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RESULTS ON THE EXISTENCE OF SOLUTIONS FOR CONFORMABLE EVOLUTION DIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITIONS

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ABSTRACT. This paper investigates the existence of solutions for conformable evolution differential equations under non-local conditions. Using fixed-point theory and analytical methods, we establish solution existence for equations involving conformable derivatives, which model systems with memory effects. An example illustrates the practical application of these results in complex dynamical systems.

*Keywords.*conformable derivatives, evolution equation, Schauder's theorem, mild solution

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1. INTRODUCTION

In many scientific and technical fields, fractional differential equations (FDEs) are now crucial for simulating complex processes, like viscoelastic materials, control systems, diffusion dynamics, and signal analysis [1, 2, 3]. Unlike classical integer-order models, FDEs effectively capture memory and hereditary properties commonly found in real-world phenomena [4, 5, 6, 7, 8, 9].

The conformable fractional derivative offers a simpler and more intuitive extension of the classical derivative, retaining many familiar properties like the chain rule, the quotient rule, the product rule, and linearity [10, 11]. Unlike classical fractional operator such as the Riemann–Liouville and Caputo derivatives, which are typically defined via integral operators and involve nonlocal memory effects, the conformable derivative is defined through a limit process that closely resembles the classical derivative's definition. This leads to easier computation and more straightforward analytical handling. Moreover, while Caputo and Riemann–Liouville derivatives need initial conditions described by of fractional integrals or derivatives, the conformable derivative allows for initial conditions specified in a form similar to classical integer-order differential equations. However, the conformable derivative may not capture all the memory and hereditary

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properties inherent in classical fractional models, which limits its use in some applications but provides a more accessible framework for others [12, 13, 14]. These characteristics enable more accessible analysis and support extending classical results specifically within the framework of conformable (or local fractional) derivatives, as introduced by Khalil and further studied in [15, 16].

The differential equations' study incorporating the conformable fractional operator has advanced swiftly. Numerous researchers have investigated the existence and uniqueness of solutions for different classes of such equations. For example, in [17], the authors employed fixed point theory to examine the existence of solutions for IVPs. Additionally, the application of conformable fractional models in fields such as control systems, biological processes, and physics underscores the importance of continued development of the associated analytical framework [18, 19, 20, 21, 22].

Nonlocal conditions in differential equations have garnered significant interest due to their natural emergence in models where a system's current state depends on both local and distributed influences, such as integrals or averages over time or space. These conditions offer a flexible and comprehensive framework to capture effects like memory and long-range interactions. While the theory of nonlocal problems has been thoroughly explored for classical and Caputo-type fractional differential equations [23, 24, 25], their investigation within the conformable fractional calculus framework is a more recent and still developing area of research [26, 27].

Inspired by these advancements, we focus on a set of nonlinear conformable fractional evolution differential equations subject to nonlocal initial conditions. In particular, we examine the following problem:

$$\begin{cases} v^{(\beta)}(t) - a(t)v(t) = \varphi(t, v(t)), \quad t \in (0, T], \\ v(0) = v_0 + g(v), \end{cases} \tag{1.1}$$

where $v^{(\beta)}(t)$ denotes the fractional derivative of order $\beta \in (0, 1]$ in the conformable sense, $a(t)$ is a given continuous function, φ is a nonlinear function, and g represents a nonlocal term involving the unknown function v. This type of equation models evolution processes that are influenced both by local interactions and accumulated historical effects.

The principal target of this work is to determine conditions under which problem (1.1) admits at least one solution. To accomplish this, we apply fixed point theorems, notably Schauder's fixed point theorem, a powerful method in nonlinear analysis for establishing existence results in infinite-dimensional settings. The interplay between the nonlinear nature of the equation and the inclusion of a nonlocal term presents both analytical challenges and theoretical interest.

The following is an outline of the paper's structure: Section 2 depicts the key definitions and mathematical concepts required for the analysis. In Section 3, we present the principal existence theorem, including the necessary conditions and a comprehensive proof. Finally, we apply our theoretical findings to an example and show how to develop a successive approximation solution for it.

2. Preliminaries

This section presents foundational material essential to the paper, beginning with an introduction to conformable fractional differentiation and integration.

Definition 2.1. [12] Assume that $v : [0, \infty) \to \mathbb{R}$ is a real-valued function. The following is the definition of the conformable fractional derivative of order $\beta \in (0, 1]$ at a position $t > 0$:

$$v^{(\beta)}(t) = \lim_{\epsilon \to 0} \frac{v(t + \epsilon t^{1-\beta}) - v(t)}{\epsilon},$$

whenever the limit exists. In such a case, v is said to be t-differentiable at t. Moreover, the value at $t = 0$ is defined by

$$v^{(\beta)}(0) = \lim_{t \to 0^+} v^{(\beta)}(t).$$

The corresponding integral operator, referred to as the conformable fractional integral, is defined by

$$I_\beta^a(v)(t) = \int_a^t \frac{v(s)}{s^{1-\beta}}\, ds.$$

Theorem 2.2. *If u, v is β-differentiable at a point $t > 0$ and $\beta \in (0, 1]$, then:*

(1) $(au + bv)^{(\beta)}(t) = au^{(\beta)}(t) + bv^{(\beta)}(t)$ *for all* $a, b \in \mathbb{R}$.
(2) $(t^p)^{(\beta)}(t) = pt^{t^{p-\beta}}$ *for all* $p \in \mathbb{R}$.
(3) $v^{(\beta)}(t) = 0$, *for every constant function* $v(t) = \lambda$.
(4) $(uv)^{(\beta)} = uv^{(\beta)} + vu^{(\beta)}$.
(5) *In addition, v is differentiable then* $v^{(\beta)}(t) = t^{1-\beta}\frac{dv}{dt}(t)$.

Proposition 2.3. *Suppose $0 < \beta \leq 1$ and $v : [a, +\infty) \to \mathbb{R}$ is a function where $a \geq 0$. If v is continuous in the domain of I_β, then*

$$(I_\beta^a(v))^{(\beta)}(t) = v(t),$$

for $t \geq a$. Furthermore, if v is differentiable, we obtain

$$I_\beta^a v^{(\beta)}(t) = v(t) - v(a).$$

Theorem 2.4. *Let $u, v : [a, b] \to \mathbb{R}$ be two β-differentiable functions for $0 < \beta \leq 1$, such that the product uv is differentiable. Then, the formula of integration by parts is expressed as*

$$\int_a^b u(s)v^{(\beta)}(s)\, s^{\beta-1} ds = \big[uv\big]_a^b - \int_a^b v(s)u^{(\beta)}(s)\, s^{\beta-1} ds.$$

Theorem 2.5. [28] ***[Schauder's Fixed Point Theorem]*** *Assume Ω is a Banach space, and let $F \subset \Omega$ be a nonempty, closed, and convex subset. Suppose $\mathfrak{B} : F \to F$ is a continuous operator that maps F into a relatively compact subset of itself. Then, $\mathfrak{B}$ has at least one fixed point in F.*

3. Main results

Let $C([0,T])$ be a Banach space of continuous functions $v:[0,T]\to\mathbb{R}$, endowed with the norm

$$\|v\| = \sup_{t\in[0,T]} |v(t)|$$

We also define $C^{(\beta)}([0,T])$, the space of β-continuously differentiable functions (in the conformable fractional sense), such that for each $v\in C^{(\beta)}([0,T])$, the conformable fractional derivative $v^{(\beta)}(t)$ exists and is continuous on $[0,T]$, where $0<\beta\le 1$.

We aim to derive the explicit solution formula for the initial value problem given in equation (1.1).

Lemma 3.1. *A solution $v\in C([0,T])$ of the conformable fractional evolution differential equation (1.1) has the explicit representation:*

$$v(t) = (v_0+g(v))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\,ds} + \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau}\varphi(s,v(s))s^{\beta-1}\,ds. \tag{3.1}$$

Proof. We use the variation of constants method. Assume the solution has the form

$$v(t)=\mu(t)u(t),$$

where $\mu(t)$ is an integrating factor, such that

$$\mu^{(\beta)}(t)=a(t)\mu(t).$$

A suitable choice is:

$$\mu(t)=e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\,ds}.$$

Using the conformable product rule,

$$v^{(\beta)}(t)=\mu^{(\beta)}(t)u(t)+\mu(t)u^{(\beta)}(t)=a(t)\mu(t)u(t)+\mu(t)u^{(\beta)}(t)=a(t)v(t)+\mu(t)u^{(\beta)}(t).$$

Substitute into the original equation:

$$v^{(\beta)}(t)-a(t)v(t)=\varphi(t,v(t))\Rightarrow \mu(t)u^{(\beta)}(t)=\varphi(t,v(t)).$$

Solving for $u^{(\beta)}(t)$:

$$u^{(\beta)}(t)=\mu(t)^{-1}\varphi(t,v(t)).$$

Apply the conformable integral:

$$u(t)=u(0)+\int_0^t \mu(s)^{-1}\varphi(s,v(s))s^{\beta-1}\,ds.$$

Since $u(0)=v(0)/\mu(0)=v_0+g(v)$, we get:

$$v(t)=\mu(t)\left[v_0+g(v)+\int_0^t \mu(s)^{-1}\varphi(s,v(s))s^{\beta-1}\,ds\right].$$

Distribute $\mu(t)$:

$$v(t)=(v_0+g(v))\,\mu(t)+\int_0^t \mu(t)\mu(s)^{-1}\varphi(s,v(s))s^{\beta-1}\,ds.$$

Since $\mu(t)\mu(s)^{-1} = e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau}$, the final result is:

$$v(t) = (v_0 + g(v))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\,ds} + \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds,$$

which finishes the proof. This technique is classical in solving linear differential equations and helps transform the original problem into an easier-to-handle integral equation. ■ □

For any function $v \in C([0,T])$, consider the operator

$$\mathfrak{B}v(t) = (v_0 + g(v))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\,ds} + \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds. \tag{3.2}$$

It can be readily verified that $\mathfrak{B}\colon C([0,T]) \to C([0,T])$. Therefore, the original problem (1.1) is equivalent to the integral equation (3.1). In other words, a function $v = v(t)$ is a solution to (1.1) if and only if it is a fixed point of the operator $\mathfrak{B}$ in the space $C([0,T])$.

Now, we establish several existence results for solutions of equation (1.1), each obtained under particular assumptions on the function φ, we begin by employing Schauder's Fixed Point Theorem to demonstrate the existence of a solution. To proceed with our analysis, we consider the following conditions:
(C1) g is continuous function in $C([0,T])$.
(C2) There exist continuous functions μ, ω, η mapping J into $[0, +\infty)$ such that, for all
$v \in C([0,T])$ and $t \in J$, the following inequalities hold:

$$\|\varphi(t, v)\| \leq \mu(t) + \omega(t)\|v\|, \quad \|g(v)\| \leq \eta(t)\|v\|.$$

Additionally, we denote by μ^*, ω^* and η^* the supremum values of the functions $\mu(t), \omega(t)$ and $\eta(t)$ over the interval $J = [0,T]$.
(C3) Assume that

$$\lambda = E(T)(\eta^* + \omega^* \frac{T^\beta}{\beta}) < 1,$$

where

$$E(t) = e^{\int_0^t \frac{|a(s)|}{s^{1-\beta}}\,ds} \quad \text{and} \quad E(t,s) = e^{\int_s^t \frac{|a(\tau)|}{\tau^{1-\beta}}\,d\tau} \leq E(T).$$

Define the set

$$B_\delta = \{v \in C([0,T]) : \|v\| \leq \delta \quad \text{and} \quad \delta \geq \frac{\theta}{1-\lambda}\},$$

where

$$\theta = E(T)(\|v_0\| + \mu^* \frac{T^\beta}{\beta}).$$

Lemma 3.2. *If* (C1)–(C3) *are satisfied, then the operator* $\mathfrak{B}$ *is continuous on* B_δ *and maps* B_δ *into itself.*

Proof. **Claim:** $\mathfrak{B}$ maps B_δ into B_δ.
Let $v \in B_\delta$ and $t \in J$. Using conditions (C2) and (C3), we estimate:

$$\begin{aligned}
\|\mathfrak{B}v(t)\| &\leq \left\|(v_0 + g(v))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\, ds}\right\| + \left\|\int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds\right\| \\
&\leq E(T)\,(\|v_0\| + \|g(v)\|) + E(T) \int_0^t \|\varphi(s, v(s))\| s^{\beta-1}\, ds \\
&\leq E(T)\,(\|v_0\| + \eta^*\|v\|) + E(T) \int_0^t (\mu(s) + \omega(s)\|v\|)\, s^{\beta-1}\, ds \\
&\leq E(T)\|v_0\| + E(T)\eta^*\|v\| + E(T)\left(\mu^*\frac{T^\beta}{\beta} + \omega^*\|v\|\frac{T^\beta}{\beta}\right) \\
&= E(T)\|v_0\| + E(T)\mu^*\frac{T^\beta}{\beta} + E(T)\left(\eta^* + \omega^*\frac{T^\beta}{\beta}\right)\|v\| \\
&= \theta + \lambda\|v\| \leq \theta + \lambda\delta \leq \delta,
\end{aligned}$$

where we used the fact that $\delta \geq \frac{\theta}{1-\lambda}$. Thus, $\mathfrak{B}(v) \in B_\delta$.

Claim: $\mathfrak{B}$ is continuous on B_δ.
Let $\{v_n\} \subset B_\delta$ be a sequence such that $v_n \to v$ in $C([0,T])$. Define $\varphi_n(s) = \varphi(s, v_n(s))$, and $\varphi(s) = \varphi(s, v(s))$. Then,

$$s^{\beta-1} e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi_n(s) \to s^{\beta-1} e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s) \quad \text{as } n \to \infty.$$

Moreover, using (C2),

$$\left\| s^{\beta-1} e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} (\varphi_n(s) - \varphi(s)) \right\| \leq 2E(T) s^{\beta-1}(\mu^* + \omega^*\delta) \in L^1([0,T]).$$

By the Dominated Convergence Theorem,

$$\left\| \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} (\varphi_n(s) - \varphi(s)) s^{\beta-1}\, ds \right\| \to 0 \quad \text{as } n \to \infty.$$

Similarly, since g is continuous (C1), we have $g(v_n) \to g(v)$. Hence,

$$\mathfrak{B}v_n(t) \to \mathfrak{B}v(t) \quad \text{uniformly in } t \in [0,T].$$

Therefore, $\mathfrak{B}$ is continuous on B_δ. □

Lemma 3.3. *Suppose that conditions* (C1)–(C3) *are satisfied. Then the image of the ball* B_δ *under the operator* $\mathfrak{B}$*, denoted* $\mathfrak{B}(B_\delta)$*, is bounded and equicontinuous.*

Proof. By Lemma 3.2, it follows that $\mathfrak{B}(B_\delta)$ is bounded.
Let $v \in B_\delta$ and let $t_1, t_2 \in J$ with $t_1 < t_2$. Consider the difference:

$$\begin{aligned}
\|\mathfrak{B}v(t_2) - \mathfrak{B}v(t_1)\| = &\left\| (v_0 + g(v)) \left[e^{\int_0^{t_2} \frac{a(s)}{s^{1-\beta}}\, ds} - e^{\int_0^{t_1} \frac{a(s)}{s^{1-\beta}}\, ds} \right] \right. \\
&\left. + \int_0^{t_2} e^{\int_s^{t_2} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds - \int_0^{t_1} e^{\int_s^{t_1} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds \right\| \\
\le\ & \|v_0 + g(v)\| \cdot \left| e^{\int_0^{t_2} \frac{a(s)}{s^{1-\beta}}\, ds} - e^{\int_0^{t_1} \frac{a(s)}{s^{1-\beta}}\, ds} \right| \\
&+ \left\| \int_{t_1}^{t_2} e^{\int_s^{t_2} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds \right\| \\
&+ \left\| \int_0^{t_1} \left(e^{\int_s^{t_2} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} - e^{\int_s^{t_1} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \right) \varphi(s, v(s)) s^{\beta-1}\, ds \right\|.
\end{aligned}$$

We estimate each term separately. First, using (C2) and boundedness of $v \in B_\delta$,

$$\|v_0 + g(v)\| \le \|v_0\| + \eta^* \delta.$$

Since the exponential function is continuous, the first term tends to zero as $t_2 \to t_1$. For the second term, we use (C2) again:

$$\begin{aligned}
\left\| \int_{t_1}^{t_2} e^{\int_s^{t_2} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v(s)) s^{\beta-1}\, ds \right\| &\le E(T)(\mu^* + \omega^* \delta) \int_{t_1}^{t_2} s^{\beta-1}\, ds \\
&= E(T)(\mu^* + \omega^* \delta) \cdot \frac{t_2^\beta - t_1^\beta}{\beta}.
\end{aligned}$$

This clearly tends to zero as $t_2 \to t_1$.

The third term also vanishes as $t_2 \to t_1$ because the exponential integrals involved are continuous in t, and the integrand is uniformly bounded by $2E(T)(\mu^* + \omega^* \delta) s^{\beta-1} \in L^1([0, T])$. Therefore, by dominated convergence,

$$\left\| \int_0^{t_1} \left(e^{\int_s^{t_2} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} - e^{\int_s^{t_1} \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \right) \varphi(s, v(s)) s^{\beta-1}\, ds \right\| \to 0 \quad \text{as } t_2 \to t_1.$$

Thus, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for $|t_2 - t_1| < \delta_0$, we have

$$\|\mathfrak{B}v(t_2) - \mathfrak{B}v(t_1)\| < \varepsilon,$$

uniformly for all $v \in B_\delta$. Hence, $\mathfrak{B}(B_\delta)$ is equicontinuous. □

Theorem 3.4. *Assume that conditions* (C1)–(C3) *hold. Then the problem* (1.1) *admits at least one mild solution in the ball* B_δ.

Proof. We observe that B_δ is a closed and convex subset. By Lemmas 3.2 and 3.3, the operator $\mathfrak{B}$ is continuous on B_δ and maps B_δ into itself, with $\mathfrak{B}(B_\delta)$ being both bounded and equicontinuous. According to the Ascoli–Arzelà Theorem, this implies that $\mathfrak{B}(B_\delta)$ is relatively compact in B_δ. Therefore, by applying Schauder's Fixed Point Theorem 2.5, we conclude that $\mathfrak{B}$ admits a fixed point in B_δ, which corresponds to a mild solution of problem (1.1). □

We now introduce the assumptions necessary to guarantee the uniqueness of the solution for the results:
(C4) There exist constants $L_\varphi, L_g > 0$ such that

$$\|\varphi(t,v)-\varphi(t,\overline{v})\| \le L_\varphi\|v-\overline{v}\|, \quad \|g(v)-g(\overline{v})\| \le L_g\|v-\overline{v}\| \quad \forall v,\overline{v} \in C([0,T]).$$

(C5) Suppose that

$$E(T)\left(L_g + L_\varphi \frac{T^\beta}{\beta}\right) < 1.$$

Theorem 3.5. *Suppose that conditions* (C4)–(C5) *are satisfied. Then the conformable fractional differential equation* (1.1) *admits a unique mild solution in* $C([0,T])$.

Proof. Let $v,\overline{v} \in C([0,T])$, and consider the operator $\mathfrak{B}$ defined by (3.2). We estimate the difference:

$$\begin{aligned}
&\|\mathfrak{B}v(t)-\mathfrak{B}\overline{v}(t)\| \\
&\quad = \left\|(g(v)-g(\overline{v}))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\,ds} + \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau}\left[\varphi(s,v(s))-\varphi(s,\overline{v}(s))\right] s^{\beta-1}ds\right\| \\
&\quad \le E(T)\|g(v)-g(\overline{v})\| + E(T)\int_0^t \|\varphi(s,v(s))-\varphi(s,\overline{v}(s))\| s^{\beta-1}ds.
\end{aligned}$$

Using assumption (C4), we have:

$$\|g(v)-g(\overline{v})\| \le L_g\|v-\overline{v}\|, \quad \|\varphi(s,v(s))-\varphi(s,\overline{v}(s))\| \le L_\varphi\|v-\overline{v}\|.$$

Therefore,

$$\begin{aligned}
\|\mathfrak{B}v(t)-\mathfrak{B}\overline{v}(t)\| &\le L_g\|v-\overline{v}\|\cdot E(T) + L_\varphi\|v-\overline{v}\|E(T)\int_0^T s^{\beta-1}ds \\
&= \left(L_g + L_\varphi\frac{T^\beta}{\beta}\right)E(T)\|v-\overline{v}\|.
\end{aligned}$$

By condition (C5), we have:

$$E(T)\left(L_g + L_\varphi \frac{T^\beta}{\beta}\right) < 1,$$

which implies that $\mathfrak{B}$ is a contraction on $C([0,T])$. Hence, by the Banach Fixed Point Theorem, the operator $\mathfrak{B}$ has a unique fixed point. This fixed point is the unique mild solution of problem (1.1). □

Example 3.6. We consider the following conformable fractional differential equation:

$$\begin{cases} v^{\left(\frac{3}{4}\right)}(t) - tv(t) = \varphi(t,v(t)), & t \in [0,1], \\ v(0) = 1 + g(v), \end{cases} \tag{3.3}$$

which corresponds to the general equation (3.3) with

$$a(t) = t, \quad \varphi(t,v(t)) = \frac{1}{3}\left(\frac{|v(t)|}{1+|v(t)|}\right), \quad g(v) = \frac{1}{9}\sin(v).$$

Verify the Lipschitz condition for φ, let $v, \bar{v} \in C([0,1])$ and fix $t \in [0,1]$. Then

$$\begin{aligned} |\varphi(t,v) - \varphi(t,\bar{v})| &= \frac{1}{3}\left|\frac{|v(t)|}{1+|v(t)|} - \frac{|\bar{v}(t)|}{1+|\bar{v}(t)|}\right| \\ &= \frac{1}{3}\left|\frac{|v(t)| - |\bar{v}(t)|}{(1+|v(t)|)(1+|\bar{v}(t)|)}\right| \\ \|\varphi(t,v) - \varphi(t,\bar{v})\| &\le \frac{1}{3}\|v - \bar{v}\|. \end{aligned}$$

So φ satisfies a Lipschitz condition with $L_\varphi = \frac{1}{3}$.
Verify the Lipschitz condition for g, we have:

$$\|g(v) - g(\bar{v})\| = \|\sin(v) - \sin(\bar{v})\| \le \frac{1}{9}\|v - \bar{v}\|,$$

so g satisfies a Lipschitz condition with $L_g = \frac{1}{9}$.
Check the contraction condition. For $\beta = \frac{3}{4}$ and $T = 1$, compute:

$$E(T) = e^{\frac{T^{\beta+1}}{\beta+1}} = e^{\frac{4}{7}} \approx 1.77.$$

Then,

$$\begin{aligned} E(T)\left(L_g + L_\varphi \cdot \frac{T^\beta}{\beta}\right) &= 1.77\left(\frac{1}{9} + \frac{1}{3}\cdot\frac{1^{3/4}}{3/4}\right) \\ &= 1.77\left(\frac{1}{9} + \frac{4}{9}\right) = 1.77 \cdot \frac{5}{9} \approx 0.9834 < 1. \end{aligned}$$

Hence, the contraction condition is satisfied. By Theorem 3.5, the conformable fractional differential equation (3.3) admits a unique solution.

We aim to construct a successive approximation (Picard-type iteration) for the conformable fractional differential equation (3.3). Let the initial guess be constant:

$$v_0(t) = v_0 = 1.$$

Now, define the sequence:

$$v_{n+1}(t) = (v_0 + g(v_n))\, e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\, ds} + \int_0^t e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau} \varphi(s, v_n(s)) s^{\beta-1}\, ds.$$

Given that:

$$a(t) = t \quad \text{and} \quad \beta = \frac{3}{4},$$

we compute the following integrals:

$$\int_0^t \frac{a(s)}{s^{1-\beta}}\, ds = \int_0^t s^\beta\, ds = \frac{t^{\beta+1}}{\beta+1}.$$

Thus:

$$e^{\int_0^t \frac{a(s)}{s^{1-\beta}}\, ds} = e^{\frac{t^{\beta+1}}{\beta+1}}.$$

Similarly, for the second exponential integral:

$$\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\, d\tau = \int_s^t \tau^\beta\, d\tau = -\frac{t^{\beta+1} - s^{\beta+1}}{\beta+1}.$$

Thus:

$$e^{\int_s^t \frac{a(\tau)}{\tau^{1-\beta}}\,d\tau} = e^{\frac{t^{\beta+1}-s^{\beta+1}}{\beta+1}}.$$

Now, we proceed with the first iteration. Start with the initial guess:

$$v_0(t) = 1.$$

Next, compute:

$$g(v_0) = \frac{1}{9}\sin(1) \approx \frac{1}{9} \times 0.8415 \approx 0.0935.$$

The function $\varphi(t, v_0(t))$ is given by:

$$\varphi(t, v_0(t)) = \frac{1}{3} \times \frac{1}{1+1} = \frac{1}{6}.$$

Now, plug these values into the iteration formula:

$$v_1(t) = (1 + 0.0935)\, e^{\frac{t^{\beta+1}}{\beta+1}} + \int_0^t e^{\frac{t^{\beta+1}-s^{\beta+1}}{\beta+1}} \cdot \frac{1}{6} s^{\beta-1}\,ds.$$

Simplify the expression:

$$v_1(t) = 1.0935 \cdot e^{\frac{t^{\beta+1}}{\beta+1}} + \frac{1}{6}\int_0^t e^{\frac{t^{\beta+1}-s^{\beta+1}}{\beta+1}} s^{\beta-1}\,ds.$$

This gives the first approximation for $v(t)$. Further iterations proceed by using $v_1(t)$ in place of $v_0(t)$, and repeating the process.

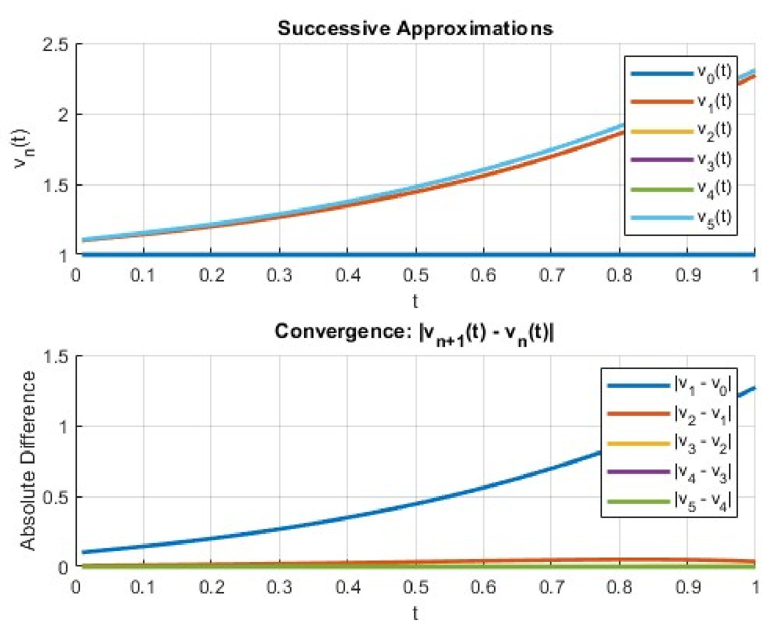

FIGURE 1. Your figure caption here

As anticipated, the sequence converges steadily and seems to reach stability with each iteration.

4. Conclusion

The findings of this study add to the growing body of research on conformable fractional differential equations by extending the theory of existence to include more general nonlocal initial conditions. By proving the existence of solutions for a specific class of nonlinear conformable evolution differential equations with nonlocal terms, we establish a strong theoretical foundation for future investigations. These results not only improve the comprehension of conformable fractional models but also provide a basis for future studies on numerical approaches and practical applications in fields where memory and hereditary effects play a crucial role.

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