Gulf Journal of Mathematics
Vol 7, Issue 3 (2019) 26-33

MAXIMAL NON-DECREASING SUBRING OF ITS QUOTIENT FIELD

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Abstract. An integral domain R is called maximal non-going down subring of its quotient field, if R is not going down, and every proper overring of R is going down. We do prove that R is a maximal non-going down subring of its quotient field if and only if R is a quasi-local domain with maximal ideal m and its integral closure $\overline{R}$ is a semi-local Prüfer domain with two maximal ideals M, N such that $M \cap N = m$, the extension $R \subset \overline{R}$ is not going down, and R is the unique quasi-local subring of $\overline{R}$

1. Introduction and preliminaries

All rings considered are assumed to be commutative (integral) domains with identity. Throughout, R denotes a domain with quotient field $qf(R)$ and integral closure $\overline{R}$. As usual, $\dim R$ denotes the (Krull) dimension of R, and by an *overring* (resp., a *proper* overring) of R, we mean a ring T such that $R \subseteq T \subseteq qf(R)$ (resp., $R \subset T \subseteq qf(R)$).

For various ring-theoretic properties P, there have been many studies of the domains all of whose overrings satisfy P [2, 3, 4, 7]. Far fewer studies have considered domains R such that R is maximal with respect to having quotient field $qf(R)$ and not satisfying P. Some recent studies of the latter kind are [1, 3, 5, 6, 8, 9, 22], which considered the property P of being treed, Jaffard, valuation, Prüfer, Noetherian, PID and ACCP subrings. Our paper is primary concerned with the property P of being going down domain. Let us recall some basic definitions and facts:

Recall from [14, 15, 24, 2] that an extension of integral domains $R \subset T$ is *going down* (or satisfies the *going down)* if, for every two prime ideals $p \subset q$ of R and Q a prime ideal of T lying over q, there exists a prime ideal P of T such that $P \subset Q$ and $P \cap R = p$. The ring R is called *going down* if $R \subset T$ is going down for every overring T of R. It is well known that, if R is going down, then R is treed in the sense that incomparable prime ideals of R are coprime [15]. But the converse is false ([23], Example 4.4). Prüfer domains, valuations domains and integral domains with Krull dimension 1 are going down domains. Other going down

Date: Received: Mar 20, 2019; Accepted: Jul 31, 2019.

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2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.

Key words and phrases. Going down domain, treed domain, pseudo-valuation domain, valuation domain, minimal overring.

domains that have attracted considerable interest are pseudo-valuation domains or locally pseudo-valuation domains.

Following Hedstrom and Houston [20], [21], R is called a *pseudo-valuaion domain* (in short, a PVD) if each prime ideal P of R is strongly prime; that is if $xy \in P$, with $x \in qf(R)$ and $y \in qf(R)$, then either $x \in P$ or $y \in P$. Recall from ([21], Proposition 1.2) that, R is a PVD if and only if there is a (uniquely determined) valuation overring V of R such that $Spec(R) = Spec(V)$ as sets. In this case, V is called the *canonically associated valuation overring* of R.

In order to globalize the PVD concept, Dobbs and Fontana introduced the following definition [12]: R is naturally defined to be a *locally pseudo-valuation domain*, (in short, an $LPVD$) in case R_M is a pseudo-valuation domain for each maximal ideal M of R. The class of LPVD domains clearly contains Prüfer domains, all PVD's domains and an abundance of other semi-local domains arising from pullback constructions (see for instance ([12], Example 2.5). For convenience, if R is a LPVD and M a maximal ideal of R, we denote by $V(R_M)$ the valuation domain that is canonically associated to the pseudo-valuation domain R_M.

In ([4], Proposition 2.2), the authors where interested in integrally closed domain R such that each of its overring is going-down (equivalently, each of its overring is treed). Recently, we have investigated conditions under which an integral domain R is not treed, but each of its proper overring is treed [1]. Such a ring R is called *maximal non-treed subring of* $qf(R)$. In the other way, D. E. Dobbs has built a non-integrally closed integral domain R such that R is treed and each proper overring of R is going down ([14], Example 2). It is worth noting that, in this example R were not going down. Our study is motivated by this example that has encouraged us to introduce the following definition: R is said to be *a maximal non-going down subring* of a field K if R is not going down while T is going down for each ring T such that $R \subset T \subseteq K$. We find that K is, in fact, the quotient field of R [Proposition 2.3], and that R must be quasi-local and not integrally closed [Proposition 2.2]. Then, we provide our main characterization [Theorem 3.1]: (R, m) is a maximal non-going subring of $qf(R)$ if and only if $R \subset \overline{R}$ does not satisfy the going down, the integral closure $\overline{R}$ of R is a semi-local Prüfer domain with two maximal ideals M, N such that $M \cap N = m$, and R is the unique quasi-local subring of $\overline{R}$. Another significant result is Theorem 3.2 which illustrates that, (R, m) is a maximal non-going down subring of $qf(R)$ if and only if R is not going down domain and has a non quasi-local minimal overring.

We will frequently use pullback constructions in this matter. Let I be a nonzero ideal of an integral domain S, $\varphi : S \to S/I$ the canonical epimorphism, and D a subring of S/I. Then $R = \varphi^{-1}(D)$ is a subring of S called the *pullback ring* $R := (S, I, D)$. The rings R and S share the ideal I, and there is a bijective correspondence (preserving inclusion) between the set of prime ideals of R which do not contain I and the set of prime ideals of S which do not contain I [10, 17].

Any unexplained material is standard, typically as in [18].

2. PRELIMINARY RESULTS

Proposition 2.1. *Let (R, m) be a quasi-local domain. If S is a treed domain and integral over R, then the following conditions are equivalent:*
(i) The extension $R \subset S$ does not satisfy the going down;
(ii) There is a maximal ideal M of S, and a prime ideal $p \subset m$ of R such that there is no prime ideal of S contained in M and lying over p.

Proof. The implication $(ii) \Longrightarrow (i)$ is obvious. Let us prove $(i) \Longrightarrow (ii)$. If $R \subset S$ does not satisfy the going down, there are two prime ideal $p \subset q$ of R and a prime ideal Q of S lying over q such that there is no prime ideal of S contained in Q and lying over p. Let M be a maximal ideal of S containing Q. If P is a prime ideal of S lying over p, then P is not contained in M, for if $P \subset M$, then P would be comparable to Q since S is treed. As $P \cap R = p \subset Q \cap R = q$, we necessarily have $P \subset Q$. But this contradicts our earler statement. □

Proposition 2.2. *If R is a maximal non-going down subring of a field K, then*
(i) R is quasi-local;
(ii) R is non-integrally closed in K; and
(iii) $R \subset \overline{R}$ does not satisfy the going down.

Proof. (i) Since R is not going down, there is an overring T of R such that $R \subset T$ does not satisfy the going down. Then there exist two prime ideals $p \subset q$ of R and a prime ideal Q of T lying over q such that there is no prime ideal of T contained in Q and lying over p. By localization of R at q, it is easy to see that $pR_q \subset qR_q$ are prime ideals of R_q and QT_q is a prime ideal of T_q lying over qR_q such that there is no prime ideal of T_q contained in QT_q and lying over pR_p. Therefore, $R_q \subset T_q$ does not satisfy the going down, and R_q is not a going down domain. But, by assumption, every ring between R and K (except R) is going down, we necessarily have $R = R_q$, and R is local.

(ii) If R is integrally closed, then every proper overring of R is going down, so R is a PVD ([2], Theorem 10), a contradiction since R is not going down.

(iv) Suppose, by way of contradiction, that $R \subset \overline{R}$ satisfies the going down. Let T be a proper overring of R. Then T is going down. Moreover, if $\overline{T}$ is the integral closure of T, then $\overline{R} \subset \overline{T}$ satisfies the going down. But the going down property is transitive, it follows that $R \subset \overline{T}$ satisfies the going down. Since, in addition $T \subset \overline{T}$ satisfies the lying over property, then $R \subset T$ is going down. As T was arbitrary, then R is going down, a contradiction. □

Proposition 2.3. *If (R, m) is a maximal non-going down subring of a field K, then K is the quotient field of R.*

Proof. We have the inclusion $qf(R) \subseteq K$. If K is not algebraic over $qf(R)$, there is an element $z \in K$ that is transcendental over $qf(R)$. But, as R is not a field, $R[z]$ is not going down and lies properly between R and K, a contradiction. Thus, $qf(R) \subseteq K$ is an algebraic extension. Assume now that $qf(R) \neq K$ and choose an element $s \in K \backslash qf(R)$. If $f(X)$ is the minimal polynomial of s over $qf(R)$, then $f(X)$ has degree $n > 1$. There is a nonzero element $r \in R$ such that $rf(X) \in R[X]$. Then $u = rs$ is integral over R and the minimal polynomial $r^n f(X/r)$ for u over $qf(R)$ is in $R[X]$. Consequently, $R[u]$ is a free $R-$module with free $R-$module basis $\{1, u, ..., u^{n-1}\}$. Let S be the integral closure of R in K. Since $R \subset \overline{R}$ does not satisfy the going down and $\overline{R} \subset S$ is an integral extension, then $R \subset S$ does not satisfy the going down. By virtue of Proposition 1, there is a maximal ideal M of S, and a prime ideal $p \subset m$ of R such that there is no prime ideal of S contained in M and lying over p. Let v be a nonzero element of p, and consider the subring $T := R + vuR + vu^2R + \cdots + vu^{n-1}R$ of K. Then $R \subset T$ since $vu \in T \backslash R$ (otherwise, we get $\frac{vu}{v} = u \in qf(R)$, absurd). Therefore, T is by assumption going down. On the other hand, we have $T \subset S$. Indeed, it is clear that $T \subseteq S$ since $u \in S$. In fact, $u \notin T$, for if $u \in T$, then $u = r_0 + vur_1 + vu^2r_2 + \cdots + vu^{n-1}r_{n-1}$ for some $r_o, r_1, r_2, \ldots, r_{n-1} \in R$. Since $\{1, u, ..., u^{n-1}\}$ is a free $R-$module basis of $R[u]$, we get $1 = vr_1 \in p$, a contradiction. Let $P := p + vuR + vu^2R + \cdots + vu^{n-1}R$. By considering the map $\varphi : T \to R/p$ that assigns every element $a_0 + vua_1 + \cdots + vu^{n-1}a_{n-1}$ to the coset $a_0 + p$ in R/p, and using the fact that $v \in p$, we find that φ is an epimorphism with $Ker(\varphi) = P$. Because of $T/P \cong R/p$, we deduce that P is a prime ideal of T. Likewise, we can show that $M' := m + vuR + vu^2R + \cdots + vu^{n-1}R$ is a maximal ideal of T. Note that $M' \subseteq M$ since $v \in m \subseteq M$. Hence, $M \cap T = M'$. Now, we have $p \subseteq P \cap R$. If $x \in P \cap R$, then $x = b_0 + vub_1 + \cdots + vu^{n-1}b_{n-1}$ for some $b_o \in p$ and $b_1, b_2, \ldots, b_{n-1} \in R$. Once again, since $\{1, u, ..., u^{n-1}\}$ is a free $R-$module basis of $R[u]$, we get $x = b_0 \in p$. Thus, $P \cap R = p$. As $T \subset S$ satisfies the going down, there is a prime ideal Q of S such that $Q \subset M$ and $Q \cap T = P$. But this implies $Q \cap R = Q \cap T \cap R = P \cap R = p$, the desired contradiction. □

Proposition 2.3 states that, if K is not the quotient field of R, then R is not a maximal non-going down subring of K. It shows, in particular, that the study of a maximal non-going down subring of its *quotient field* is justifiable.

3. Main results

For the following statements, we need the unibranched concept (in the sense of [11]). Let $R \subset T$ be an extension of rings. We say that a prime ideal p of R is *unibranched* in T if there is a unique prime ideal q of T such that $q \cap R = p$. We say that R is *unibranched* in T if, every prime ideal of R is unibranched in T.

Theorem 3.1. (R, m) *is a maximal non-going down subring of* $qf(R)$, *if and only if*

(i) $R \subset \overline{R}$ *does not satify the going down;*

(ii) $\overline{R}$ *is a semi-local Prüfer domain with two maximal ideals* M, N *such that* $M \cap N = m$, *and*

(iii) R *is the unique non-quasi-local subring of* $\overline{R}$.

Proof. Assume that (R, m) is a maximal non-going down subring of $qf(R)$. According to Proposition 2.2, $R \subset \overline{R}$ does not satisfy the going down. In view of Proposition 2.1, there is a maximal ideal M of $\overline{R}$, and a prime ideal $p \subset m$ of R such that there is no prime ideal of $\overline{R}$ contained in M and lying over p. By going up, there is a chain of prime ideals $P \subset N$ of $\overline{R}$ such that $P \cap R = p$ and $N \cap R = m$. We necessarily have $P \nsubseteq M$ and N is a maximal ideal of $\overline{R}$ different from M. Set $I := M \cap N$, and consider the pullback ring $T = R + I$. Then T is an intermediate ring between R and $\overline{R}$. Set $P_1 := P \cap T$, then $P_1 \subset I$ is a chain of prime ideals of T and $M \cap T = I$. If there is a prime ideal Q of $\overline{R}$ contained in M and lying over P_1, then $Q \cap R = Q \cap T \cap R = P_1 \cap R = p$, a contradiction since there is no prime ideal of $\overline{R}$ contained in M and lying over p. This proves that T is not going down. Hence, $T = R$, and R is the pullback ring $(\overline{R}, m, R/m)$. Moreover, $I = m = M \cap N$, and thus $\overline{R}$ has only two maximal ideals M and N. Since every overring of $\overline{R}$ is going down, then $\overline{R}$ is a LPVD ([2], Theorem 10). Let $V(\overline{R}_M)$ (resp., $V(\overline{R}_N)$) be the valuation overring canonically associated to the PVD $\overline{R}_M$ (resp., $\overline{R}_N$). By virtue of ([21], Proposition 1.2), we have $Spec(\overline{R}_M) = Spec(V(\overline{R}_M))$ and $Spec(\overline{R}_N) = Spec(V(\overline{R}_N))$. We shall prove that $\overline{R}$ is a Prüfer domain. Suppose, by way of contradiction, that $\overline{R}$ is not a Prüfer domain. According to ([18], Theorem 26.2), there is an overring W of $\overline{R}$ and two prime ideals $P' \subset Q'$ of W such that $P' \cap \overline{R} = Q' \cap \overline{R} = P_o$. In light of ([18], Corollary 19.7), we may assume that W is a valuation overring of $\overline{R}$ with maximal ideal Q'. Four cases may happen:

Case 1: $P_o \subset M$. Then $\overline{R}_M \subset \overline{R}_{P_o} \subseteq W$. As $\overline{R}_M$ is a PVD, then $\overline{R}_{P_o}$ is comparable to $V(\overline{R}_M)$ under containment ([16], Proposition 1.3). But the inclusion $\overline{R}_{Po} \subseteq V(\overline{R}_M)$ leads to the contradiction $M = (M\overline{R}_M \cap \overline{R}_{P_o}) \cap \overline{R} \subseteq P_o\overline{R}_{P_o} \cap \overline{R} = P_o$. It follows that $V(\overline{R}_M) \subset \overline{R}_{P_o} \subseteq W$, and $\overline{R}_{P_o}$ is a valuation domain. Thus, $P' \subset Q' \subseteq P_o\overline{R}_{P_o}$, a contradicton since $P_o\overline{R}_{P_o} \subseteq P'W_{P'} = P'$.

Case 2: $P_o \subset N$. This case can be treated similarly as Case 1.

Case 3: $P_o = M$. Then $M\overline{R}_M \subseteq P' \subset Q'$ and $W \subset V(\overline{R}_M)$. We claim that W and $V(\overline{R}_N)$ are incomprable. Indeed, if $W \subseteq V(\overline{R}_N)$, then $N\overline{R}_N \subseteq Q'$ and this implies that $M\overline{R}_M$ and $N\overline{R}_N$ are comparable, while if $V(\overline{R}_N) \subseteq W$, then $M\overline{R}_M \subset Q' \subseteq N\overline{R}_N$. In both cases, we find a contradiction. Now, let $S := W \cap V(\overline{R}_N)$. Then S is a semi-local Prüfer overring of $\overline{R}$ with two maximal ideals $Q' \cap S$ and $N\overline{R}_N \cap S$. Moreover, $P' \cap S$ and $P\overline{R}_N \cap S$ are two incomparable prime ideals of S such that $P' \cap S \subset Q' \cap S$ and $P\overline{R}_N \cap S \subset N\overline{R}_N \cap S$. Let $J := (Q' \cap S) \cap (N\overline{R}_N \cap S)$ and $H := R + J$. Then (H, J) is a quasi-local domain lying between R and S. Since $P' \cap H$ and $P\overline{R}_N \cap H$ are two incomparable prime ideals of H contained properly in J, then H is not treed, so H is not going down. But, by assumption, this happens only when $H = R$. It follows that $m = J = (Q' \cap S) \cap (N\overline{R}_N \cap S)$. However, we have shown that $S \subset W \subset V(\overline{R}_M)$, then $m \subseteq M\overline{R}_M \cap S$. As $N\overline{R}_N \cap S \nsubseteq M\overline{R}_M \cap S$, then $Q' \cap S \subseteq M\overline{R}_M \cap S$, so $W = S_{Q' \cap S} \supseteq V(\overline{R}_M) = S_{M\overline{R}_M \cap S}$, a contradiction since $W \subset V(\overline{R}_M)$.

Case 4: $P_o = N$. Then $N\overline{R}_N \subseteq P' \subset Q'$ and $W \subset V(\overline{R}_N)$. By using a similar argument to Case 3, we can show that W and $V(\overline{R}_M)$ are incomprable. Set $S_1 := W \cap V(\overline{R}_M)$, then S_1 is a semi-local Prüfer domain with two maximal ideals $Q' \cap S_1$ and $M\overline{R}_M \cap S_1$. Let $J_1 := (Q' \cap S_1) \cap (M\overline{R}_M \cap S_1)$ and $H_1 := R + J_1$. Then (H_1, J_1) is a quasi-local domain lying between R and S_1. Furthermore, $P\overline{R}_N \cap H_1 \subset N\overline{R}_N \cap H_1 \subset Q' \cap H_1 = J_1$ is a chain of prime ideals of H_1

and $M\overline{R}_M \cap H_1 = (M\overline{R}_M \cap S_1) \cap H_1 = J_1$. If $H_1 \subset V(\overline{R}_M)$ satifies the going down, then there is a prime ideal Q'' of $V(\overline{R}_M)$ such that $Q'' \subset M\overline{R}_M$ and $Q'' \cap H_1 = P\overline{R}_N \cap H_1$. Therefore, by considering the prime ideal $P'' = Q'' \cap \overline{R}$ of $\overline{R}$, we find that $P'' \subset M$ and $P'' \cap R = (Q'' \cap H_1) \cap \overline{R} \cap R = P \cap R = p$, an another contradiction. Thus, H_1 is not going down. Afortiori, we have $H_1 = R$ and $m = J_1 := (Q' \cap S_1) \cap (M\overline{R}_M \cap S_1)$. Once again, from the inclusions $S_1 \subset W \subset V(\overline{R}_N)$, we get $m \subseteq N\overline{R}_N \cap S_1$. As $M\overline{R}_N \cap S_1 \nsubseteq N\overline{R}_N \cap S_1$, then $Q' \cap S_1 \subseteq N\overline{R}_N \cap S_1$, so $W = (S_1)_{Q' \cap S} \supseteq V(\overline{R}_N) = (S_1)_{N\overline{R}_N \cap S_1}$, a contradiction since $W \subset V(\overline{R}_N)$.

It remains to show that R is the unique non-quasi-local subring of $\overline{R}$. We have already seen that R is the pullback ring $R := (\overline{R}, m, R/m)$. Let $\varphi : \overline{R} \to \overline{R}/m$ be the canonical epimorphism and let R_1 be a quasi-local subring of $\overline{R}$. Then m is the maximal ideal of R_1, and $p_1 := P \cap R_1 \subset m$. If $R_1 \subset \overline{R}$ satisfies the going down, then there exists a prime ideal $\overline{P} \subset M$ such that $\overline{P} \cap R_1 = p_1$. But this proves the existence of a prime ideal $\overline{P}$ of $\overline{R}$ such that $\overline{P} \subset M$ and $\overline{P} \cap R = \overline{P} \cap R_1 \cap R = p_1 \cap R = P \cap R = p$, and this is absurd. Hence, R_1 is not going down, and this happens only when $R_1 = R$.

Conversely, assume that (i), (ii) and (iii) are satisfied. Then R is not going down. We will show that each proper overring T of R is going down. As $\overline{R} \subseteq \overline{T}$, then $\overline{T}$ is a Prüfer overring of $\overline{R}$ with at most two maximal ideals. If $\overline{T}$ is local, then $\overline{T}$ is a valuation ring, and so T is going down ([13], Corollary 2.5). Suppose that $\overline{T}$ is not local, and denote by Q_1 and Q_2 its maximal ideals. Set $P_1 := Q_1 \cap \overline{R}$ and $P_2 := Q_2 \cap \overline{R}$. Then $\overline{R}_{P_1} = \overline{T}_{Q_1}$ and $\overline{R}_{P_2} = \overline{T}_{Q_2}$, moreover, P_1 and P_2 are not both contained in M (otherwise, we get $\overline{R}_M \subseteq \overline{R}_{P_1} \cap \overline{R}_{P_2} = \overline{T}_{Q_1} \cap \overline{T}_{Q_2} = \overline{T}$, so $\overline{T}$ would be a valuation domain). Likewise, P_1 and P_2 are not both contained in N. Therefore, two cases may happen:

Case 1: $P_1 = M$ and $P_2 = N$. Then $\overline{R}_M = \overline{T}_{Q_1}$ and $\overline{R}_N = \overline{T}_{Q_2}$, so $\overline{R} = \overline{T}$ and T is an intermediate ring between R and $\overline{R}$. Therefore, T is integral over R and shares m with it. As $T \neq R$, then T is semi-local with two maximal ideals $M \cap T$ and $N \cap T$. Since every maximal ideal of T is unibranched in the going down domain $\overline{T}$, then T is also going down ([13], Lemma 2.2).

Case 2: $P_1 \subset M$ or $P_2 \subset N$. By symmetry of the problem, we will treat only the case $P_1 \subset M$. Note that $R_{P_1 \cap R} = \overline{R}_{P_1} = \overline{T}_{Q_1}$, so $T_{Q_1 \cap T} = \overline{T}_{Q_1} \nsubseteq \overline{T}_{Q_2}$, whereas $T_{Q_2 \cap T} \subseteq \overline{T}_{Q_2}$. Consequently, $T_{Q_1 \cap T} \neq T_{Q_2 \cap T}$, and thus $Q_1 \cap T$ and $Q_2 \cap T$ are two distinct maximal ideals of T. Again by ([13], Lemma 2.2), as every maximal ideal of T is unibranched in the going down domain $\overline{T}$, then T is also going down. □

Our next characterization of maximal non-going down subrings uses the concept of minimal overring (in the sense of [19]). Recall that R_o is called a *minimal overring* of R if $R \subset R_o$ and $R_o \subseteq T$ for every proper overring T of R.

Theorem 3.2. *R is a maximal non-going down subring of $qf(R)$, if and only if R is not going down and has a non-quasi-local minimal overring.*

Proof. Assume that R is a maximal non-going down subring of $qf(R)$. Then R is quasi-local, say with maximal ideal m [Proposition 2.2] and R is not going down. According to Theorem 3.1, R is the pullback ring $(\overline{R}, m, R/m)$, where $\overline{R}$ is a semi-local Prüfer domain with two maximal ideals M, N such that $M \cap N = m$. Moreover, R is the unique non quasi-local subring of $\overline{R}$. Let $\varphi : \overline{R} \to \overline{R}/m$ be the canonical epimorphism. Then the inclusions

$$R/m \subset R/m \times R/m \subseteq \overline{R}/m = \overline{R}/M \times \overline{R}/N,$$

permit us to consider the intermediate ring $R_o := \varphi^{-1}(R/m \times R/m)$ between R and $\overline{R}$. Clearly, R_o is semi-local with two maximal ideals $M \cap R_o$ and $N \cap R_o$. We shall prove that R_o is a minimal overring of R. Let T be a proper overring of R. Two cases have to be considered:

Case 1: $T \subseteq \overline{R}$. In this case T has exactly two maximal ideals $M_1 = M \cap T$ and $N_1 = N \cap T$ such that $R/m \subseteq T/M_1$ and $R/m \subseteq T/N_1$. From the inclusions

$$R/m \subset R/m \times R/m \subseteq T/m = T/M_1 \times T/N_1,$$

we deduce that $R_o = \varphi^{-1}(R/m \times R/m) \subseteq \varphi^{-1}(T/m) = T$.

Case 2: $T \nsubseteq \overline{R}$. Since $\overline{R}$ is Prüfer, $\overline{R}_M$ and $\overline{R}_M$ are valuation rings. For $t \in T \backslash \overline{R}$, we have $t \notin \overline{R}_M$ or $t \notin \overline{R}_N$. Assume that $t \notin \overline{R}_M$, then $1/t \in M\overline{R}_M$. There are $\alpha \in M$ and $\beta \in \overline{R} \backslash M$ such that $1/t = \alpha/\beta$. Take an element $\lambda \in N \backslash M$, then $1/t = (\lambda\alpha)/(\lambda\beta)$, $\lambda\beta \in \overline{R} \backslash M$ and $\lambda\alpha \in M \cap N = m \subset R$. Thus, $\lambda\alpha t = \lambda\beta \in \overline{R} \backslash R$ (in fact, $\lambda\beta \in R$ yields the contradiction $\lambda\beta \in N \cap R = m \subset M$). Therefore, $R \subset R[\lambda\beta] \subseteq \overline{R}$, and so $R_o \subseteq R[\lambda\beta]$, by case 1. Hence, $R_o \subseteq R[\lambda\beta] \subseteq R[t] \subseteq T$.

Conversely, assume that R is non-going down and has a non-quasi-local minimal overring R_o. From the fact that an integral domain is the intersection of the localizations taken with respect to its set of maximal ideals, it follows that R is necessarily quasi-local. If R is integrally closed, then R is an intersection of valuation domains, so R is a valuation domain, a contradiction since R is not going down Thus, R is not integrally closed and $R \subset R_o \subseteq \overline{R}$. Because R_o is, by assumption, not quasi-local, then R_o has exactly two maximal ideals ([19], Corollary 2.2). Furthermore, $\overline{R}$ is a semilocal Prüfer domain with two maximal ideals ([19], Proposition 2.5). Since R has a minimal overring R_o, we need only prove that every overring of R_o is going down. In fact, as the integral closure $\overline{R}$ of R_o is a Prüfer domain and every maximal ideal of R_o is unibranched in $\overline{R}$, then R_o is unibranched in $\overline{R}$ ([13], Lemma 2.2). Finally, by application of ([26], Corollary 2.13) and ([26], Proposition 2.14), we can conclude that every overring of R_o is going down. □

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