Gulf Journal of Mathematics
Vol 20, Issue 1 (2025) 176-189

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i.3021

ON THE WELL-POSEDNESS OF THE VON KÁRMÁN PLATE SYSTEM WITH VISCOELASTIC BOUNDARY DAMPING

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ABSTRACT. In this article, we consider a von Kármán plate system with viscoelastic damping localized on the boundary. Using the multiplier method and exploiting key properties of convex functions, we establish an explicit and general decay rate result without imposing $u_0 = 0$ on Γ as in [5]. This allows a larger class of relaxation functions and initial data, hence, generalizes some previous results existing in the literature.

1. INTRODUCTION

This paper concerns the following problem:

$$
\begin{gathered}
u_{tt} + \Delta^2 u = [u, v], \text{in } \Omega \times (0, \infty) \\
\Delta^2 v = -[u, u], \text{in } \Omega \times (0, \infty) \\
\frac{\partial v}{\partial n} = v = 0, \text{on } \Gamma \times (0, \infty) \\
\frac{\partial u}{\partial n} + \int_0^t g_1(t-a)\, \mathfrak{B}_1 u(a)\, da = 0, \text{on } \Gamma \times (0, \infty) \\
u - \int_0^t g_2(t-a)\, \mathfrak{B}_2 u(a)\, da = 0, \text{on } \Gamma \times (0, \infty) \\
u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), x \in \Omega.
\end{gathered}
\tag{1.1}
$$

The spatial domain $\Omega \subset \mathbb{R}^2$ is assumed to be bounded with a sufficiently smooth boundary Γ. The damping is modeled through the relaxation functions g_1 and g_2, which are continuously differentiable, nonincreasing, and positive functions on $[0, \infty)$. The von Kármán bracket is given by:

$$[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}.$$

The differential operators $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined as follows:

$$\mathfrak{B}_1 u = \Delta u + (1-\nu) B_1 u, \quad \mathfrak{B}_2 u = \frac{\partial \Delta u}{\partial n} + (1-\nu) \frac{\partial B_2 u}{\partial \eta}.$$

Date: Received: May 13, 2025; Accepted: June 8, 2025.
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2020 *Mathematics Subject Classification.* Primary 74D99; Secondary 35B40.
Key words and phrases. General decay, memory term, von Kármán plate system, relaxation function.

where

$$B_1u = 2n_1n_2u_{xy} - n_1^2u_{yy} - n_2^2u_{xx}, \quad B_2u = \left(n_1^2 - n_2^2\right) u_{xy} + n_1n_2 \left(u_{yy} - u_{xx}\right).$$

In this context, Γ is equipped with the outward unit normal vector $n = (n_1, n_2)$, while the associated tangential unit vector is given by $\eta = (-n_2, n_1)$. The parameter $\nu \in (0, \frac{1}{2})$ represents the Poisson ratio.

Physically, the variable u denotes the transverse displacement, and v represents the Airy stress function of a vibrating plate subjected to boundary viscoelastic damping. The memory effect described by the integral equations $(1.1)_4$, $(1.1)_5$ arises from interaction with an additional viscoelastic element. This equation predicts the plate's deformation or shape, which is critical for designing structural components such as aircraft wings, building supports, and mechanical systems. Moreover, von Kármán plates with boundary memory effects are widely used in communication and signal processing. Specifically, understanding energy decay at the boundary can improve signal clarity in communication systems.

We now review some established results for the von Kármán plate system with a memory condition imposed at the boundary. In [13], Rivera and Menzala analyzed the asymptotic behavior of solutions to the following von Kármán plate system with memory:

$$\begin{gathered} u_{tt} - h\Delta^2 u_{tt} + \Delta^2 u - \int_0^t g\left(t-a\right) \Delta^2 u\left(a\right) da = \left[u, v\right], \text{ in } \Omega \times (0, \infty) \\ \Delta^2 v = -\left[u, u\right], \text{ in } \Omega \times (0, \infty), \end{gathered} \tag{1.2}$$

where h is a constant representing the plate's thickness. More recently, Kang [4] investigated the general decay rates for the von Kármán plate model (1.2) under the condition:

$$g'\left(t\right) \leq -\xi\left(t\right) H\left(g\left(t\right)\right), \quad t \geq 0, \tag{1.3}$$

where ξ is a positive, nonincreasing, differentiable function, and H satisfies appropriate conditions. This result significantly extends the findings in [5, 11, 13]. Subsequently, Balegh et al. [1] derived a general energy decay result for the system in [12], incorporating a nonlinear boundary delay term, assuming g satisfies (1.3). The stability of viscoelastic equations with relaxation functions g satisfying (1.3) has been extensively studied in [3, 6, 8, 9]. In [5], Kang examined the system (1.1) for functions k_i $(i = 1, 2)$ satisfying:

$$k_i\left(0\right) > 0, \quad \lim_{t\to\infty} k_i\left(t\right) = 0, \quad k_i'\left(t\right) \leq 0, \tag{1.4}$$

$$k_i''\left(t\right) \geq H\left(-k_i'\left(t\right)\right), \quad \forall t > 0, \tag{1.5}$$

where H is a positive function that is either linear or strictly increasing, strictly convex, and in $C^2(0, r]$, with $r < 1$, and $H\left(0\right) = 0$. For the case $u_0 = 0$ on the boundary Γ, an explicit energy decay formula was established, which does not necessarily exhibit exponential or polynomial decay.

Our aim is to examine problem (1.1) for resolvent kernels satisfying (1.4) and (1.5), particularly for the boundary condition $u_0 \neq 0$ on Γ. The proof uses the multiplier method and estimates from [5], with adaptations to derive our results. This paper is structured as follows: Section 2 introduces notation and preliminaries, and establishes several technical lemmas. In Section 3, we present the main theorem and prove the general decay of solutions for the von Kármán plate system.

2. Preliminaries

In this section, we introduce the material needed to establish our main result. Throughout this paper, the L^2-norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma)}$ are abbreviated as $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively, and C denotes a positive constant. We define

$$\Gamma = \{x \in \Gamma : n(x) \cdot m(x) > 0\},$$

where $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^2$. We define the bilinear form $a(.,.)$ as follows

$$a(u,v) = \int_\Omega \{u_{xx}v_{xx} + u_{yy}v_{yy} + \nu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\nu)u_{xy}v_{xy}\}\, dxdy,$$

and

$$(g * u)(t) := \int_0^t g(t-a)\, u(a)\, da,$$

$$(g\Box u)(t) := \int_0^t g(t-a)\, |u(t) - u(a)|^2\, da.$$

As in [12, 14], we use the boundary conditions $(1.1)_4$ and $(1.1)_5$ to estimate the terms $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Differentiating $(1.1)_4$ and $(1.1)_5$ and applying Volterra's inverse operator, we have

$$\mathfrak{B}_1 u = -\eta_1 \left\{ \frac{\partial u_t}{\partial n} - k_1(t) \frac{\partial u_0}{\partial n} + k_1(0) \frac{\partial u}{\partial n} + k_1' * \frac{\partial u}{\partial n} \right\}, \tag{2.1}$$

$$\mathfrak{B}_2 u = \eta_2 \{u_t - k_2(t)\, u_0 + k_2(0)\, u + k_2' * u\}, \tag{2.2}$$

where $\eta_i = \dfrac{1}{g_i(0)}, (i = 1, 2)$ and the resolvent kernels $k_i, (i = 1, 2)$ satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i'.$$

Thus, we use boundary conditions (2.1) and (2.2) instead of $(1.1)_4$ and $(1.1)_5$.

We introduce the following assumption on k_1 and k_2.

(A) The resolvent kernels $k_i : \mathbb{R}^+ \to \mathbb{R}^+ (i = 1, 2)$ are twice differentiable functions such that

$$k_i(0) > 0, \quad \lim_{t\to\infty} k_i(t) = 0, \quad k_i'(t) \le 0,$$

and there exists a positive function $H \in C^1(\mathbb{R}^+)$, and H is a linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$k_i''(t) \ge H(-k_i'(t)), \quad t > 0. \tag{2.3}$$

As in [5, 7]. From (A) we easily see that there exists $t_0 > 0$ large enough such that for all $t \ge t_0$

$$max\{k_i(t), -k_i'(t), k_i''(t)\} < min\{r, H(r), H_0(r)\}, \tag{2.4}$$

and

$$\tau_1 \leq H\left(-k_i'(t)\right) \leq \tau_2, \quad \forall t \in [0, t_0], \tag{2.5}$$

for some positive constants τ_1 and τ_2. Since k_i' is nondecreasing $k_i'(0) < 0$ and $k_i'(t_0) < 0$, we have

$$0 < -k_i'(t_0) \leq -k_i'(t) \leq k_i'(0), \quad \forall t \in [0, t_0]. \tag{2.6}$$

Hence, by (2.3), (2.5) and (2.6),

$$k_i''(t) \geq \frac{\tau_1}{k_i'(0)} k_i'(0) \geq \frac{\tau_1}{k_i'(0)} k_i'(t),$$

which gives

$$k_i''(t) \geq \tau_3 k_i'(t), \quad \forall t \in [0, t_0], i = 1, 2. \tag{2.7}$$

Theorem 2.1. [12] *Let $k_i \in C^2(\mathbb{R}_+)$ be such that $k_i, -k_i', k_i'' \geq 0$ for $i = 1, 2$. If $(u_0, u_1) \in H^2(\Omega) \cap L^2(\Omega)$, then there exists a unique weak solution for system (1.1). Moreover, if $(u_0, u_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$, then the solution of (1.1) has the following regularity:*

$$u \in C^1\left([0, T), H^2(\Omega)\right) \cap C^2\left([0, T), H^4(\Omega)\right), v \in C^0\left([0, T), H^4(\Omega) \cap H^2(\Omega)\right).$$

The energy function of system (1.1) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}a(u, u) + \frac{1}{4}\|\Delta v\|^2 + \frac{\eta_1}{2}k_1(t)\left\|\frac{\partial u}{\partial n}\right\|_\Gamma^2 + \frac{\eta_2}{2}k_2(t)\|u\|_\Gamma^2 \\ &\quad -\frac{\eta_1}{2}\int_\Gamma k_1' \Box \frac{\partial u}{\partial n} d\mu - \frac{\eta_2}{2}\int_\Gamma k_2' \Box u d\mu. \end{aligned}$$

Remark 2.2. Since H_0 is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\lambda x) \leq \lambda H_0(x), \tag{2.8}$$

provided that $0 \leq \lambda \leq 1$, $x \in (0, r]$.

Lemma 2.3. [12] *Given $(u_0, u_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$, the energy of the solution of (1.1) satisfies*

$$\begin{aligned} E'(t) &\leq -\frac{\eta_2}{2}\|u_t\|_\Gamma^2 + \frac{\eta_2}{2}k_2'(t)\|u\|_\Gamma^2 + \frac{\eta_1}{2}P_1 k_1^2(t) + \frac{\eta_2}{2}P_2 k_2^2(t) - \frac{\eta_1}{2}\left\|\frac{\partial u_t}{\partial n}\right\|_\Gamma^2 \\ &\quad + \frac{\eta_1}{2}k_1'(t)\left\|\frac{\partial u}{\partial n}\right\|_\Gamma^2 - \frac{\eta_1}{2}\int_\Gamma k_1'' \Box \frac{\partial u}{\partial n} d\mu - \frac{\eta_2}{2}\int_\Gamma k_2'' \Box u d\mu, \end{aligned} \tag{2.9}$$

where

$$P_1 = \int_\Gamma \left|\frac{\partial u_0}{\partial n}\right|^2 d\mu, \text{ and } P_2 = \int_\Gamma |u_0|^2 d\mu. \tag{2.10}$$

Since $u_0 \neq 0$ on Γ, Lemma 2.3 says that $E(t)$ may not be nonincreasing. So, we introduce the modified energy functional $E_e(t)$ by

$$E_e(t) = E(t) + \frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\, da + \frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\, da. \tag{2.11}$$

Then, from (2.9), we have

$$E_e'(t) = E'(t) - \frac{\eta_1}{2}P_1 k_1^2(t) - \frac{\eta_2}{2}P_2 k_2^2(t) \leq 0. \tag{2.12}$$

Remark 2.4. If $u_0 = 0$ on Γ, then $E'(t) \leq 0$, hence $E'(t) \leq E(0)$. If $u_0 \neq 0$ on Γ, then

$$E(t) \leq E(0) + \frac{\eta_1}{2} P_1 k_1^2(t) + \frac{\eta_2}{2} P_2 k_2^2(t) < A, \tag{2.13}$$

for some $A > 0$.

3. Decay of solutions

In this section, we establish our main result. For this, we use the following lemma from [5, 14], cited without proof.

Lemma 3.1. *There exists $C > 0$ such that, for $N > 0$, the functional*

$$\mathcal{L}(t) = NE(t) + \Phi(t),$$

is equivalent to E_e and satisfies

$$\begin{aligned}\mathcal{L}'(t) \quad \leq \quad & -\beta E(t) + Ck_1^2(t) \int_\Gamma \left| \frac{\partial u_0}{\partial n} \right|^2 d\mu + Ck_2^2(t) \int_\Gamma |u_0|^2 d\mu \\ & -C \int_\Gamma k_1' \Box \frac{\partial u}{\partial n} d\mu - C \int_\Gamma k_2' \Box u d\mu, \end{aligned} \tag{3.1}$$

where

$$\Phi(t) = \int_\Omega \left\{ m(x) . \nabla u + \frac{1}{2} u \right\} u_t dx.$$

The following theorem establishes well-posedness and general decay for the von Kármán plate system with memory-type boundary conditions.

Theorem 3.2. *Given $(u_0, u_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$, we assume that (A) hold. Then there exist positive constants $\rho_0, \rho_1, \theta_0, C$ and t_0 such that:*
I) In the special case $H(t) = ct^p$, where $p \in \left[1, \frac{3}{2}\right)$, the solution of (1.1) satisfies

$$\begin{aligned} E(t) \leq E_e(0) & \left(\frac{\rho_0 (1 + \int_{t_0}^t ((P_1 k_1(a))^{2p-1} + (P_2 k_2(a))^{2p-1}) da)}{t}\right)^{\frac{1}{2p-1}} \\ & -\frac{\eta_1}{2} P_1 \int_t^\infty k_1^2(a) da - \frac{\eta_2}{2} P_2 \int_t^\infty k_2^2(a) da, \quad \textit{for all } t \geq t_0. \end{aligned} \tag{3.2}$$

II) In the general case, the solution corresponding to (1.1) satisfies

$$\begin{aligned} E(t) \leq E_e(0) & H_1^{-1}\left(\frac{\rho_1 (1 + \int_{t_0}^t (P_1 H_0(k_1(a)) + P_2 H_0(k_2(a))) da)}{t}\right) \\ & -\frac{\eta_1}{2} P_1 \int_t^\infty k_1^2(a) da - \frac{\eta_2}{2} P_2 \int_t^\infty k_2^2(a) da, \quad \textit{for all } t \geq t_0, \end{aligned} \tag{3.3}$$

where

$$H_1(a) = aH_0'(\theta_0 a), \quad H_0(a) = H(D(a)),$$

provided that D is a positive C^1 function satisfying $D(0) = 0$, and H_0 is a strictly increasing and strictly convex C^2 function defined on $(0, r]$, and

$$\int_0^{+\infty} \frac{-k_i'(a)}{H_0^{-1}(k_i''(a))} da < +\infty, \quad i = 1, 2. \tag{3.4}$$

Remark 3.3. 1. If $u_0 = 0$ in Γ, we then obtain the results in Kang [5].
2. If $\int_0^{+\infty} H_0(k_i(a))\,da < +\infty, (i = 1, 2)$, then (3.3) reduce to

$$E(t) \le k_0 H_1^{-1}\left(\frac{C}{t}\right) - \frac{\eta_1}{2} P_1 \int_t^{\infty} k_1^2(a)\,da - \frac{\eta_2}{2} P_2 \int_t^{\infty} k_2^2(a)\,da,$$

which clearly shows that $\lim_{t\to+\infty} E(t) = 0$, in this case.
3. The usual decay rate estimate, already proved for $k_i\,(i = 1, 2)$ satisfying the condition $k_i''(a) \ge \varkappa(-k_i'(a))^p$ with $p \in \left[1, \frac{3}{2}\right)$, is a special case of our result. We will provide a "simple" proof for this particular case.
4. The condition $k_i''(a) \ge \varkappa(-k_i'(a))^p, p \in \left[1, \frac{3}{2}\right)$, assumes $-k_i'(t) \le \varkappa e^{-\hat{c}t}$ when $p = 1$ and $-k_i'(t) \le \frac{\varkappa}{t^{\frac{1}{p-1}}}$, when $p \in \left(1, \frac{3}{2}\right)$.
5. For $k_1'(t) = k_2'(t) = -re^{-\sqrt{t}}, r \in (0, 1)$, we obtain

$$k_i''(t) = H(-k_i'(t)), (i = 1, 2) \text{ for } 0 < t \le r, \text{ and } r < 1,$$

where

$$H(t) = \frac{t}{2\ln\left(\frac{r}{t}\right)}.$$

Since

$$H'(t) = \frac{\ln\left(\frac{r}{t}\right) + 1}{2\left[\ln\left(\frac{r}{t}\right)\right]^2} \quad \text{and} \quad H''(t) = \frac{\ln\left(\frac{r}{t}\right) - 2}{2t\left[\ln\left(\frac{r}{t}\right)\right]^3},$$

the function H satisfies hypothesis (A) on the interval $(0, r]$ for any $0 < r < 1$. Furthermore, by choosing $D(t) = t^\gamma$, condition (3.4) is satisfied for any $\gamma > 0$. Consequently, an explicit rate of decay can be determined using Theorem 3.2. The function $H_0(t) = H(t^\gamma)$ possess derivative

$$H_0'(t) = \frac{\gamma t^{\gamma-1}\left[1 + \ln\left(\frac{r}{t^\gamma}\right)\right]}{2\left[\ln\left(\frac{r}{t^\gamma}\right)\right]^2}.$$

Subsequently, we perform straightforward calculations and apply (3.3) to conclude that

$$E(t) \le k_1\left(\frac{k_2 + k_3\int_{t_0}^t (P_1 H_0(k_1(a)) + P_2 H_0(k_2(a)))\,da}{t}\right)^{\frac{1}{2\gamma}}$$
$$-\frac{\eta_1}{2} P_1 \int_t^{\infty} k_1^2(a)\,da - \frac{\eta_2}{2} P_2 \int_t^{\infty} k_2^2(a)\,da, \quad \text{for all } t \ge t_0,$$

for each $\gamma > 1$, where

$$H_0(k_i(a)) = \frac{(k_i(a))^\gamma}{2\ln\left(\frac{r}{(k_i(a))^\gamma}\right)}, \quad i = 1, 2.$$

Hence, as $\gamma \to 1$, the energy diminishes at the following rate

$$E(t) \le k_1\left(\frac{k_2 + k_3\int_{t_0}^t (P_1 H_0(k_1(a)) + P_2 H_0(k_2(a)))\,da}{t}\right)^{\frac{1}{2}}$$
$$-\frac{\eta_1}{2} P_1 \int_t^{\infty} k_1^2(a)\,da - \frac{\eta_2}{2} P_2 \int_t^{\infty} k_2^2(a)\,da, \quad \text{for all } t \ge t_0. \quad (3.5)$$

If $\int_0^\infty H\left(k_i\left(a\right)\right)da < +\infty, \left(i = 1, 2\right)$, then the equation (3.5) reduce to

$$E\left(t\right) \leq \frac{C}{t^{\frac{1}{2}}} - \frac{\eta_1}{2}P_1\int_t^\infty k_1^2\left(a\right)da - \frac{\eta_2}{2}P_2\int_t^\infty k_2^2\left(a\right)da.$$

Proof of Theorem 3.2. Using (2.7), (2.12), and (3.1) with the notation (2.10), we obtain

$$\begin{aligned}\mathcal{L}'\left(t\right) &\leq -\beta E\left(t\right) - CE'\left(t\right) + CP_1k_1^2\left(t\right) + CP_2k_2^2\left(t\right)\\ &\quad -C\int_{t_0}^t k_1'\left(a\right)\int_\Gamma\left|u\left(t\right) - u\left(t-a\right)\right|^2 d\mu da\\ &\quad -C\int_{t_0}^t k_2'\left(a\right)\int_\Gamma\left|\frac{\partial u\left(t\right)}{\partial n} - \frac{\partial u\left(t-a\right)}{\partial n}\right|^2 d\mu da. \end{aligned}\tag{3.6}$$

We take $L(t) = \mathcal{L}(t) + CE(t)$, which is equivalent to $E(t)$, and we use (3.6) to obtain:

$$\begin{aligned}L'\left(t\right) &\leq -\beta E\left(t\right) + CP_1k_1^2\left(t\right) + CP_2k_2^2\left(t\right)\\ &\quad -C\int_{t_0}^t k_1'\left(a\right)\int_\Gamma\left|u\left(t\right) - u\left(t-a\right)\right|^2 d\mu da\\ &\quad -C\int_{t_0}^t k_2'\left(a\right)\int_\Gamma\left|\frac{\partial u\left(t\right)}{\partial n} - \frac{\partial u\left(t-a\right)}{\partial n}\right|^2 d\mu da. \end{aligned}\tag{3.7}$$

Similar to [5] we consider two cases:
I) The special case: $H\left(t\right) = ct^p$ and $p \in \left[1, \frac{3}{2}\right)$. If $p \in \left(1, \frac{3}{2}\right)$, one can easily show that $\int_0^\infty\left[-k_i'\left(a\right)\right]^{1-\delta_0}da < +\infty$ for any $\delta_0 < 2 - p$. Using (2.9) and (2.13) and choosing t_0 even larger if needed, we deduce that, for all $t \geq t_0$

$$\begin{aligned}I\left(t\right) &= \int_{t_0}^t\left(k_2'\left(a\right)\right)^{1-\delta_0}\int_\Gamma\left|u\left(t\right) - u\left(t-a\right)\right|^2 d\mu da\\ &\leq 2\int_{t_0}^t\left(k_2'\left(a\right)\right)^{1-\delta_0}\int_\Gamma\left[\left|u\left(t\right)\right|^2 + \left|u\left(t-a\right)\right|^2\right]d\mu da\\ &\leq cA\int_{t_0}^t\left(k_2'\left(a\right)\right)^{1-\delta_0}da < 1. \end{aligned}\tag{3.8}$$

Similarly, we obtain

$$\begin{aligned}J\left(t\right) &= \int_{t_0}^t\left(k_1'\left(a\right)\right)^{1-\delta_0}\int_\Gamma\left|\frac{\partial u\left(t\right)}{\partial n} - \frac{\partial u\left(t-a\right)}{\partial n}\right|^2 d\mu da\\ &\leq cA\int_{t_0}^t\left(k_1'\left(a\right)\right)^{1-\delta_0}da < 1. \end{aligned}\tag{3.9}$$

From Hölder inequality, Jensen's inequality, (2.3), (2.12) and (3.8), we get

$$\begin{aligned}
&\int_{t_0}^{t} \left(-k_2'(a)\right)^{1-\delta_0} \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da \\
&= \int_{t_0}^{t} \left(-k_2'(a)\right)^{\delta_0} \left(-k_2'(a)\right)^{1-\delta_0} \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da \\
&= \int_{t_0}^{t} \left(-k_2'(a)\right)^{1-p+\delta_0\left(\frac{\delta_0}{p-1+\delta_0}\right)} \left(-k_2'(a)\right)^{1-\delta_0} \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da \\
&\leq I(t) \left[\frac{1}{I(t)} \int_{t_0}^{t} \left(-k_2'(a)\right)^{p-1+\delta_0} \left(-k_2'(a)\right)^{1-\delta_0} \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\leq \left[\int_{t_0}^{t} \left(-k_2'(a)\right)^{p} \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\leq C \left[\int_{t_0}^{t} k_2''(a) \int_\Gamma |u(t)-u(t-a)|^2 \, d\mu da\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\leq C \left(-E_e'(t)\right)^{\frac{\delta_0}{p-1+\delta_0}}. \qquad (3.10)
\end{aligned}$$

Similarly, we obtain

$$\int_{t_0}^{t} \left(-k_1'(a)\right)^{1-\delta_0} \int_\Gamma \left|\frac{\partial u(t)}{\partial n} - \frac{\partial u(t-a)}{\partial n}\right|^2 d\mu da \leq C \left(-E_e'(t)\right)^{\frac{\delta_0}{p-1+\delta_0}}. \qquad (3.11)$$

Therefore, using (3.10), (3.11) we see that (3.7) yield for $\delta_0 = \frac{1}{2}$

$$L'(t) \leq -\beta E(t) + C\left(-E_e'(t)\right)^{\frac{1}{2p-1}} + CP_1 k_1^2(t) + CP_2 k_2^2(t), \quad \text{for all } t \geq t_0.$$

However,

$$\begin{aligned}
&\left[L(t) + \frac{\eta_1}{2} P_1 \int_t^\infty k_1^2(a)\, da + \frac{\eta_2}{2} P_2 \int_t^\infty k_2^2(a)\, da\right]' \\
\leq\ & L'(t) \\
\leq\ & -\beta E(t) + C\left(-E_e'(t)\right)^{\frac{1}{2p-1}} + CP_1 k_1^2(t) + CP_2 k_2^2(t),
\end{aligned}$$

hence, for all $t \geq t_0$, we have

$$\begin{aligned}
&\left[L(t) + \frac{\eta_1}{2} P_1 \int_t^\infty k_1^2(a)\, da + \frac{\eta_2}{2} P_2 \int_t^\infty k_2^2(a)\, da\right]' \\
\leq\ & -\beta E_e(t) + C\left(-E_e'(t)\right)^{\frac{1}{2p-1}} + CP_1 k_1^2(t) + CP_2 k_2^2(t) \\
& + \beta \frac{\eta_1}{2} P_1 \int_t^\infty k_1^2(a)\, da + \beta \frac{\eta_2}{2} P_2 \int_t^\infty k_2^2(a)\, da. \qquad (3.12)
\end{aligned}$$

Using assumption (A), for all $t \geq t_0$, we have

$$\int_t^\infty k_i^2(a)\, da \leq k_i(a) \int_0^\infty k_i(a)\, da, \quad \text{and} \quad k_i^2(a) \leq c' k_i(a), \qquad (3.13)$$

then (3.12) becomes, for some positive constant C,

$$\begin{aligned}&\left[L(t)+\frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\,da+\frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\,da\right]' \\ \leq\;& -\beta E_e(t)+C\left(-E_e'(t)\right)^{\frac{1}{2p-1}}+CP_1k_1(t)+CP_2k_2(t).\end{aligned} \tag{3.14}$$

Now we multiply by $E_e^{2p-2}(t)$, and using (2.12), we obtain

$$\begin{aligned}&\left[\left(L(t)+\frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\,da+\frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\,da\right)E_e^{2p-2}(t)\right]' \\ \leq\;& \left[L(t)+\frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\,da+\frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\,da\right]'E_e^{2p-2}(t) \\ \leq\;& -\beta E_e^{2p-1}(t)+CP_1k_1(t)E_e^{2p-2}(t)+CP_2k_2(t)E_e^{2p-2}(t) \\ &+CE_e^{2p-2}(t)\left(-E_e'(t)\right)^{\frac{1}{2p-1}}.\end{aligned}$$

Then, applying Young's inequality, with $q=2p-1$, and $q'=\frac{2p-1}{2p-2}$, for

$$E_e^{2p-2}(t)\left(-E_e'(t)\right)^{\frac{1}{2p-1}},$$

and

$$P_ik_i(t)E_e^{2p-2}(t),\quad \text{for } i=1,2.$$

We obtain for $C>0$,

$$\begin{aligned}&\left[\left(L(t)+\frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\,da+\frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\,da\right)E_e^{2p-2}(t)\right]' \\ \leq\;& -(\beta-3\sigma)E_e^{2p-1}(t)+C\left(P_1k_1(t)\right)^{2p-1}+C\left(P_2k_2(t)\right)^{2p-1}-C_\sigma E_e'(t).\end{aligned}$$

Consequently, for $3\sigma\leq\beta$, we obtain

$$L_0'(t)\leq-\beta' E_e^{2p-1}(t)+C\left(P_1k_1(t)\right)^{2p-1}+C\left(P_2k_2(t)\right)^{2p-1}.$$

where β' is some positive constant and

$$L_0(t)=\left(L(t)+\frac{\eta_1}{2}P_1\int_t^\infty k_1^2(a)\,da+\frac{\eta_2}{2}P_2\int_t^\infty k_2^2(a)\,da\right)E_e^{2p-2}(t)+C_\sigma E_e(t).$$

Also, it's easy to show that this inequality is true for $p=1$. Once again, we use (2.12) to deduce that

$$\begin{aligned}\left[tE_e^{2p-1}(t)\right]' &\leq E_e^{2p-1}(t) \\ &\leq -\frac{1}{\beta'}L_0'(t)+\frac{C}{\beta'}\left(P_1k_1(t)\right)^{2p-1}+\frac{C}{\beta'}\left(P_2k_2(t)\right)^{2p-1},\end{aligned}$$

for all $t\geq t_0$. A simple integration over (t_0,t) yields

$$\begin{aligned}tE_e^{2p-1}(t) &\leq \frac{1}{\beta'}L_0(t_0)+\frac{C}{\beta'}\int_{t_0}^t\left(\left(P_1k_1(a)\right)^{2p-1}+\left(P_2k_2(a)\right)^{2p-1}\right)da+t_0E_e^{2p-1}(t_0), \\ &\leq C_1+\frac{C}{\beta'}\int_{t_0}^t\left(\left(P_1k_1(a)\right)^{2p-1}+\left(P_2k_2(a)\right)^{2p-1}\right)da,\end{aligned}$$

hence,

$$E_e^{2p-1}(t) \leq \frac{C_1}{t} + \frac{C}{\beta' t}\int_{t_0}^{t}((P_1 k_1(a))^{2p-1} + (P_2 k_2(a))^{2p-1})da.$$

Therefore,

$$E_e(t) \leq \left(\frac{\rho_0(1+\int_{t_0}^{t}((P_1 k_1(a))^{2p-1} + (P_2 k_2(a))^{2p-1})da)}{t}\right)^{\frac{1}{2p-1}}. \tag{3.15}$$

II) The general case: As in [5], we define $\eta(t)$ and $\xi(t)$ by

$$\eta(t) := \int_{t_0}^{t} \frac{-k_2'(a)}{H_0^{-1}(k_2''(a))}\int_\Gamma |u(t)-u(t-a)|^2 d\mu da,$$
$$\xi(t) := \int_{t_0}^{t} \frac{-k_1'(a)}{H_0^{-1}(k_1''(a))}\int_\Gamma \left|\frac{\partial u(t)}{\partial n} - \frac{\partial u(t-a)}{\partial n}\right|^2 d\mu da,$$

where H_0 is such that (3.4) is satisfied. From (3.4), (2.12) and the trace theory, and choosing t_0 even larger if needed, we find that $\eta(t)$ and $\xi(t)$ satisfy

$$\eta(t) < 1, \quad \xi(t) < 1, \quad \forall t \geq t_0. \tag{3.16}$$

Additionally, we define $\kappa(t)$ and $\gamma(t)$ as follows:

$$\kappa(t) := \int_{t_0}^{t} k_2''(a)\frac{-k_2'(a)}{H_0^{-1}(k_2''(a))}\int_\Gamma |u(t)-u(t-a)|^2 d\mu da,$$
$$\gamma(t) := \int_{t_0}^{t} k_1''(a)\frac{-k_1'(a)}{H_0^{-1}(k_1''(a))}\int_\Gamma \left|\frac{\partial u(t)}{\partial n} - \frac{\partial u(t-a)}{\partial n}\right|^2 d\mu da.$$

By (2.3) and the properties of H_0 and D, we obtain

$$\frac{-k_i'(a)}{H_0^{-1}(k_i''(a))} \leq \frac{-k_i'(a)}{H_0^{-1}(H(-k_i'(a)))} = \frac{-k_i'(a)}{D^{-1}(-k_i'(a))} \leq \alpha_0, \tag{3.17}$$

for some positive constant α_0. Then using (2.4), (2.12) and (3.17) and choosing t_0 even larger, we can see that $\kappa(t)$ satisfies, for all $t \geq t_0$,

$$\kappa(t) \leq \alpha_0 \int_{t_0}^{t} k_2''(a)\int_\Gamma |u(t)-u(t-a)|^2 d\mu da \leq min\{r, H(r), H_0(r)\}. \tag{3.18}$$

Similarly, we deduce that

$$\gamma(t) \leq min\{r, H(r), H_0(r)\}.$$

Using (3.16), (2.8) and Jensen's inequality, we have

$$\begin{aligned}\kappa(t) &= \frac{1}{\eta(t)}\int_{t_0}^{t}\eta(t)H_0[H_0^{-1}(k_2''(a))]\frac{-k_2'(a)}{H_0^{-1}(k_2''(a))}\int_\Gamma |u(t)-u(t-a)|^2\,d\mu da\\ &\geq \frac{1}{\eta(t)}\int_{t_0}^{t}H_0[\eta(t)H_0^{-1}(k_2''(a))]\frac{-k_2'(a)}{H_0^{-1}(k_2''(a))}\int_\Gamma |u(t)-u(t-a)|^2\,d\mu da\\ &\geq H_0\left(\frac{1}{\eta(t)}\int_{t_0}^{t}\eta(t)H_0^{-1}(k_2''(a))\frac{-k_2'(a)}{H_0^{-1}(k_2''(a))}\int_\Gamma |u(t)-u(t-a)|^2\,d\mu da\right)\\ &\geq H_0\left(-\int_{t_0}^{t}k_2'(a)\int_\Gamma |u(t)-u(t-a)|^2\,d\mu da\right),\end{aligned}$$

this implies

$$-\int_{t_0}^{t}k_2'(a)\int_\Gamma |u(t)-u(t-a)|^2\,d\mu da \leq H_0^{-1}(\kappa(t)),\quad \forall t\geq t_0. \tag{3.19}$$

Similarly, we get

$$-\int_{t_0}^{t}k_1'(a)\int_\Gamma \left|\frac{\partial u(t)}{\partial n}-\frac{\partial u(t-a)}{\partial n}\right|^2 d\mu da \leq H_0^{-1}(\gamma(t)),\quad \forall t\geq t_0. \tag{3.20}$$

From (3.1), (3.19) and (3.20) we deduce that

$$\begin{aligned}\mathcal{L}'(t) &\leq -\beta E(t)+CP_1k_1^2(t)+CP_2k_2^2(t)+CH_0^{-1}(\kappa(t))+CH_0^{-1}(\gamma(t))\\ &\leq -\beta E(t)+CP_1k_1^2(t)+CP_2k_2^2(t)+CH_0^{-1}(\Theta(t)),\end{aligned} \tag{3.21}$$

where

$$\Theta(t)=\max\{\kappa(t),\gamma(t)\}\leq \min\{r,H(r),H_0(r)\}.$$

For $\theta_1>0$ and $\theta_0<r$, we define the functional,

$$G(t):=H_0'(\theta_0F(t))\mathcal{L}(t)+\theta_1E_e(t),$$

where $F(t)=\frac{E_e(t)}{E_e(0)}$, and $E_e(t)$ is the modified energy satisfies

$$\alpha_1G(t)\leq E_e(t)\leq \alpha_2G(t),$$

and

$$\begin{aligned}G'(t) &\leq \theta_0F'(t)H_0''(\theta_0F(t))\mathcal{L}(t)+H_0'(\theta_0F(t))\mathcal{L}'(t)+\theta_1E_e'(t)\\ &\leq H_0'(\theta_0F(t))\mathcal{L}'(t)+\theta_1E_e'(t).\end{aligned}$$

Hence using (3.21), we obtain for all $t\geq t_0$

$$\begin{aligned}G'(t) &\leq -\beta E(t)H_0'(\theta_0F(t))+C\left(P_1k_1^2(t)+P_2k_2^2(t)\right)H_0'(\theta_0F(t))\\ &\quad +CH_0'(\theta_0F(t))H_0^{-1}(\Theta(t))+\theta_1E_e'(t).\end{aligned} \tag{3.22}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young

$$H_0^*\left(\tilde{\vartheta}\right)=\tilde{\vartheta}\left(H_0'\right)^{-1}\left(\tilde{\vartheta}\right)-H_0\left[\left(H_0'\right)^{-1}\left(\tilde{\vartheta}\right)\right],\quad \text{if } \tilde{\vartheta}\in(0,H_0'(r)] \tag{3.23}$$

and H_0^* satisfies the following Young's inequality

$$\vartheta\tilde{\vartheta}\leq H_0(\vartheta)+H_0^*(\tilde{\vartheta}),\quad (\vartheta,\tilde{\vartheta})\in(0,r]\times(0,H_0'(r)]. \tag{3.24}$$

Using (3.23) and (3.24) with

$$\vartheta = H_0^{-1}(\Theta(t)) \text{ and } \tilde{\vartheta} = H_0'(\theta_0 F(t)).$$

So (3.22) becomes

$$\begin{aligned} G'(t) &\leq -\beta E(t) H_0'(\theta_0 F(t)) + C\left(P_1 k_1^2(t) + P_2 k_2^2(t)\right) H_0'(\theta_0 F(t)) \\ &\quad + C H_0^*(H_0'(\theta_0 F(t))) + C H_0(H_0^{-1}(\Theta(t))) + \theta_1 E_e'(t) \\ &\leq -\beta E(t) H_0'(\theta_0 F(t)) + C(P_1 k_1^2(t) + P_2 k_2^2(t)) H_0'(\theta_0 F(t)) \\ &\quad + C F(t) H_0'(\theta_0 F(t)) - (C - \theta_1) E_e'(t). \end{aligned}$$

Thus, by selecting θ_1 appropriately, we have that, for all $t \geq t_0$,

$$\begin{aligned} G'(t) &\leq -\beta E_e(t) H_0'(\theta_0 F(t)) + C\theta_0 F(t) H_0'(\theta_0 F(t)) \\ &\quad + C\left(P_1 k_1^2(t) + P_2 k_2^2(t)\right) H_0'(\theta_0 F(t)) \\ &\quad + \beta \frac{\eta_1}{2} P_1 \left(\int_t^\infty k_1^2(a)\, da\right) H_0'(\theta_0 F(t)) \\ &\quad + \beta \frac{\eta_2}{2} P_2 \left(\int_t^\infty k_2^2(a)\, da\right) H_0'(\theta_0 F(t)). \end{aligned} \tag{3.25}$$

Then, using (3.13), for some positive constant C, (3.25) becomes

$$\begin{aligned} G'(t) &\leq -\beta E_e(t) H_0'(\theta_0 F(t)) + C\theta_0 F(t) H_0'(\theta_0 F(t)) \\ &\quad + C(P_1 k_1(t) + P_2 k_2(t)) H_0'(\theta_0 F(t)). \end{aligned} \tag{3.26}$$

Similarly, using (3.23) and (3.24), we find that, for t_0 even larger (if needed),

$$k_i(t) H_0'(\theta_0 F(t)) \leq \theta_0 F(t) H_0'(\theta_0 F(t)) + H_0(k_i(t)), \quad i = 1, 2.$$

So with a suitable choice of θ_0, (3.26) becomes

$$\begin{aligned} G'(t) &\leq -P_3 F(t) H_0'(\theta_0 F(t)) + C P_1 H_0(k_1(t)) + C P_2 H_0(k_2(t)) \\ &= -P_3 H_1'(F(t)) + C P_1 H_0(k_1(t)) + C P_2 H_0(k_2(t)), \end{aligned} \tag{3.27}$$

where $H_1(t) = t H_0'(\theta_0 t)$.

Given that $H_1'(t) = H_0'(\theta_0 t) + \theta_0 t H_0''(\theta_0 t)$, and considering the strict convexity of H_0 on $(0, r]$, it follows that $H_1'(t)$ and $H_1(t) > 0$ for $t \in (0, 1]$. Therefore, taking into account that

$$E'(t) \leq \frac{\eta_1}{2} P_1 k_1^2(t) + \frac{\eta_2}{2} P_2 k_2^2(t),$$

and (3.27) for all $t \geq t_0$, we have

$$\begin{aligned} [t H_1(F(t))]' &\leq H_1(F(t)) \\ &\leq -\frac{1}{P_3} G'(t) + \frac{C}{P_3}(P_1 H_0(k_1(t)) + P_2 H_0(k_2(t))). \end{aligned}$$

A direct integration over (t_0, t) result in

$$\begin{aligned} t H_1(F(t)) &\leq t_0 H_1(F(t_0)) \\ &\quad + \frac{1}{P_3} G'(t_0) + \frac{C}{P_3} \int_{t_0}^t (P_1 H_0(k_1(a)) + P_2 H_0(k_2(a)))\, da. \end{aligned}$$

This implies, for some positive constant k_2 and for all $t \geq t_0$,

$$H_1\left(F\left(t\right)\right) \leq \frac{k_2}{t} + \frac{C}{P_3 t}\int_{t_0}^{t}\left(P_1 H_0\left(k_1\left(a\right)\right) + P_2 H_0\left(k_2\left(a\right)\right)\right) da.$$

Therefore, for some positive constant ρ_1, we obtain

$$F\left(t\right) \leq H_1^{-1}\left(\frac{\rho_1(1 + \int_{t_0}^{t}\left(P_1 H_0\left(k_1\left(a\right)\right) + P_2 H_0\left(k_2\left(a\right)\right)\right) da)}{t}\right).$$

Hence, applying (2.11), (3.3) is established. □

4. Conclusion

In this paper, we consider a von Kármán plate system with viscoelastic damping localized on the boundary. We establish general energy decay rates using the multiplier method and properties of convex functions, particularly for cases where $u_0 \neq 0$ on Γ, thereby extending previous results such as those in Kang [5]. The study of von Kármán systems with memory effects is an important area in the theory of PDEs, with applications in mechanics. The use of the multiplier method combined with properties of convex functions related to the function H governing the decay of resolvent kernels is a standard and powerful approach for establishing energy decay rates. This paper provides decay rates for both a common class of polynomial H functions and more general forms of H, thereby offering broader applicability. The work represents a solid piece of analysis.

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