2.3365 & 1.2587 \end{bmatrix}$$

The problem can be reformulated as a QEP with the help of quadratification as derived in the equation (3.2). The localization sets are plotted in plane by considering the reformulated QEP. It is seen that all eigenvalues of $\mathbb{Q}(\lambda)$ are finite as the leading coefficient matrix $\mathbb{Q}_2$ is nonsingular. We plot block Gersgorin sets associated with second companion linearization in Figure 1. In the Figure 2, we plot the set with the help of first companion form. All plotting are done using 2-norm. Figure 3 represents Gersgorin set corresponding to linearizations in $\mathbb{L}_2(Q)$. We take $v_2 = \sigma_{min}(\mathbb{Q}_2)$, $v_1 = 2\|\mathbb{Q}_2\|_2$, $U_1 = \mathbb{Q}_1$ and $U_2 = \mathbb{Q}_2$.

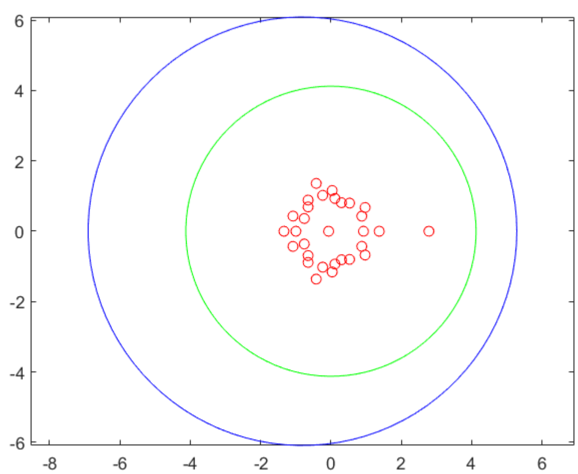

FIGURE 1. Block Gersgorin set corresponding to second companion form

Theorem 3.3. [28] *Let $\mathbb{Q}(\lambda) \in \mathbb{P}_n$ be quadratic matrix polynomial as defined in (3.2), then for any strong linearization $L(\lambda)$ in $\mathbb{L}_1(\mathbb{Q})$ or $\mathbb{DL}(\mathbb{Q})$.*

$$\{\lambda \in \Lambda(Q) : |\lambda| \geq 1\} \subseteq S_1 \, and \, \{\lambda \in \Lambda(Q) : |\lambda| \leq 1\} \subseteq S_2.$$

Lemma 3.4. [28] *Let $\mathbb{Q}(\lambda)$ be quadratic matrix polynomial defined in (3.2) with block Gersgorin set S_1 and S_2 given by (3.20) and (3.21) respectively. Let, $K_1, K_2 \in \mathbb{C}^{n,n}$ such that $K_1 - K_2$ is non-singular. Then,*

$$S_1 \subseteq \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{\|\mathbb{Q}_0\|_p + \|\mathbb{Q}_1 + K_1\|_p}{\left\|\mathbb{Q}_2^{-1}\right\|_p^{-1} - \|K_1\|_p} \right\} \tag{3.23}$$

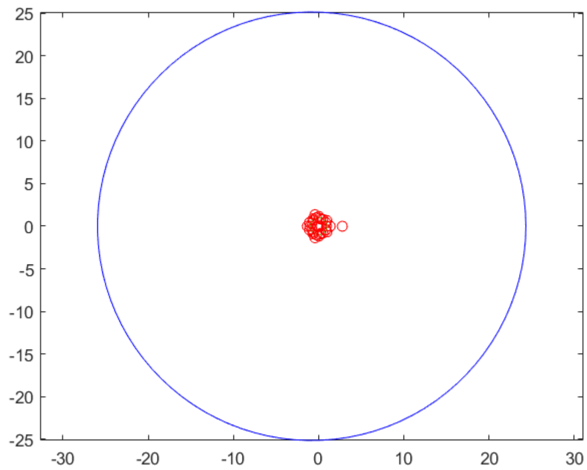

FIGURE 2. Block Gersgorin set corresponding to first companion form $\mathbb{L}_1(Q)$

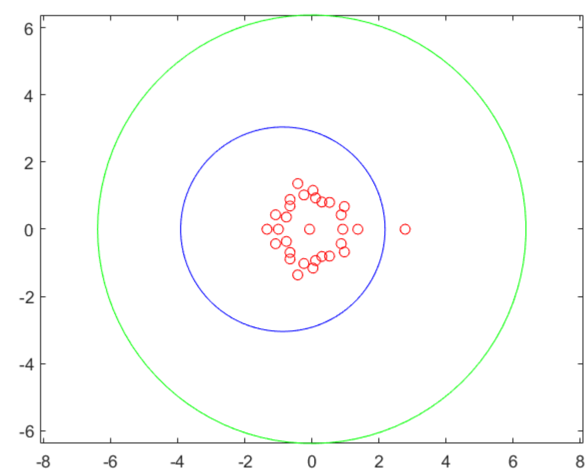

FIGURE 3. Block Gersgorin set corresponding to linearizations in $\mathbb{L}_2(Q)$

whenever $\left\|\mathbb{Q}_2^{-1}\right\|_p^{-1} > \|K_1\|_p$, *and*

$$S_2 \subseteq \left\{ \lambda \in \mathbb{C} : |\lambda| \geq \frac{\left\|\mathbb{Q}_0^{-1}\right\|_p^{-1} - \|\mathbb{Q}_1 + K_2\|_p}{\|\mathbb{Q}_2\|_p + \|K_2\|_p} \right\}. \tag{3.24}$$

Setting $K_1 = 0$ in equation (3.23) and $K_2 = -\mathbb{Q}_1$ in equation (3.24) along with theorem (3.3) we obtain the upper and lower bound for the eigenvalues of $\mathbb{Q}(\lambda)$ given by $\tilde{U} \geq |\lambda|$ and $\tilde{L} \leq |\lambda|$ respectively, where

$$\tilde{U} = min\left\{\tilde{u_1}, \tilde{u_2}\right\} \tag{3.25}$$

with

$$\tilde{u_1} = max\left\{1, \|\mathbb{Q}_2^{-1}\|_p(\|\mathbb{Q}_1\|_p + \|\mathbb{Q}_0\|_p)\right\},$$

$$\tilde{u_2} = \|Q_2^{-1}\|_p max\left\{\|\mathbb{Q}_0\|_p, (\|Q_2\|_p + \|\mathbb{Q}_1\|_p)\right\},$$

and

$$\tilde{L} = min\left\{1, \frac{1}{\|\mathbb{Q}_0^{-1}\|_p(\|\mathbb{Q}_2\|_p + \|\mathbb{Q}_1\|_p)}\right\}. \tag{3.26}$$

For the example 1, the eigenvalues and their respective absolute values are obtained by using standard solver *polyeig* available in MATLAB and are given in the Table 1. The calculated condition number $K_{\mathbb{L}}(\lambda)$ in Table 1 for linearized problem class, the common lineraization $\mathbb{C}_1(\lambda)$ is used. The eigenvalue bounds calculated from (3.25) and (3.26) are $\tilde{U} = 85.9364$ and $\tilde{L} = 0.0157$. It is worth mentioning that the absolute values of eigenvalues of example (3.22) are lying inside the calculated bound. Therefore, this techniques can be adopted to select initial guess of eigenvalues in applying numerical techniques to solve even grade PEP.

4. Bounds of condition numbers of $\mathbb{L}(\lambda)$ relative of $\mathbb{P}(\lambda)$ and $\mathbb{Q}(\lambda)$

In this section, condition number bounds of $\mathbb{P}(\lambda)$ with respect to its linearization classes specified in equations (3.3)-(3.6) were introduced. We derive upper bounds of condition number ratios of $\mathbb{P}(\lambda)$ with respect to that of linearization classes with the help of moduli of eigenvalues and norms of coefficient matrices. In the derivation of results, the 2-norm of the matrices are considered. For simple, nonzero and finite eigenvalue λ of QEP defined in (3.2) with corresponding right and left eigenvectors x and y, the norm wise condition number is defined in [26] as follows

$$\kappa_{\mathbb{Q}}(\lambda) = \lim_{\epsilon\to 0}\left\{\frac{|\Delta\lambda|}{\epsilon|\lambda|} : (\mathbb{Q}(\lambda+\Delta\lambda) + \Delta\mathbb{Q}(\lambda+\Delta\lambda))(x+\delta x) = 0; \|\Delta\mathbb{Q}_i\|_2 \le \epsilon\|\mathbb{Q}_i\|_2\right\} \tag{4.1}$$

where $i = 0:2$ and $\Delta\mathbb{Q}(\lambda) = \sum_{j=0}^{2}\lambda^i\Delta\mathbb{Q}_i$ with $\Delta\mathbb{Q}_i$ as the perturbation to $\mathbb{Q}_i$. Tisseur [26] also provided formula to compute the condition number of $\mathbb{Q}(\lambda)$ and its linereaization classes of $\mathbb{L}(\lambda)$ and are given by:

$$\kappa_{\mathbb{P}}(\lambda) = \frac{\left(\sum_{j=0}^{2m}|\lambda|^j\|\Delta\mathbb{P}_j\|_2\right)\|v\|_2\|u\|_2}{|\lambda||v^*\mathbb{P}'(\lambda)u|} \tag{4.2}$$

$$\kappa_{\mathbb{Q}}(\lambda) = \frac{\left(\sum_{i=1}^{2}|\lambda|^i\|\Delta\mathbb{Q}_i\|_2\right)\|y\|_2\|x\|_2}{|\lambda||y^*\mathbb{Q}'(\lambda)x|} \tag{4.3}$$

$$\kappa_{\mathbb{L}}(\lambda) = \frac{(|\lambda|\|X\|_2 + \|Y\|_2)\|w\|_2\|z\|_2}{|\lambda||w^*\mathbb{L}'(\lambda)z|} \tag{4.4}$$

where, $\mathbb{L}(\lambda)$ represents any linearization classes defined in equations (3.3)-(3.6) and X, Y represents their coefficient matrices in the generalised matrix pencils.

Bounding of the condition number ratios $\frac{\kappa_{\mathbb{L}}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$ and $\frac{\kappa_{\mathbb{L}}(\lambda)}{\kappa_{\mathbb{Q}}(\lambda)}$ are important in numerical computation of eigenvalues [26], which follows from the construction of one sided transformation of the linearization satisfying equations (4.5) and (4.6).

$$\mathbb{L}(\lambda)\mathbb{H}(\lambda) = h \otimes \mathbb{Q}(\lambda) \tag{4.5}$$

$$\mathbb{G}(\lambda)\mathbb{L}(\lambda) = g^T \otimes \mathbb{Q}(\lambda) \tag{4.6}$$

where, $\mathbb{H}(\lambda)$ and $\mathbb{G}(\lambda)$ are $2mn \times n$ matrix polynomials and $h, g^T \in \mathbb{C}^{2m}$. Also, we have from [16],

$$z = \mathbb{H}(\lambda)x, w = \mathbb{G}^*(\lambda)y \tag{4.7}$$

Now, we are ready to the derive bounds of the condition number ratios.

Theorem 4.1. *Consider the first common linearization for $\mathbb{C}_1(\lambda)$ of the QEP defined in (3.3) and assume that λ is nonzero, finite and simple. Then,*

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \begin{cases} \dfrac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2 & if |\lambda| \geq 1, \|P_{2m}\| \geq 1, \\ \dfrac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2 & if |\lambda| \geq 1, \|P_{2m}\| \leq 1, \\ \dfrac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1 & if |\lambda| \leq 1, \|P_{2m}\| \geq 1, \\ \dfrac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1 & if |\lambda| \leq 1, \|P_{2m}\| \leq 1. \end{cases}$$

where,

$$\Phi_1 = \sqrt{\begin{matrix} 1 + (|\lambda|\|P_{2m}\| + \|P_{2m-1}\|)^2 + (|\lambda|^2\|P_{2m}\| + |\lambda|\|P_{2m-1}\| + \|P_{2m-2}\|)^2 \\ + \ldots + (|\lambda|^{2m-1}\|P_{2m}\| + |\lambda|^{2m-2}\|P_{2m-1}\| + |\lambda|^{2m-3}\|P_{2m-2}\| + \ldots + \|P_1\|)^2 \end{matrix}}$$

$$\Phi_2 = \sqrt{\begin{matrix} 1 + \left(\dfrac{\|P_{2m-2}\|}{|\lambda|} + \dfrac{\|P_{2m-3}\|}{|\lambda|^2} + \ldots + \dfrac{\|P_0\|}{|\lambda|^{2m-1}}\right)^2 + \\ \left(\dfrac{\|P_{2m-3}\|}{|\lambda|} + \dfrac{\|P_{2m-4}\|}{|\lambda|^2} + \ldots + \dfrac{\|P_0\|}{|\lambda|^{2m-2}}\right)^2 + \ldots + \left(\dfrac{\|P_0\|}{|\lambda|}\right)^2 \end{matrix}}$$

and $\Psi = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4} + \ldots + |\lambda|^2 + 1}$

Proof. Based on the structure of $\mathbb{C}_1(\lambda)$, we can construct the following

$$\mathbb{C}_1(\lambda)\mathbb{H}(\lambda) = e_1 \otimes \mathbb{P}(\lambda) \tag{4.8}$$

$$\mathbb{G}(\lambda)\mathbb{C}_1(\lambda) = e_{2m}^T \otimes \mathbb{P}(\lambda) \tag{4.9}$$

where, $\mathbb{H}(\lambda) = \begin{bmatrix}\lambda^{2m-1}I_n & \lambda^{2m-2}I_n & \ldots & \lambda^2 I_n & \lambda I_n & I_n\end{bmatrix}^T$ and

$$\mathbb{G}(\lambda) = \begin{bmatrix} I_n \\ \lambda P_{2m} + P_{2m-1} \\ \lambda^2 P_{2m} + \lambda P_{2m-1} + P_{2m-2} \\ \ldots \\ \lambda^{2m-1}P_{2m} + \lambda^{2m-2}P_{2m-1} + \ldots + P_1 \end{bmatrix}^T$$

Combining equation (4.7) with the equations (4.8) and (4.9), we have

$$z = \begin{bmatrix} \lambda^{2m-1} I_n \\ \lambda^{2m-2} I_n \\ \vdots \\ \lambda I_n \\ I_n \end{bmatrix} x, \; w = \begin{bmatrix} I_n \\ (\lambda P_{2m} + P_{2m-1})^* \\ (\lambda^2 P_{2m} + \lambda P_{2m-1} + P_{2m-2})^* \\ \vdots \\ (\lambda^{2m-1} P_{2m} + \lambda^{2m-2} P_{2m-1} + \ldots + P_1)^* \end{bmatrix} y \tag{4.10}$$

Based on the works of [15], we derive the core identity to prove the results. Since $\mathbb{C}_1(\lambda) \in \mathbb{L}_1(\mathbb{Q})$, therefore from equation (3.5) we have

$$\mathbb{C}_1(\lambda)(\Lambda \otimes I_n) = v \otimes \mathbb{P}(\lambda)$$

Differentiating with respect to λ we get

$$\mathbb{C}_1^{'}(\lambda)(\Lambda \otimes I_n) + \mathbb{C}_1(\lambda)(\Lambda^{'} \otimes I_n) = v \otimes \mathbb{P}^{'}(\lambda)$$

At an eigenvalue λ, pre and post multiplying the above equation by w^* and $1 \otimes x$ respectively, we get

$$\begin{aligned} w^* \mathbb{C}_1^{'}(\lambda) z &= w^* \left(v \otimes \mathbb{P}^{'}(\lambda)\right)(1 \otimes x) \\ \implies w^* \mathbb{C}_1^{'}(\lambda) z &= \left(\mathbb{G}(\lambda)^T \otimes y^*\right) . \left(v \otimes \mathbb{P}^{'}(\lambda) x\right) \\ \implies w^* \mathbb{C}_1^{'}(\lambda) z &= \left(\mathbb{G}(\lambda)^T v\right) . \left(y^* \mathbb{P}^{'}(\lambda) x\right) \end{aligned}$$

Using equation (4.10) and the fact that $v = e_1$ the following identity holds

$$w^* \mathbb{C}_1^{'}(\lambda) z = y^* \mathbb{P}^{'}(\lambda) x$$

The condition number ratio of $\mathbb{C}_1(\lambda)$ relative to that of $\mathbb{P}(\lambda)$ is given by

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} = \frac{(|\lambda|\|X\| + \|Y\|)\|w\|\|z\|}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \ldots + |\lambda|\|P_1\| + \|P_0\|)\|y\|\|x\|}.$$

We now compare the bound the ratio $\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$ in terms of coefficient matrices of $\mathbb{P}(\lambda)$ and $\mathbb{C}_1(\lambda)$ and eigenvectors. Using Lemma 3.5 in [16], we have

$$\|X\| = max(\|P_{2m}\|, 1), \; \|Y\| \leq 2m \; max(max_{i=0:2m-1}\|P_i\|, 1).$$

From eigenvector recovery property (4.9), the relation between the norms of the right eigenvector x and z is given by

$$\|z\| = \| \begin{bmatrix} \lambda^{2m-1} I_n \\ \lambda^{2m-2} I_n \\ \vdots \\ \lambda^2 I_n \\ \lambda I_n \\ I_n \end{bmatrix} x\| = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4} + \ldots + |\lambda|^2 + 1}\|x\|,$$

which implies $\dfrac{\|z\|}{\|x\|} = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4} + \ldots + |\lambda|^2 + 1} = \Psi \ (say)$. The norm of y relative to that of w can be obtained by using the equation (4.10). From equation (4.10) we have

$$\|w\| = \left\| \begin{bmatrix} I_n \\ (\lambda P_{2m} + P_{2m-1})^* \\ (\lambda^2 P_{2m} + \lambda P_{2m-1} + P_{2m-2})^* \\ \vdots \\ (\lambda^{2m-1} P_{2m} + \lambda^{2m-2} P_{2m-1} + \lambda^{2m-3} P_{2m-2} + \ldots + P_1)^* \end{bmatrix} y \right\|$$

$$= \left\| \begin{bmatrix} I_n \\ (\lambda^{-1} P_{2m-2} + \lambda^{-2} P_{2m-3} + \ldots + \lambda^{-2m+1} P_1 + P_0)^* \\ (\lambda^{-1} P_{2m-3} + \lambda^{-2} P_{2m-4} + \ldots + \lambda^{-2m+3} P_1 + \lambda^{-2m+2} P_0)^* \\ \vdots \\ (\lambda^{-1} P_0)^* \end{bmatrix} y \right\|$$

then

$$\frac{\|w\|}{\|y\|} \leq min\,(\Phi_1, \Phi_2)$$

where,

$$\Phi_1 = \sqrt{\begin{aligned} &1 + (|\lambda|\|P_{2m}\| + \|P_{2m-1}\|)^2 + (|\lambda|^2\|P_{2m}\| + |\lambda|\|P_{2m-1}\| + \|P_{2m-2}\|)^2 \\ &\quad + \ldots + (|\lambda|^{2m-1}\|P_{2m}\| + |\lambda|^{2m-2}\|P_{2m-1}\| + |\lambda|^{2m-3}\|P_{2m-2}\| + \ldots + \|P_1\|)^2 \end{aligned}}$$

and

$$\Phi_2 = \sqrt{\begin{aligned} &1 + \left(\frac{\|P_{2m-2}\|}{|\lambda|} + \frac{\|P_{2m-3}\|}{|\lambda|^2} + \ldots + \frac{\|P_0\|}{|\lambda|^{2m-1}}\right)^2 \\ &\quad + \left(\frac{\|P_{2m-3}\|}{|\lambda|} + \frac{\|P_{2m-4}\|}{|\lambda|^2} + \ldots + \frac{\|P_0\|}{|\lambda|^{2m-2}}\right)^2 + \ldots + \left(\frac{\|P_0\|}{|\lambda|}\right)^2 \end{aligned}}$$

Also,

$$min\,(\Phi_1, \Phi_2) = \begin{cases} \Phi_1 & if |\lambda| \leq 1 \\ \Phi_2 & if |\lambda| \geq 1 \end{cases}$$

Thus, for the upper bound of $\dfrac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$, the following four cases arises.

Case 1 : if $|\lambda| \geq 1, \|P_{2m}\| \geq 1$

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2.$$

Case 2 : if $|\lambda| \geq 1, \|P_{2m}\| \leq 1$

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2.$$

Case 3 : if $|\lambda| \leq 1, \|P_{2m}\| \geq 1$

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1.$$

Case 4 : if $|\lambda| \leq 1, \|P_{2m}\| \leq 1$

$$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1.$$

□

Theorem 4.2. *Consider the second common linearization for $\mathbb{C}_2(\lambda)$ of the QEP defined in (3.4) and assume that λ is nonzero, finite and simple. Then,*

$$\frac{\kappa_{\mathbb{C}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \begin{cases} \dfrac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2 & if |\lambda| \geq 1, \|P_{2m}\| \geq 1, \\ \dfrac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_2 & if |\lambda| \geq 1, \|P_{2m}\| \leq 1, \\ \dfrac{|\lambda|\|P_{2m}\| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1 & if |\lambda| \leq 1, \|P_{2m}\| \geq 1, \\ \dfrac{|\lambda| + 2m\ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi_1 & if |\lambda| \leq 1, \|P_{2m}\| \leq 1. \end{cases}$$

where,

$$\Phi_1 = \sqrt{\begin{array}{l} 1 + (|\lambda|\|P_{2m}\| + \|P_{2m-1}\|)^2 + (|\lambda|^2\|P_{2m}\| + |\lambda|\|P_{2m-1}\| + \|P_{2m-2}\|)^2 \\ \quad + \ldots + (|\lambda|^{2m-1}\|P_{2m}\| + |\lambda|^{2m-2}\|P_{2m-1}\| + |\lambda|^{2m-3}\|P_{2m-2}\| + \ldots + \|P_1\|)^2 \end{array}}$$

$$\Phi_2 = \sqrt{\begin{array}{l} 1 + \left(\dfrac{\|P_{2m-2}\|}{|\lambda|} + \dfrac{\|P_{2m-3}\|}{|\lambda|^2} + ... + \dfrac{\|P_0\|}{|\lambda|^{2m-1}}\right)^2 \\ \quad + \left(\dfrac{\|P_{2m-3}\|}{|\lambda|} + \dfrac{\|P_{2m-4}\|}{|\lambda|^2} + \ldots + \dfrac{\|P_0\|}{|\lambda|^{2m-2}}\right)^2 + \ldots + \left(\dfrac{\|P_0\|}{|\lambda|}\right)^2 \end{array}}$$

and

$$\Psi = \sqrt{|\lambda|^{4m-2} + |\lambda^{4m-4}| + \ldots + |\lambda|^2 + 1}.$$

Proof. The one-sided transformation of $\mathbb{C}_2(\lambda)$ is gained by

$$\mathbb{C}_2(\lambda)\mathbb{H}(\lambda) = e_{2m} \otimes \mathbb{P}(\lambda)$$

$$\mathbb{G}(\lambda)\mathbb{C}_2(\lambda) = e_1^T \otimes \mathbb{P}(\lambda)$$

where,

$$\mathbb{H}(\lambda) = \begin{bmatrix} I_n \\ \lambda P_{2m} + P_{2m-1} \\ \lambda^2 P_{2m} + \lambda P_{2m-1} + P_{2m-2} \\ \vdots \\ \lambda^{2m-1}P_{2m} + \lambda^{2m-2}P_{2m-1} + \lambda^{2m-3}P_{2m-2} + \ldots + P_1 \end{bmatrix}$$

and $\mathbb{G}(\lambda) = \begin{bmatrix}\lambda^{2m-1}I_n & \lambda^{2m-2}I_n & \lambda^{2m-3}I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n\end{bmatrix}$.
Then,

$$z = \begin{bmatrix} I_n \\ \lambda P_{2m} + P_{2m-1} \\ \lambda^2 P_{2m} + \lambda P_{2m-1} + P_{2m-2} \\ \vdots \\ \lambda^{2m-1}P_{2m} + \lambda^{2m-2}P_{2m-1} + \lambda^{2m-3}P_{2m-2} + \ldots + P_1 \end{bmatrix} x, \ w = \begin{bmatrix} \lambda^{2m-1}I_n \\ \lambda^{2m-2}I_n \\ \vdots \\ \lambda^2 I_n \\ \lambda I_n \\ I_n \end{bmatrix} y.$$

Again, for $\mathbb{C}_2(\lambda)$, and from [15] the following equation holds

$$w^* \mathbb{C}_2^{'}(\lambda) z = y^* \mathbb{P}^{'}(\lambda) x$$

The derivation of the above identity is same as in theorem 4.1. The condition number of $\mathbb{C}_2(\lambda)$ relative to that of $\mathbb{P}(\lambda)$ gives

$$\frac{\kappa_{\mathbb{C}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} = \frac{(|\lambda|\|X\| + \|Y\|)\|w\|\|z\|}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \ldots + |\lambda|\|P_1\| + \|P_0\|)\|y\|\|x\|}.$$

From Lemma 3.5 in [16], we have

$$\|X\| = max(\|P_{2m}\|, 1), \ \|Y\| \leq 2m \ max(max_{i=0:2m-1}\|P_i\|, 1).$$

The norms of eigenvectors of $\mathbb{C}_2(\lambda)$ relative to those of $\mathbb{Q}_(\lambda)$ satisfy

$$\frac{\|w\|}{\|y\|} = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4} + \ldots + |\lambda|^2 + 1} = \Psi \ (say),$$

and

$$\frac{\|z\|}{\|x\|} \leq min\,(\Phi_1, \Phi_2) = \Phi \ (say),$$

where, Φ_1 and Φ_2 are defined in Theorem 4.1. Hence, for $\frac{\kappa_{\mathbb{C}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$ the following ratio of upper bound can be established

$$\frac{\kappa_{\mathbb{C}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{|\lambda| \ max(\|P_{2m}\|, 1) + 2m \ max(1, max_{i=0:2m-1}\|P_i\|)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + ... + |\lambda|\|P_1\| + \|P_0\|}\Psi\Phi.$$

The remaining part of case-by-case discussion is same as that of Theorem 4.1. □

Theorem 4.3. *Let λ be a nonzero, finite and simple eigenvalue of $\mathbb{P}(\lambda)$ and consider the first symmetric linearization $\hat{\mathbb{C}}_1(\lambda)$. Then,*

$$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \begin{cases} \dfrac{(|\lambda|\|P_{2m}\| + 2m \ max(max_{i=0:2m-1}\|P_i\|, 1))}{(|\lambda|^{2m}\|P_{2m}\| + \ldots + |\lambda|\|P_1\| + \|P_0\|)} \dfrac{|\lambda|^{4m-2} + \ldots + 1}{|\lambda|^{2m-1}} & if \|P_{2m}\| \geq \|P_0\| \\ \dfrac{(|\lambda|\|P_0\| + 2m \ max(max_{i=0:2m-1}\|P_i\|, 1))}{(|\lambda|^{2m}\|P_{2m}\| + \ldots + |\lambda|\|P_1\| + \|P_0\|)} \dfrac{|\lambda^{2m-1}|^2 + \ldots + 1}{|\lambda|^{2m-1}} & if \|P_{2m}\| \leq \|P_0\| \end{cases}$$

Proof. The one-sided transformation of $\hat{\mathbb{C}}_1(\lambda)$ is gained by

$$\hat{\mathbb{C}}_1(\lambda)\mathbb{H}(\lambda) = e_1 \otimes \mathbb{P}(\lambda)$$
$$\mathbb{G}(\lambda)\hat{\mathbb{C}}_1(\lambda) = e_1^T \otimes \mathbb{P}(\lambda) \tag{4.11}$$

where,

$$\mathbb{H}(\lambda) = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}^T,$$

$$\mathbb{G}(\lambda) = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}^T.$$

The right and left eigenvectors of $\hat{\mathbb{C}}_1(\lambda)$ takes the following forms

$$\begin{aligned} z &= \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix} x, \\ w &= \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix} y. \end{aligned} \tag{4.12}$$

For $\hat{\mathbb{C}}_1(\lambda)$, based on equations (4.11) and (4.12), it is proved in [16] that

$$w^* \hat{\mathbb{C}}_1'(\lambda) z = \lambda^{2m-1} y^* \mathbb{P}'(\lambda) x$$

The derivation is same as in theorem 4.1. For the condition number of $\hat{\mathbb{C}}_1(\lambda)$ relative to that of $\mathbb{P}(\lambda)$, it notes that

$$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} = \frac{(|\lambda|\|X\| + \|Y\|)\|w\|\|z\|}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)\|y\|\|x\|} \frac{1}{|\lambda|^{2m-1}}.$$

From Lemma 3.5 in [16], we have

$$\|X\| = max(\|P_{2m}\|, \|P_0\|), \;\; \|Y\| \leq 2m \; max(max_{i=0:2m-1}\|P_i\|, 1),$$

and

$$\frac{\|w\|}{\|y\|} = \frac{\|z\|}{\|x\|} = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4} + \dots + |\lambda|^2 + 1}.$$

Then we can bound the ratio $\dfrac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$ as follows

$$\frac{\kappa_{\mathbb{L}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{(|\lambda| \; max(\|P_{2m}\|, \|P_0\|) + 2m \; max(max_{i=0:2m-1}\|P_i\|, 1))}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)} \frac{|\lambda|^{4m-2} + \dots + |\lambda|^2 + 1}{|\lambda|^{2m-1}}.$$

Hence, the following cases holds

Case 1 : If $\|P_{2m}\| \geq \|P_0\|$

$$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{(|\lambda|\|P_{2m}\| + 2m \; max(max_{i=0:2m-1}\|P_i\|, 1))}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)} \frac{|\lambda|^{4m-2} + \dots + |\lambda|^2 + 1}{|\lambda|^{2m-1}}$$

Case 2 : $\|P_{2m}\| \leq \|P_0\|$

$$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \frac{(|\lambda|\|P_0\| + 2m \; max(max_{i=0:2m-1}\|P_i\|, 1))}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)} \frac{|\lambda^{2m-1}|^2 + \dots + |\lambda|^2 + 1}{|\lambda|^{2m-1}}.$$

□

Theorem 4.4. *Let λ be a nonzero, finite and simple eigenvalue of $\mathbb{P}(\lambda)$ and consider the second symmetric linearization $\hat{\mathbb{C}}_2(\lambda)$. Then,*

$$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \leq \begin{cases} \dfrac{(|\lambda| \; 2m \; max(max_{i=0:2m-1}\|P_i\|, 1) + \|P_{2m}\|)(|\lambda^{4m-2}| + \dots + 1)}{|\lambda|^{2m}\|P_{2m}\| + \dots + |\lambda|\|P_1\| + \|P_0\|} & if \|P_{2m}\| \geq \|P_0\| \\ \dfrac{(|\lambda| \; 2m \; max(max_{i=0:2m-1}\|P_i\|, 1) + \|P_0\|)(|\lambda|^{4m-2} + \dots + 1)}{|\lambda|^{2m}\|P_{2m}\| + \dots + |\lambda|\|P_1\| + \|P_0\|} & if \|P_{2m}\| \leq \|P_0\|. \end{cases}$$

Proof. The one-sided transformation of $\hat{\mathbb{C}}_2(\lambda)$ is obtained by

$$\mathbb{L}_2(\lambda)\mathbb{H}(\lambda) = e_{2m} \otimes \mathbb{P}(\lambda)$$

$$\mathbb{G}(\lambda)\hat{\mathbb{C}}_2(\lambda) = e_{2m}^T \otimes \mathbb{P}(\lambda)$$

where,

$$\mathbb{H}(\lambda) = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}^T$$

and

$$\mathbb{G}(\lambda) = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}.$$

The right and left eigenvectors of $\hat{\mathbb{C}}_2(\lambda)$ takes the following forms

$$z = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}^T x$$

and

$$w = \begin{bmatrix} \lambda^{2m-1} I_n & \lambda^{2m-2} I_n & \dots & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix}^T y.$$

Again, we have

$$w^* \hat{\mathbb{C}}_2'(\lambda) z = y^* \mathbb{P}'(\lambda) x$$

The derivation of the above result is same as in theorem 4.1. For the condition number of $\hat{\mathbb{C}}_2(\lambda)$ relative to that of $\mathbb{P}(\lambda)$, it follows that

$$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} = \frac{(|\lambda|\|X\| + \|Y\|)\|w\|\|z\|}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)\|y\|\|x\|}.$$

From Lemma 3.5 in [16], we have

$$\|X\| \le 2m\ max(max_{i=0:2m-1}\|P_i\|, 1), \|Y\| = max(\|P_{2m}\|, \|P_0\|),$$

and

$$\frac{\|w\|}{\|y\|} = \frac{\|z\|}{\|x\|} = \sqrt{|\lambda|^{4m-2} + |\lambda|^{4m-4}| + \dots + |\lambda|^2 + 1}.$$

Then we can bound the ratio $\dfrac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$ as follows

$$\frac{\kappa_{\mathbb{L}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \le \frac{(|\lambda|\ 2m\ max(max_{i=0:2m-1}\|P_i\|, 1) +\ max(\|P_{2m}\|, \|P_0\|))(|\lambda^{2m-1}|^2 + \dots + |\lambda|^2 + 1)}{(|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|)}.$$

Hence, the following cases holds

Case 1 : If $\|P_{2m}\| \ge \|P_0\|$

$$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \le \frac{(|\lambda|\ 2m\ max(max_{i=0:2m-1}\|P_i\|, 1) + \|P_{2m}\|)(|\lambda^{2m-1}|^2 + \dots + |\lambda|^2 + 1)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|}$$

Case 2 : $\|P_{2m}\| \le \|P_0\|$

$$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)} \le \frac{(|\lambda|\ 2m\ max(max_{i=0:2m-1}\|P_i\|, 1) + \|P_0\|)(|\lambda^{2m-1}|^2 + \dots + |\lambda|^2 + 1)}{|\lambda|^{2m}\|P_{2m}\| + |\lambda|^{2m-1}\|P_{2m-1}\| + \dots + |\lambda|\|P_1\| + \|P_0\|}.$$

□

4.1. **Numerical experiments.** We present the numerical example to compare the upper bounds of condition number ratios of linearization classes with respect to original even grade PEP and induced QEP. All the computations are performed in the environment of MATLAB $R2019a$ with windows 11 operating system, Intel (R) i3-1011oU CPU, 2.10 GHZ processor. We consider the same example as taken in equation (3.22) to find the location of the eigenvalues. Numerical experiments are performed for $\mathbb{C}_1(\lambda)$, $\mathbb{C}_2(\lambda)$, $\hat{\mathbb{C}}_1(\lambda)$ and $\hat{\mathbb{C}}_2(\lambda)$ by developing matlab code. Since the results are same for $\mathbb{C}_1(\lambda)$ and $\mathbb{C}_2(\lambda)$, we omit it for $\mathbb{C}_2(\lambda)$.

From Table 2 it is clear that the upper bounds of condition number ratios of linearization classes such as $\mathbb{C}_1(\lambda)$, $\hat{\mathbb{C}}_1(\lambda)$ and $\hat{\mathbb{C}}_2(\lambda)$ with respect to QEP are good and tight enough relative to the original PEP.

5. Conclusion

We provided a framework of the quadratification techniques of even grade PEP which converts the problem into a QEP of high dimension. We provided eigenvalue location of the regular QEP by constructing Block Gersgorin sets from few linearization classes of QEP. Upper bounds of condition number bounding ratio of $\mathbb{P}(\lambda)$ with respect to that of linearization classes are also derived. It is to be noted that after quadratification, an even grade PEP may becomes a singular QEP if $\det(P_{2m}) = \det(P_0) = 0$. Again, computation of eigenvalues of singular PEP is a complex task due to the fact that small perturbations to the coefficients matrices in $\mathbb{P}(\lambda)$ may have arbitrarily bad effects on the accuracy of eigenvalues. Therefore, further research works on singular QEP are required to enrich the PEP of grade k.

Acknowledgement. The authors are thankful to anonymous reviewer for providing valuable feedback to improve the paper. The authors also thank Dibrugarh University, Assam, India for providing all facility during the research works.

Conflict of interest. The authors declare that there is no conflict of interest in this research work.

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Table 1. Eigenvalues,their absolute values and condition numbers

Sl.No.	λ	$\lvert\lambda\rvert$	$K_{\mathbb{Q}}(\lambda)$	$K_{\mathbb{L}}(\lambda)$
1	2.7843 + 0.0000i	2.7843	21.6204	10.6684
2	1.3716 + 0.0000i	1.3716	13.6795	162.0287
3	0.9710 + 0.6761i	1.1832	8.7680	193.9411
4	0.9710 - 0.6761i	1.1832	8.7680	193.9411
5	0.9243 + 0.0000i	0.9243	5.5931	162.0287
6	0.8765 + 0.4291i	0.9759	5.7835	878.0135
7	0.8765 - 0.4291i	0.9759	5.7835	878.0135
8	-0.4184 + 1.3612i	1.4240	13.1287	84.8810
9	-0.4184 - 1.3612i	1.4240	13.1287	84.8810
10	0.5285 + 0.8007i	0.9594	9.4996	216.9904
11	0.5285 - 0.8007i	0.9594	9.4996	216.9904
12	0.0346 + 1.1581i	1.1587	11.3768	145.4600
13	0.0346 - 1.1581i	1.1587	11.3768	145.4600
14	0.2992 + 0.8116i	0.8650	9.6492	111.0292
15	0.2992- 0.8116i	0.8650	9.6492	111.0292
16	0.1116 + 0.9327i	0.9394	8.6147	421.3473
17	0.1116 - 0.9327i	0.9394	8.6147	421.3473
18	-0.2350 + 1.0214i	1.0481	8.7910	296.9963
19	-0.2350 - 1.0214i	1.0481	8.7910	296.9963
20	-1.3368 + 0.0000i	1.3368	15.1479	40.8037
21	-0.6492 + 0.8865i	1.0989	12.7835	105.4639
22	-0.6492- 0.8865i	1.0989	12.7835	105.4639
23	-1.0795 + 0.4312i	1.1625	10.5871	48.3007
24	-1.0795 - 0.4312i	1.1625	10.5871	48.3007
25	-0.6499 + 0.6918i	0.9493	10.9849	274.0706
26	-0.6499 - 0.6918i	0.9493	10.9849	274.0706
27	-0.9958 + 0.0000i	0.9958	12.8569	482.8105
28	-0.7577 + 0.3632i	0.8403	9.0121	103.6983
29	-0.7577 - 0.3632i	0.8403	9.0121	103.6983
30	-0.0665 + 0.0000i	0.0665	1.9305	471.6395

TABLE 2. Absolute eigenvalues and upper bounds of condition number ratios for $\mathbb{C}_1(\lambda)$, $\hat{\mathbb{C}}_1(\lambda)$ and $\hat{\mathbb{C}}_2(\lambda)$

$\lvert\lambda\rvert$	$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{Q}}(\lambda)}$	$\frac{\kappa_{\mathbb{C}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$	$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{Q}}(\lambda)}$	$\frac{\kappa_{\hat{\mathbb{C}}_1}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$	$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{Q}}(\lambda)}$	$\frac{\kappa_{\hat{\mathbb{C}}_2}(\lambda)}{\kappa_{\mathbb{P}}(\lambda)}$
2.7843	3.1869	37.1291	1.4232	5.5685	0.6958	1.3711e+05
1.3716	5.8178	151.5538	1.9877	8.2125	1.0894	186.8977
1.1832	6.6130	215.4774	2.2100	9.6056	1.1748	50.7249
1.1832	6.6130	215.4774	2.2100	9.6056	1.1748	50.7249
0.9243	31.5310	336.5423	2.7283	24.7247	1.3127	11.3503
0.9759	31.5733	388.0591	2.5954	17.9661	1.2832	14.1144
0.9759	31.5733	388.0591	2.5954	17.9661	1.2832	14.1144
1.4240	5.6300	139.7439	1.9395	8.0120	1.0676	264.1500
1.4240	5.6300	139.7439	1.9395	8.0120	1.0676	264.1500
0.9594	31.5558	369.5317	2.6358	19.7216	1.2925	13.0907
0.9594	31.5558	369.5317	2.6358	19.7216	1.2925	13.0907
1.1587	6.7328	228.2349	2.2461	9.9651	1.1868	42.7770
1.1587	6.7328	228.2349	2.2461	9.9651	1.1868	42.7770
0.8650	31.5311	296.7396	2.9074	39.4600	1.3480	9.4049
0.8650	31.5311	296.7396	2.9074	39.4600	1.3480	9.4049
0.9394	31.5395	349.7492	2.6873	22.3265	1.3040	12.0339
0.9394	31.5395	349.7492	2.6873	22.3265	1.3040	12.0339
1.0481	7.3297	315.9797	2.4379	13.0640	1.2436	20.7999
1.0481	7.3297	315.9797	2.4379	13.0640	1.2436	20.7999
1.3368	5.9498	160.4163	2.0226	8.3704	1.1043	147.7494
1.0989	7.0435	267.8624	2.3435	11.2441	1.2170	28.5761
1.0989	7.0435	267.8624	2.3435	11.2441	1.2170	28.5761
1.1625	6.7140	226.1537	2.2404	9.9046	1.1849	43.9136
1.1625	6.7140	226.1537	2.2404	9.9046	1.1849	43.9136
0.9493	31.5469	359.1622	2.6615	20.9687	1.2983	12.5319
0.9493	31.5469	359.1622	2.6615	20.9687	1.2983	12.5319
0.9958	31.5991	413.4669	2.5488	16.2209	1.2720	15.5690
0.8403	31.5484	284.6049	2.9921	49.4626	1.3632	8.8481
0.8403	31.5484	284.6049	2.9921	49.4626	1.3632	8.8481
0.0665	44.7770	233.6004	64.0168	6.2275e+11	1.8497	1.9782