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LORENTZ-TYPE SPACES AND COMPACTNESS OF GENERALIZED DIFFERENCE OPERATORS

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Abstract. In this article, we characterize the sequences u and v that define continuous and compact generalized difference operators acting on the broad class of quasi-Banach Lorentz-type sequence spaces. Our results provide genuine extensions and generalizations of known results about the continuity and compactness for multiplication-type operators defined over traditional Lorentz sequence spaces.

1. Introduction

Compactness is one of the most significant topological properties desired in a subset of a topological space. In normed spaces, this property enables the effective application of mathematics to other areas of knowledge, particularly by facilitating the establishment of existence results of solutions in optimization problems. In the context of continuous linear operators, compactness allows the Bolzano–Weierstrass property to be transferred to the image of a bounded sequence. As a result, it finds further applications in problems modeled by linear transformations defined by infinite matrices acting on Banach sequence spaces. This article focuses on such problems, specifically through the study of generalized difference operators acting on certain weighted Lorentz sequence spaces.

Let X be a vector space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The space X is called a *quasi-normed space* if there exists a function $\|\cdot\| : X \to [0,\infty)$ satisfying the following properties:

(1) $\|\boldsymbol{\xi}\| = 0$ occurs if and only if the vector $\boldsymbol{\xi}$ is the zero vector;
(2) For any scalar $\lambda \in \mathbb{K}$ and any vector $\boldsymbol{\xi} \in X$, it holds that $\|\lambda\boldsymbol{\xi}\| = |\lambda| \cdot \|\boldsymbol{\xi}\|$;
(3) For all vectors $\boldsymbol{\xi}, \boldsymbol{\varsigma} \in X$, the inequality
$$\|\boldsymbol{\xi} + \boldsymbol{\varsigma}\| \leq C(\|\boldsymbol{\xi}\| + \|\boldsymbol{\varsigma}\|) \tag{1.1}$$
holds, where $C \geq 1$ is a constant independent of $\boldsymbol{\xi}$ and $\boldsymbol{\varsigma}$

The pair $(X, \|\cdot\|)$ is then referred to as a quasi-normed space. The smallest constant $C \geq 1$ for which the inequality (1.1) holds is called the *modulus of*

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concavity of the quasi-norm. Clearly, in the particular case $C = 1$, a quasi-norm $\|\cdot\|$ reduces to the usual notion of norm but, in general, a quasi-norm is not a norm. However, an important result by Aoki [2] and Rolewicz [18] establishes that if $(X, \|\cdot\|)$ is a quasi-normed space, then we can find a number $p \in (0, 1]$ and a quasi-norm $\|\cdot\|_p$, equivalent to the original one on X, such that

$$\|\xi + \varsigma\|_p^p \leq \|\xi\|_p^p + \|\varsigma\|_p^p,$$

for all $\xi, \varsigma \in X$ and consequently, X is endowed with a metric given by the relation $d_p(\xi, \varsigma) = \|\xi - \varsigma\|_p^p$. Hence, the study of convergence and topological properties in a quasi-normed space $(X, \|\cdot\|)$ is inherited from the metric space (X, d_p). For instance, $\xi_n \to \xi$ in $(X, \|\cdot\|)$ if and only if $d_p(\xi_n, \xi) \to 0$ which is equivalent to $\|\xi_n - \xi\| \to 0$. In particular, a quasi-normed space $(X, \|\cdot\|)$ is called a quasi-Banach space if the pair (X, d_p) is a complete metric space.

The study of quasi-Banach spaces has a long history, as evidenced by the references of Aoki [2]. A significant contribution is found in the work of Kalton, presented in Chapter 25 of the *Handbook of the Geometry of Banach Spaces*, Vol. 2 [14], where a wide range of problems concerning these spaces is discussed. In particular, Kalton highlights that fundamental results from Banach space theory—such as the Uniform Boundedness Principle, which ensures that pointwise bounded families of operators are uniformly bounded; the Open Mapping Theorem, which guarantees that surjective bounded linear operators between complete spaces are open mappings; and the Closed Graph Theorem—remain applicable to this type of space, as these results rely solely on completeness.

Recently, many researchers have shown interest in various issues related to quasi-Banach spaces. Some have focused on the geometric aspects of interpolated spaces, a line of study that began with Krée's paper [15]. Other researchers have examined the problem of characterizing the compactness of certain subsets in specific quasi-Banach function spaces [5, 11], while others have directed their attention to the characterization of the topological properties of certain classical operators acting on quasi-Banach sequence spaces, including continuity, compactness, and the closedness of their range, among others (see, for example, [1]).

This work focuses on the study of the continuity and compactness of generalized difference operators acting on certain quasi-Banach sequence spaces which will be defined in terms of weighted Lorentz-type sequence spaces in the next section. We recall that for fixed sequences u and v, the generalized difference operator $\Delta_{u,v}$ maps the sequence $\xi = \{\xi(n)\}$ to the sequence $\mathbf{y} = \Delta_{u,v}(\xi)$, where the components $y(n)$ are defined by

$$y(n) = v(n-1)\xi(n-1) + u(n)\xi(n), \tag{1.2}$$

with the convention that $\xi(0) = v(0) = 0$. This operator is a special case of a transformation defined by an infinite matrix, in fact, it can be viewed as the multiplication by an infinite bidiagonal matrix, and its study has received significant attention in classical Banach sequence spaces, as evidenced by the recent work of El-Shabrawy [12]. Moreover, when v is the zero sequence, the operator $\Delta_{u,v}$

becomes the well-known multiplication operator M_u which is a subject of great interest for many researchers (see, for instance, [8, 17] or more recently [7] and the numerous references therein). Additionally, the generalized difference operators are special cases of tridiagonal operators which have been studied recently by Caicedo et al. in [4]).

The main objective of this article is to characterize the continuity and compactness of this operator in the general setting of weighted Lorentz-type sequence spaces. Finally, throughout the article, we denote by $\boldsymbol{\alpha} = \{\alpha(k)\}$ a sequence of real or complex numbers. In contrast, (ξ_n) will be used to represent a sequence whose elements are themselves sequences.

2. Lorentz-type sequence spaces

In this section we introduce the weighted Lorentz-type sequence spaces and we give some properties. First of all, we denote by $\#(A)$ the cardinality of the set $A \subseteq \mathbb{N}$, which is the counting measure of the set A in $\mathbb{N}$, the set of strictly positive integers. With this notation, the *distribution* of the complex sequence $\xi = \{\xi(n)\}$ is the real-valued extended function $\#_\xi : [0, \infty) \to [0, \infty]$ defined by

$$\#_\xi(\lambda) = \#(\{n \in \mathbb{N} : |\xi(n)| > \lambda\}).$$

From this definition, it is easy to see that $\#_\xi$ is a nonincreasing function satisfying the following properties:

Monotonicity. If $|\xi| \leq |\varsigma|$, then $\#_\xi(\lambda) \leq \#_\varsigma(\lambda)$ for all $\lambda > 0$,

Additivity. Given any $\lambda_1, \lambda_2 > 0$, the following inequality holds:

$$\#_{\boldsymbol{\xi}+\boldsymbol{\varsigma}}(\lambda_1 + \lambda_2) \leq \#_{\boldsymbol{\xi}}(\lambda_1) + \#_{\boldsymbol{\varsigma}}(\lambda_2),$$

where $\boldsymbol{\xi} = \{\xi(n)\}$ and $\boldsymbol{\varsigma} = \{\varsigma(n)\}$ are sequences of complex numbers. Moreover, the notation $|\boldsymbol{\xi}| \leq |\boldsymbol{\varsigma}|$ is understood to mean that $|\xi(n)| \leq |\varsigma(n)|$ for all $n \in \mathbb{N}$.

The *nonincreasing rearrangement* of the complex sequence $\xi = \{\xi(m)\}$ is the real-valued extended sequence ξ^*, defined by

$$\xi^*(m) = \inf\{\lambda > 0 : \#_\xi(\lambda) \leq m - 1\},$$

for all $m \in \mathbb{N}$, where we define $\inf(\emptyset) = +\infty$. In fact, for $\lambda > 0$, we define $\xi^*(\lambda) = \xi^*(m)$, where $m \in \mathbb{N}$ satisfies $m - 1 \leq \lambda < m$.

The properties of the nonincreasing rearrangement are similar to those of the distribution function. Observe that ξ^* is a nonincreasing sequence, with

$$\xi^*(1) = \|\xi\|_\infty = \sup_{m\in\mathbb{N}} |\xi(m)|,$$

and that $\xi^* \leq \varsigma^*$ for all complex sequences ξ, ς such that $|\xi| \leq |\varsigma|$. We can think of ξ^* as the sequence $|\xi| = \{|\xi(n)|\}$, but ordered in decreasing order. For instance, if $\mathbf{e}_n$ denotes the n-th canonical sequence defined by $e_n(k) = 0$ for $k \neq n$ and $e_n(n) = 1$, then its nonincreasing rearrangement is $\mathbf{e}_n^* = \mathbf{e}_1$ for all $n \in \mathbb{N}$.

Furthermore, we can observe that if $\xi^*(m) < \infty$, then $\#_\xi(\xi^*(m)) \leq m-1$. Also, for any two complex sequences ξ and ς, we have

$$(\xi+\varsigma)^* (m_1+m_2) \leq \xi^* (m_1) + \varsigma^* (m_2)$$

for all $m_1, m_2 > 0$. We recommend the recent book by Castillo and Chaparro [6] for a more comprehensive study, including detailed proofs related to the distribution function and the nonincreasing rearrangement.

Next, we will consider weighted sequence spaces that include the nonincreasing rearrangement ξ^* of the complex sequence ξ. A real sequence $\omega = \{\omega(m)\}$, where $\omega(m) > 0$ for every $m \in \mathbb{N}$, is referred to as a *weight*. The global Δ_2-condition for a weight $\boldsymbol{\omega}$ holds if there exists a constant $K_\Delta > 0$ such that

$$\omega(2n) \leq K_\Delta \omega(n) \tag{2.1}$$

for all $n \in \mathbb{N}$. Given a weight ω and $q \geq 1$ fixed, we can define the weighted l^q space as the set $l^q(\omega)$ of all complex sequences $\xi = \{\xi(m)\}$ such that

$$\|\xi\|_{l^q(\omega)} = \left(\sum_{m=1}^{\infty} |\xi(m)|^q \, \omega(m)\right)^{\frac{1}{q}} < \infty.$$

These kinds of spaces naturally arise in the literature in the context of studying operator properties in Banach sequence spaces. For instance, for $p > 1$, the Cesàro space ces_q is contained in $l^q(\omega)$ with $\omega(n) = n^{1-q}$, which implies that every evaluation functional on ces_q is continuous.

Next, for a fixed $q \geq 1$ and a weight ω satisfying the global Δ_2 condition, We define the Lorentz-type space $l(\omega, q)$ as the set of all complex sequences $\boldsymbol{\xi}$ such that $\boldsymbol{\xi}^* \in l^q(\omega)$. In other words,

$$\boldsymbol{\xi} \in l(\omega, q) \iff \|\boldsymbol{\xi}\|_{(\omega,q)} := \|\boldsymbol{\xi}^*\|_{l^q(\omega)} = \left(\sum_{m=1}^{\infty} [\xi^*(m)]^q \, \omega(m)\right)^{1/q} < \infty.$$

Clearly, when $\omega(m) = m^{\frac{q}{p}-1}$ with $p > 1$ fixed, we recover the classical Lorentz sequence space $l(p, q)$ (see [16] and [13]). The global Δ_2-condition ensures that the weight sequence ω grows in a controlled manner, which allows for the proper definition and desirable properties of the weighted Lorentz-type sequence spaces. In particular, the global Δ_2 condition on ω implies that $l(\omega, q)$ is a linear space and that $\|\cdot\|_{(\omega,q)}$ is a quasi-norm for this space. In particular, we can find a constant $L > 0$ ensuring that

$$\|\xi+\varsigma\|_{(w,q)} \leq L \left(\|\xi\|_{(w,q)} + \|\varsigma\|_{(w,q)} \right),$$

for all $\xi, \varsigma \in l(\omega, q)$. It is important to note that for any $m \in \mathbb{N}$, the canonical sequence $\mathbf{e}_m \in l(\omega, q)$ satisfies $\|\mathbf{e}_m\|_{(\omega,q)} = \omega(1)^{\frac{1}{q}}$. Additionally, this space is solid in the sense that if $\varsigma \in l(\omega, q)$ and $|\xi(m)| \leq |\varsigma(m)|$ for all $m \in \mathbb{N}$, then

$$\xi \in l(\omega, q) \, \wedge \|\xi\|_{(\omega,q)} \leq \|\varsigma\|_{(\omega,q)} \, . \tag{2.2}$$

In fact, the following result holds:

Theorem 2.1. *Let ω be a weight and suppose that $q \geq 1$. The Lorentz-type space $\left(l(\omega, q), \|\cdot\|_{(\omega,q)}\right)$ is a quasi-Banach space.*

Proof. The proof is standard, and we provide the details. Let (ξ_n) be any Cauchy sequence in $\left(l(\omega, q), \|\cdot\|_{(\omega,q)}\right)$ and let $\varepsilon > 0$ be arbitrary. Since (ξ_n) is a Cauchy sequence, we can find an $N \in \mathbb{N}$ such that

$$[(\xi_n - \xi_m)^* (1)]^q \omega(1) \leq \|\xi_n - \xi_m\|^q_{(\omega,q)} \leq \varepsilon^q \omega(1),$$

for all $n, m \geq N$. Thus,

$$(\xi_n - \xi_m)^* (1) = \inf \{\lambda > 0 : \#_{\xi_n - \xi_m}(\lambda) \leq 0\} < \varepsilon.$$

The definition of infimum implies that there exists $\lambda_0 > 0$ such that $\lambda_0 < \varepsilon$ and $\#_{\xi_n - \xi_m} (\lambda_0) = 0$. Consequently, since $\#_{\xi_n - \xi_m}$ is a decreasing and nonnegative function, we also have $\#_{\xi_n - \xi_m} (\varepsilon) = 0$. In other words,

$$\#_{\xi_n - \xi_m} (\varepsilon) = \# (\{k \in \mathbb{N} : |\xi_n(k) - \xi_m(k)| > \varepsilon\}) = 0,$$

for all $n, m \geq N$. This last relation implies that (ξ_n) is a Cauchy sequence in measure. By Riesz's theorem, we can find a subsequence (ξ_{n_k}) of the sequence (ξ_n) that converges pointwise to some complex sequence $\xi = \{\xi(k)\}$. Furthermore, for a fixed $H \in \mathbb{N}$, the properties of non-increasing rearrangement give us the following estimate:

$$\begin{aligned}
\sum_{j=1}^{H} [(\xi - \xi_N)^* (j)]^q \, \omega(j) &\leq \sum_{j=1}^{H} \left[\left(\liminf_{k\to\infty} \xi_{n_k} - \xi_N\right)^* (j)\right]^q \omega(j) \\
&\leq \liminf_{k\to\infty} \sum_{j=1}^{H} [(\xi_{n_k} - \xi_N)^* (j)]^q \, \omega(j) \\
&\leq \liminf_{k\to\infty} \|\xi_{n_k} - \xi_N\|^q_{(\omega,q)} \leq \varepsilon^q \omega(1)
\end{aligned}$$

and $\xi - \xi_N \in l(\omega, q)$ which implies $\xi \in l(\omega, q)$. Similarly, for any $m \geq N$, we have $\|\xi_m - \xi\| \leq \varepsilon\omega(1)^{\frac{1}{q}}$, which prove that $\left(l(\omega, q), \|\cdot\|_{(\omega,q)}\right)$ is a quasi-Banach space. □

We conclude this section by mentioning that this type of space has been recently considered within the framework of quasi-Banach function spaces by [5], where the problem of compactness is addressed. Additionally, other authors have studied the compactness problem in new quasi-Banach sequence spaces, as seen in [19].

3. Continuous generalized difference operators on $l(\omega, q)$

In this section, we characterize all complex sequences **u** and **v** that define continuous difference operators $\Delta_{u,v}$ on $l(\omega, q)$. We recall that if X is a quasi-Banach space, a linear operator T is said to be continuous on X if a constant $M > 0$ can be found such that

$$\|T(\xi)\|_X \leq M\|\xi\|_X,$$

for all $\xi \in X$. As usual, the set of all continuous operators on X is denoted by $\mathcal{B}(X)$ and it forms a quasi-normed space with the relation

$$\|T\| = \sup_{x \neq 0} \frac{\|T(\xi)\|_X}{\|\xi\|_X},$$

which is referred as the operator quasi-norm.

As mentioned in the introduction, the generalized difference operator $\Delta_{u,v}$ is the transformation that maps an element $\xi = \{\xi(k)\}$ belonging to a sequence space to the sequence $\mathbf{y} = \Delta_{u,v}(\xi)$ defined by

$$y(k) = v(k-1)\xi(k-1) + u(k)\xi(k),$$

where $k \in \mathbb{N}$ and we set $\xi(0) = v(0) = 0$. This operator generalizes the well known multiplication operator M_u, which is obtained when $\mathbf{v} = \mathbf{0}$, the null sequence and it is a particular case of the tridiagonal operator studied by Caicedo et al. in [4]. Furthermore, if R is the right-shift operator defined by

$$R(\xi) = \{0, \xi(1), \xi(2), \xi(3), \cdots\},$$

where $\xi = \{\xi(n)\}$, then the generalized difference operator is given by

$$\Delta_{u,v}(\xi) = RM_v(\xi) + M_u(\xi), \tag{3.1}$$

for all $\xi \in X$.

Additionally, for any $\xi \in l(\omega, q)$, we have $\mathbf{y}^* = \xi^*$, where $\mathbf{y} = R(\xi)$. Thus, we obtain

$$\|R(\xi)\|_{(\omega,q)} = \|\xi\|_{(\omega,q)}.$$

This relation indicates that R is an isomorphism, although not surjective, on $l(\omega, q)$, meaning that R is an embedding. Furthermore, it is known (see, for instance, [8] and the references therein) that M_u is continuous on $l(\omega, q)$ if and only if u is bounded sequence. Therefore, this fact, combined with the equation (3.1), shows that if u and v are bounded complex sequences, then $\Delta_{u,v}$ is a continuous operator on $l(\omega, q)$. The converse is also true, as we will demonstrate in the following result.

Theorem 3.1. *Let $q \geq 1$ be fixed and let ω be a weight satisfying the global Δ_2-condition. Suppose that $\mathbf{u} = \{u(m)\}$ and $\mathbf{v} = \{v(m)\}$ are complex sequences. The continuity of $\Delta_{u,v}$ on $l(\omega, q)$ is equivalent to the boundedness of both $\boldsymbol{u}$ and $\boldsymbol{v}$.*

Proof. We focus on proving the direct implication. We observe that for any $m \in \mathbb{N}$, the canonical sequence $\mathbf{e}_m$ belongs to $l(\omega, q)$ and its image $\mathbf{y}_m = \Delta_{u,v}(\mathbf{e}_m)$ is the sequence defined by $y_m(m) = u(m)$, $y_m(m+1) = v(m)$ and $y_m(k) = 0$ for all $k \notin \{m, m+1\}$. Hence, for $m \geq 2$ and $\lambda > 0$, the distribution function of $\mathbf{y}_m$ is given by:

$$\#_{\mathbf{y}_m}(\lambda) = \begin{cases} 0, & \max\{|v(m)|, |u(m)|\} \leq \lambda, \\ 1, & \min\{|v(m)|, |u(m)|\} \leq \lambda, < \max\{|v(m)|, |u(m)|\} \\ 2, & 0 < \lambda < \min\{|v(m)|, |u(m)|\}. \end{cases}$$

Therefore, its decreasing rearrangement is given by:

$$\mathbf{y}_m^*(k) = \begin{cases} \max\{|v(m)|, |u(m)|\}, & k = 1, \\ \min\{|v(m)|, |u(m)|\}, & k = 2, \\ 0, & k > 2. \end{cases}$$

From this, we deduce that

$$\|\Delta_{u,v}(\mathbf{e}_m)\|_{(\omega,q)}^q = \max\{|v(m)|^q, |u(m)|^q\} w(1) + \min\{|v(m)|^q, |u(m)|^q\} w(2). \tag{3.2}$$

Hence, if $\Delta_{u,v}$ is a continuous operator on $l(\omega,q)$, then we can find a constant $M > 0$ such that

$$\|\Delta_{u,v}(\mathbf{e}_m)\|_{(\omega,q)} \leq M \|\mathbf{e}_m\|_{(\omega,q)},$$

for all $m \in \mathbb{N}$ and in particular,

$$\max\{|v(m)|, |u(m)|\} \leq M,$$

for all $m \in \mathbb{N}$ which prove the result. $\square$

The following statement follows directly from the preceding result and concerns the continuity of generalized difference operators on classical Lorentz sequence spaces, which are Banach spaces.

Corollary 3.2. *Suppose that $p > 1$ and $q \geq 1$, and let $\mathbf{u} = \{u(n)\}$ and $\mathbf{v} = \{v(n)\}$ be complex sequences. The generalized difference operator $\Delta_{u,v}$ is continuous on $l(p,q)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are bounded sequences.*

Note that when $v = 0$, the null sequence, we recover a well-known result concerning the continuity of the multiplication operator acting on the Lorentz sequence spaces (see [3]).

4. Compact generalized difference operators on $l(\omega, q)$

In this section, we characterize all complex sequences $\mathbf{u}$ and $\mathbf{v}$ that define compact difference operators $\Delta_{u,v}$ on $l(\omega,q)$. We recall that if X is a quasi-Banach spaces, a continuous and linear operator $K : X \to X$ is said to be compact if the image of the closed unit ball $B_X = \{\xi \in X : \|\xi\|_X \leq 1\}$ is a relatively compact subset of X. This means that $\overline{K(B_X)}$ is a compact set of the metric space (X, d_p). In particular, an operator $K : X \to X$ is compact precisely when, given any bounded sequence $\{\xi_n\} \subseteq X$, the image sequence $\{K(\xi_n)\}$ has a convergent subsequence in X.

The properties of compact operators on quasi-Banach spaces are similar to those on Banach spaces. To keep the presentation self-contained, we include the proofs of two of these properties, which are used in establishing the main result of this section.

Lemma 4.1. *Let $(X, \|\cdot\|)$ be a quasi-Banach space. If $K : X \to X$ is a continuous operator and* $\dim(K(X)) < \infty$ *(i.e., K has finite rank), then $K : X \to X$ is a compact operator.*

Proof. In this case, it is sufficient to prove that the quasi-norm $\|\cdot\|$ is equivalent to a norm on X. Then, the result follows by Bolzano-Weierstrass theorem. Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be a fixed basis for X, if $\xi = c_1v_1 + c_2v_2 + \cdots + c_nv_n \in X$, then we set

$$\|\xi\|_1 = \sum_{k=1}^{n} |c_k| .$$

That is, $\|\xi\|_1$ is the norm-1 of the vector $\hat{\xi} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$, which is equivalent to the Euclidean norm. In particular, by the properties of the quasi-norm $\|\cdot\|$, for any $\xi \in X$,

$$\|\xi\| = \left\| \sum_{k=1}^{n} c_k v_k \right\| \leq C^{n-1} M \|\xi\|_1 , \tag{4.1}$$

where $C \geq 1$ is the modulus of concavity and $M = \max\{\|v_1\|, \|v_2\|, \cdots, \|v_n\|\}$. Additionally, we can prove that there exists a constant $c > 0$ such that

$$\|\xi\| \geq c \|\xi\|_1 , \tag{4.2}$$

for all $\xi \in X$. Indeed, if (4.2) is false, then there exists a sequence $(\xi_m) \subset X$ such that

$$\|\xi_m\| < \frac{1}{m} \|\xi_m\|_1 ,$$

for all $m \in \mathbb{N}$. Thus the sequence $(\mathbf{y}_m)$ defined by $\mathbf{y}_m = \frac{1}{\|\xi_m\|_1}\xi_m$ satisfies $\|\mathbf{y}_m\| \to 0$ as $m \to \infty$. Furthermore, we observe that $\|\mathbf{y}_m\|_1 = 1$ for all $m \in \mathbb{N}$, this leads us to conclude that the corresponding sequence $(\hat{y}_m) \subset \mathbb{R}^n$ is bounded in the norm-1 and consequently in the Euclidean norm. Therefore, by Bolzano-Weierstrass theorem, we can find a subsequence $\left(\hat{y}_{m_j}\right)$ of $(\hat{y}_m)$ and $\hat{y} \in \mathbb{R}^n$ such that $\lim_{j\to\infty} \left\|\hat{y}_{m_j} - \hat{y}\right\|_1 = 0$. Hence, the corresponding vectors in X satisfy $\lim_{j\to\infty} \left\|\mathbf{y}_{m_j} - \mathbf{y}\right\|_1 = 0$. Additionally, since $\|\cdot\|_1$ is a norm on X, we have

$$\left| \left\|\mathbf{y}_{m_j}\right\|_1 - \|\mathbf{y}\|_1 \right| \leq \left\|\mathbf{y}_{m_j} - \mathbf{y}\right\|_1 ,$$

for all $j \in \mathbb{N}$, which implies that $\|\mathbf{y}\|_1 = 1$ and, in particular, $\mathbf{y} \neq 0$. Furthermore, the inequality (4.1) also allows us to write

$$\left\|\mathbf{y}_{m_j} - \mathbf{y}\right\| \leq C^{n-1} M \left\|\mathbf{y}_{m_j} - \mathbf{y}\right\|_1 ,$$

which implies that $\mathbf{y}_{m_j} \to \mathbf{y} \neq 0$ as $j \to \infty$ in $(X, \|\cdot\|)$, leading to a contradiction with the fact that $\lim_{m\to\infty} \|\mathbf{y}_m\| = 0$. In conclusion, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|\xi\|_1 \leq \|\xi\| \leq c_2 \|\xi\|_1$$

for all $\xi \in X$ and the quasi-norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_1$. $\square$

Additionally, we require the following property:

Lemma 4.2. *If $(X, \|\cdot\|)$ is a quasi-Banach space and if $\{K_m\}$ is a sequence of compact operators on X satisfying $\lim_{m\to\infty} \|K - K_m\| = 0$, then $K : X \to X$ is a compact operator.*

Proof. Let (ξ_n) be any bounded sequence in $(X, \|\cdot\|)$. The boundedness of the sequence implies the existence of a constant $M > 0$ with $\|\xi_n\| \leq M$ for every $n \in \mathbb{N}$. By definition of the quasi-norm of an operator, we then have

$$\|K_m(\xi_n) - K(\xi_n)\| \leq M \|K_m - K\|$$

for each $n \in \mathbb{N}$. Therefore, by the hypothesis, we obtain

$$\lim_{m\to\infty} \left(\sup_{n\in\mathbb{N}} \|K_m(\xi_n) - K(\xi_n)\| \right) = 0.$$

Thus, for any $\varepsilon > 0$, we can find an index $H \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \|K_H(\xi_n) - K(\xi_n)\| < \frac{\varepsilon}{3C^2}, \tag{4.3}$$

where $C \geq 1$ is the modulus of concavity of the quasi-norm.

Next, since $K_H : X \to X$ is a compact operator, it follows that there exists a subsequence $(K_H(\xi_{n_k}))$ of $(K_H(\xi_n))$ that converges in X, and thus this subsequence is also a Cauchy sequence in X. Therefore, we can find some index $N \in \mathbb{N}$ for which

$$\left\| K_H\left(\xi_{n_{k_i}}\right) - K_H\left(\xi_{n_{k_j}}\right) \right\| < \frac{\varepsilon}{3C^2}, \tag{4.4}$$

for all $k_i, k_j > N$. Hence, for any $k_i, k_j > N$, the inequalities (4.3) and (4.4) allow us to write:

$$\begin{aligned}
&\left\| K\left(\xi_{n_{k_i}}\right) - K\left(\xi_{n_{k_j}}\right) \right\| \\
&= \left\| K\left(\xi_{n_{k_i}}\right) - K_H\left(\xi_{n_{k_i}}\right) + K_H\left(\xi_{n_{k_i}}\right) - K_H\left(\xi_{n_{k_j}}\right) + K_H\left(\xi_{n_{k_j}}\right) - K\left(\xi_{n_{k_j}}\right) \right\| \\
&\leq C^2 \left(\left\| K\left(\xi_{n_{k_i}}\right) - K_H\left(\xi_{n_{k_i}}\right) \right\| + \left\| K_H\left(\xi_{n_{k_i}}\right) - K_M\left(\xi_{n_{k_j}}\right) \right\| \right. \\
&\quad \left. + \left\| K_H\left(\xi_{n_{k_j}}\right) - K\left(\xi_{n_{k_j}}\right) \right\| \right) \\
&< C^2 \left(\frac{\varepsilon}{3C^2} + \frac{\varepsilon}{3C^2} + \frac{\varepsilon}{3C^2} \right) = \varepsilon.
\end{aligned}$$

We conclude that $(K(x_{n_k}))$ is a Cauchy sequence in the quasi-Banach space $(X, \|\cdot\|)$ and therefore it is a convergent sequence in X. This completes the proof. $\square$

For more properties of compact operators, we recommend the excellent book by Conway [10]. With this definition and these properties, we present the following result:

Theorem 4.3. *Let $q \geq 1$ and ω be a weight satisfying the global Δ_2-condition. Suppose that $\mathbf{u} = \{u(m)\}$ and $\mathbf{v} = \{v(m)\}$ are bounded sequences. The operator $\Delta_{u,v}$ is compact on $l(\omega, q)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ belong to c_0; that is, if and only if*

$$\lim_{m\to\infty} \max\{|u(m)|, |v(m)|\} = 0.$$

Proof. Suppose first that $\mathbf{u}$ and $\mathbf{v}$ belong to c_0. For a given $\varepsilon > 0$, we can find an index $N \in \mathbb{N}$ such that

$$|u(k)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |v(k)| \leq \frac{\varepsilon}{2}$$

for all $k \geq N$. Next, for a fixed $H \in \mathbb{N}$, we consider the sequences $\mathbf{u}_H$ and $\mathbf{v}_H$ defined by

$$u_H(k) = \begin{cases} u(k), & k \leq H, \\ 0, & k > H. \end{cases} \quad \text{and} \quad v_H(k) = \begin{cases} v(k), & k \leq H, \\ 0, & k > H. \end{cases}$$

We observe that $\mathbf{u}_H$ and $\mathbf{v}_H$ are bounded sequences, implying that Δ_{u_H, v_H} is a bounded operator on $l(\omega, q)$. Furthermore, we have $\dim \left(\Delta_{u_H, v_H}\left(l(\omega, q)\right)\right) \leq H + 1$, which indicates that the operator Δ_{u_H, v_H} is compact on $l(\omega, q)$.

Finally, for any $\xi = \{\xi(k)\} \in l(\omega, q)$, we set $\mathbf{y} = R(\xi)$ and for any $H \geq N + 1$ and all $k \in \mathbb{N}$, we have

$$\left|\left(v - v_H\right)(k-1)\xi(k-1) + \left(u - u_H\right)(k)\xi(k)\right| \leq \frac{\varepsilon}{2}\left(|y(k)| + |\xi(k)|\right),$$

where, as before, we set $v(0) = \xi(0) = 0$. Thus using the property (2.2), we can write

$$\left\|\left(\Delta_{u,v} - \Delta_{u_H, v_H}\right)(\xi)\right)\|_{(\omega,q)} \leq \frac{\varepsilon}{2}\left\|(|\mathbf{y}| + |\xi|)\right\|_{(\omega,q)} \leq \frac{\varepsilon}{2} L \left(\|R(\xi)\|_{(\omega,q)} + \|\xi\|_{(\omega,q)}\right),$$

where $|\xi| = \{|\xi(k)|\}$. Therefore, we obtain

$$\left\|\left(\Delta_{u,v} - \Delta_{u_H, v_H}\right)(\xi)\right)\|_{(\omega,q)} \leq \epsilon L \|\xi\|_{(\omega,q)},$$

for all $H \geq N + 1$. Thus, we conclude that

$$\lim_{H \to \infty} \|\Delta_{u,v} - \Delta_{u_H, v_H}\| = 0$$

and consequently, $\Delta_{u,v}$ is a compact operator on $l(\omega, q)$ since it is limit of compact operators on $l(\omega, q)$.

Next, we suppose that $\mathbf{u}$, $\mathbf{v}$ are complex bounded sequences and that $\Delta_{u,v}$ is a compact operator on $l(\omega, q)$. By way of contradiction, we assume that $\mathbf{u} = \{u(k)\} \notin c_0$. Then, there exists an $\varepsilon_0 > 0$ and an increasing, unbounded sequence $\{n_k\}$ of positive integers such that $|u(n_k)| \geq \varepsilon_0$ for all $k \in \mathbb{N}$. Without loss of generality and for convenience, we can further assume that

$$|u(k)| \geq \varepsilon_0$$

for all $k \in \mathbb{N}$. Next, we consider the sequence $(\mathbf{e}_k)$ of canonical basis sequences in $l(\omega, q)$. We know that $\|\mathbf{e}_k\|_{(\omega,q)} = \omega(1)^{\frac{1}{q}}$ for all $k \in \mathbb{N}$ which indicates that this sequence is bounded in $l(\omega, q)$. Furthermore, for $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$, we define $\hat{\mathbf{e}}_{k_1,k_2} = \mathbf{e}_{k_1} - \mathbf{e}_{k_2}$, which satisfies $\hat{\mathbf{e}}_{k_1,k_2}(k_1) = 1$, $\hat{\mathbf{e}}_{k_1,k_2}(k_2) = -1$ and $\hat{\mathbf{e}}_{k_1,k_2}(n) = 0$ for $n \neq k_1$ and $n \neq k_2$. Therefore $\mathbf{y}_{k_1,k_2} = \Delta_{u,v}(\hat{\mathbf{e}}_{k_1,k_2})$ is the sequence in $l(\omega, q)$ that satisfies $\mathbf{y}_{k_1,k_2}(k_1) = u(k_1)$, $\mathbf{y}_{k_1,k_2}(k_2 + 1) = -v(k_2)$. For this reason, we have

$$\|\mathbf{y}_{k_1,k_2}\|_{(\omega,q)} \geq \max\left\{|u(k_1)|, |v(k_2)|\right\}\omega^{\frac{1}{q}}(1) \geq \varepsilon_0\, \omega^{\frac{1}{q}}(1).$$

Thus, the sequence $(\Delta_{u,v}(\mathbf{e}_k))$ satisfies

$$\|\Delta_{u,v}(\mathbf{e}_{k_1}) - \Delta_{u,v}(\mathbf{e}_{k_2})\|_{(\omega,q)} \geq \varepsilon_0\, \omega^{\frac{1}{q}}(1),$$

for all $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$, which implies that it does not have a convergent subsequence. Therefore, $\Delta_{u,v}$ is not a compact operator on $l(\omega, q)$. This conclude the proof of the result. □

As a direct consequence of the preceding result, we obtain the following result regarding the compactness of the generalized difference operators on the classical Lorentz sequence spaces.

Corollary 4.4. *Suppose that $p > 1$ and $q \geq 1$, and let $\mathbf{u} = \{u(n)\}$ and $\mathbf{v} = \{v(n)\}$ be complex sequences. The generalized difference operator $\Delta_{u,v}$ is compact on $l(p, q)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ belong to c_0 .*

Note that when $v = 0$, the null sequence, we recover a well-known result concerning the compactness of the multiplication operator acting on the Lorentz sequence spaces (see [3] and [9]).

Final Remark

We conclude this article by noting that the results presented above also hold for the case $q = \infty$. Specifically, for a given weight ω, we define the space $l(\omega, \infty)$ as the collection of all sequences $\xi = \{\xi(n)\}$ satisfying

$$\|\xi\|_{(\omega,\infty)} = \sup_n \xi^*(n)\, \omega(n) < \infty.$$

Clearly, when $\omega(n) = n^{\frac{1}{p}}$ with $p > 1$, we recover the classical Lorentz sequence spaces $l_{(p,\infty)}$ (see [16], [13] and [9]). As before, if the weight ω satisfies the global Δ_2-condition, then $\big(l(\omega, \infty), \|\cdot\|_{(\omega,\infty)}\big)$ constitutes a quasi-Banach space. In this space, the study of continuity and compactness of the generalized difference operators is analogous to the analysis carried out in the previous sections. In fact, the same arguments allow us to establish the following results:

(1) Let ω be a weight satisfying the global Δ_2-condition. Suppose $\mathbf{u} = \{u(n)\}$ and $\mathbf{v} = \{v(n)\}$ are complex sequences. The operator $\Delta_{u,v}$ is continuous on $l(\omega, \infty)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are bounded sequences.

(2) Let ω be a weight satisfying the global Δ_2-condition. Suppose $\mathbf{u} = \{u(n)\}$ and $\mathbf{v} = \{v(n)\}$ are bounded sequences. The operator $\Delta_{u,v}$ is compact on $l(\omega, \infty)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ belong to c_0. That is, for $q \geq 1$ fixed, we can conclude that $\Delta_{u,v} : l(\omega, \infty) \to l(\omega, \infty)$ is compact if and only if $\Delta_{u,v} : l(\omega, q) \to l(\omega, q)$ is compact.

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