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ON HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS VIA ABC DERIVATIVE

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Abstract. In this article, we study the existence of solutions to a fractional hybrid integro-differential equation with nonlocal boundary conditions. Our approach is based on a generalized version of Dhage's hybrid fixed point theorem, adapted to the sum of three fractional operators. To illustrate the applicability of our main result, we conclude with a concrete example, accompanied by numerical results and graphical representations generated using Python.

1. Introduction

Fractional differential equations emerge in the mathematical modeling of systems and processes in various engineering and scientific fields, including physics, chemistry, aerodynamics, electrodynamics of complex media, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, and more (see[2, 9, 6, 21, 17]). This represents a key advantage of fractional differential equations over classical integer-order models. In particular, boundary value problems involving nonlinear fractional differential equations have garnered significant attention in recent years, owing to their ability to effectively model and describe non-homogeneous physical phenomena with complex structural behavior.

Hybrid fractional differential equations have also been explored by several researchers. This class of equations involves the fractional derivative of an unknown function, combined with a nonlinearity that depends on it. Dhage and Lakshmikantham [8] initiated the study of first-order hybrid differential equations and established fundamental results concerning the existence and uniqueness of solutions. In addition, they developed differential inequalities associated with hybrid ordinary differential equations (ODEs), which were employed to derive comparison results and investigate qualitative properties of solutions. Building upon the approach in [8], Zhao et al. [28] extended the analysis of first-order hybrid differential equations to encompass hybrid fractional differential equations (FDEs) involving the Riemann–Liouville (RL) fractional derivative. Additionally, various types of hybrid FDEs under different initial and boundary conditions have been investigated by several researchers [1]-[26].

Motivated by the above works, in this manuscript we develop the theory of nonlocal nonlinear ABC-hybrid FDEs of the form

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$$\begin{cases} {}^{ABC}D_a^{\nu}\left(\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}\right)=\chi(\xi,\sigma(\xi)), \xi\in\mathcal{T}=[a,b],\\ \frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}=\sigma_0+\lambda(\sigma) \end{cases} \tag{1.1}$$

where ${}^{ABC}D_a^{\nu}$ denotes the left Caputo AB-derivative of order ν, with $0<\nu<1$, $\sigma_0\in\mathbb{R}$, $\sigma\in C(\mathcal{T})$, $\phi\in C(\mathcal{T}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\varphi\in C(\mathcal{T}\times\mathbb{R},\mathbb{R})$, ${}^{ABC}D_a^{\nu}\left(\frac{\sigma-\varphi(\cdot,\sigma)}{\phi(\cdot,\sigma)}\right)\in C(\mathcal{T}\times\mathbb{R},\mathbb{R})$, $\chi\in C(\mathcal{T}\times\mathbb{R},\mathbb{R})$ and $\chi(a,\sigma(a))=0$.

This paper is structured as follows: In Section 2, we review essential definitions and key results concerning the ABC-fractional derivative. Section 3 is devoted to establishing equivalent fractional integral equations and deriving an existence result. As an application, an illustrative example with numerical results and figures is presented in Section 4 to demonstrate the main findings.

2. Preliminary results

Let $\mathcal{T}=[a,b]$ be an interval in $\mathbb{R}$, $\mathcal{C}=C(\mathcal{T},\mathbb{R})$ be the Banach space of continuous real-valued function with the supremum norm $\|\sigma\|=\sup_{\xi\in\mathcal{T}}|\sigma(\xi)|$, and the multiplication action on $\mathcal{C}$ by $(\sigma_1\cdot\sigma_2)(\xi)=\sigma_1(\xi)\sigma_2(\xi)$ for all σ_1, $\sigma_2\in\mathcal{C}$. Then an ordered triple $(\mathcal{C};\|\cdot\|;\cdot)$ is a Banach algebra.

In the following, we recall the basic definitions and results that will be used later.

Definition 2.1. [27] Let $p\in[1,+\infty)$, the Sobolev space $H^p(a,b)$ is defined by

$$H^p(a,b)=\{f\in L^2(a,b): D^{\beta}f\in L^2(a,b),\quad \text{for all } |\beta|\le p\}$$

Definition 2.2. [3, 4] Let $\sigma\in H^1(a,b)$, where $a\le b$, and let $0<\nu<1$. The Caputo Atangana–Baleanu fractional derivative of order ν is defined as

$$\left({}^{ABC}D_a^{\nu}\sigma\right)(\xi)=\frac{B(\nu)}{1-\nu}\int_a^{\xi}\sigma'(z)E_{\nu}\left(-\nu\frac{(\xi-z)^{\nu}}{1-\nu}\right)dz$$

where E_ν is the Mittag-Leffler function defined by $E_\nu(\xi)=\Sigma_{n=0}^{\infty}\frac{\xi^n}{\Gamma(n\nu+1)}$ and $B(\nu)$ is a normalizing positive function satisfying $B(0)=B(1)=1$. The left Riemann Atangana–Baleanu fractional derivative of σ of order ν is defined by

$$\left({}^{ABR}D_a^{\nu}\sigma\right)(\xi)=\frac{B(\nu)}{1-\nu}\frac{d}{d\xi}\int_a^{\xi}\sigma(z)E_{\nu}\left(-\nu\frac{(\xi-z)^{\nu}}{1-\nu}\right)dz,$$

The associated fractional integral is defined by

$${}^{AB}I_a^{\nu}(\xi)=\frac{1-\nu}{B(\nu)}\sigma(\xi)+\frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^{\xi}(\xi-z)^{\nu-1}\sigma(z)dz$$

Theorem 2.3. [3] *Let $0\le\nu\le1$ and $\sigma\in H^1(a,b)$. Then the following relation holds.*

$$\left({}^{ABC}D_a^{\nu}\sigma\right)(\xi)=\left({}_a^{ABR}D^{\nu}\sigma\right)(\xi)-\frac{B(\nu)}{1-\nu}\sigma(a)E_{\nu}\left(-\nu\frac{(\xi-a)^{\nu}}{1-\nu}\right),$$

Lemma 2.4. [3, 4] *For $0<\nu<1$and $\sigma\in H^1(a,b)$, then we have*

$$\left({}^{AB}I_a^{\nu}\left({}^{ABC}D_a^{\nu}\sigma\right)\right)(\xi)=\sigma(\xi)-\sigma(a)$$

Theorem 2.5. *([7] B.C. Dhage) Let $\mathcal{K}$ be a nonempty, closed convex and bounded subset of a Banach algebra $\mathcal{V}$ and let $\mathfrak{T}_1,\mathfrak{T}_3:\mathcal{V}\longrightarrow\mathcal{V}$ and $\mathfrak{T}_2:\mathcal{K}\longrightarrow\mathcal{V}$ be three operators satisfying the following conditions:*

(1) *$\mathfrak{T}_1$ and $\mathfrak{T}_3$ are Lipschitzian with Lipschitz constants Λ and Λ^*, respectively,*

(2) $\mathfrak{T}_2$ *is compact and continuous,*
(3) $\Lambda M+\Lambda^*<1$, *where* $M=\sup\{\|\mathfrak{T}_2(\sigma)\|:\sigma\in\mathcal{K}\}$.
(4) $\sigma=\mathfrak{T}_1\sigma\mathfrak{T}_2\mu+\mathfrak{T}_3\sigma\Rightarrow\sigma\in\mathcal{K}$, *for all* $\mu\in\mathcal{K}$.

Then the operator equation $\mathfrak{T}_1\sigma\mathfrak{T}_2\sigma+\mathfrak{T}_3\sigma=\sigma$ *has a solution in* $\mathcal{K}$.

3. Main results

Before presenting the main result of this paper, we begin by defining what is meant by a solution to problem (1.1) and by establishing the fundamental Lemma 3.2.

Definition 3.1. A function σ belonging to $\mathcal{C}$ such that ${}^{ABC}D_a^\nu\left(\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}\right)\in\mathcal{C}$ is considered a solution to the problem (1.1) if satisfies the equation ${}^{ABC}D_a^\nu\left(\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}\right)=\chi(\xi,\sigma(\xi))$ and the condition
$\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}=\sigma_0+\lambda(\sigma)$.

Lemma 3.2. *A function $\vartheta\in\mathcal{C}$ satisfies (1.1) if and only if ϑ satisfies the following fractional integral equation*

$$\sigma(\xi)=\varphi(\xi,\sigma(\xi))+\phi(\xi,\sigma(\xi))\left[\sigma_0+\lambda(\sigma)+\frac{1-\nu}{B(\nu)}\chi(\xi,\sigma(\xi))+\frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^\xi(\xi-z)^{\nu-1}\chi(z,\sigma(z))dz\right]. \tag{3.1}$$

Proof. First, we assume that $\vartheta\in\mathcal{C}$ satisfies (1.1). Applying the AB-fractional integral to both sides of (1.1) and making use of Lemma 2.4, we get

$$\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}-\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}=\frac{1-\nu}{B(\nu)}\chi(\xi,\sigma(\xi))+\frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^\xi(\xi-z)^{\nu-1}\chi(z,\sigma(z))dz \tag{3.2}$$

Therefore, by substituting the previously defined value of $\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}$, we get (3.1). Now, assuming that σ satisfies (3.1) putting $\xi=a$ in (3.1), since $\chi(a,\sigma(a))=0$ it is evidently that $\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}=\sigma_0+\lambda(\sigma)$, then (3.1) can be written in the form of (3.2). Applying ${}^{ABR}D_a^\nu$ to both sides of (3.2), we obtain

$$\begin{aligned}{}^{ABR}D_a^\nu\left(\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}\right)&={}^{ABR}D_a^\nu\left(\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}\right)+\chi(\xi,\sigma(\xi))\\&=\left(\frac{\sigma(a)-\varphi(a,\sigma(a))}{\phi(a,\sigma(a))}\right)\frac{B(\nu)}{1-\nu}E_\nu\left(-\frac{\nu}{1-\nu}(\xi-a)^\nu\right)\end{aligned}$$

By benefiting from Theorem 2.3, we get ${}^{ABC}D_a^\nu\left(\frac{\sigma(\xi)-\varphi(\xi,\sigma(\xi))}{\phi(\xi,\sigma(\xi))}\right)=\chi(\xi,\sigma(\xi))$, which establishes the desired result. This completes the proof. □

The following existence result for problem (1.1) is based on fixed point techniques. To establish the result, we require the following hypotheses:

$\mathcal{H}_1$) There exist two constants $\Lambda_\phi>0,\Lambda_\varphi>0$ such that for all ξ and $\sigma_1,\sigma_2\in\mathcal{C}$

$$|\phi(\xi,\sigma_1)-\phi(\xi,\sigma_2)|\le\Lambda_\phi|\sigma_1-\sigma_2| \text{ and } |\varphi(\xi,\sigma_1)-\varphi(\xi,\sigma_2)|\le\Lambda_\varphi|\sigma_1-\sigma_2|$$

$\mathcal{H}_2$) The function $\lambda:\mathcal{C}\longrightarrow\mathbb{R}$ is continuous and bounded. That is, there exists a constant $\lambda^*>0$ such that

$$|\lambda(\sigma)|\le\lambda^*$$

$\mathcal{H}_3$) The function $\chi \in \mathcal{C}([a,b]\times\mathbb{R},\mathbb{R})$ and there exist $\varpi \in \mathcal{C}$ and a nonincreasing function $\Omega \in C(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$|\chi(\xi,\sigma)| \leq |\varpi(\xi)|\Omega(|\sigma|) \text{ for each } \xi \in \mathcal{T} \text{ and all } \sigma \text{ in } \mathcal{C}.$$

$\mathcal{H}_4$) There exist $\varrho > 0$ such that

$$\varrho \geq \frac{\varphi^* + \phi^*\left(\sigma_0 + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)}{1-\Lambda_\varphi - \Lambda_\phi\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)} \tag{3.3}$$

where $\varphi^* = \sup_{\xi\in[a,b]}|\varphi(\xi,0)|, \phi^* = \sup_{\xi\in[a,b]}|\phi(\xi,0)|$, and $G^* = \frac{1-\nu}{B(\nu)} + \frac{(b-a)^\nu}{B(\nu)\Gamma(\nu)}$

In the following, we consider three operators $\mathfrak{T}_1,\mathfrak{T}_2,\mathfrak{T}_3 : \mathcal{C}\to\mathcal{C}$, defined as follows:

$$\mathfrak{T}_1\sigma(\xi) = \phi(\xi,\sigma(\xi)), \quad \xi\in[a,b] \tag{3.4}$$

$$\mathfrak{T}_2\sigma(\xi) = \sigma_0 + \lambda(\sigma) + \frac{1-\nu}{B(\nu)}\chi(\xi,\sigma(\xi)) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^\xi(\xi-z)^{\nu-1}\chi(z,\sigma(z))dz, \quad \xi\in[a,b] \tag{3.5}$$

$$\mathfrak{T}_3\sigma(\xi) = \varphi(\xi,\sigma(\xi)), \quad \xi\in[a,b] \tag{3.6}$$

We also consider the ball

$$\mathcal{B}_\varrho = \{\sigma\in\mathcal{C}/\|\sigma\|\leq\varrho\} \text{ where } \varrho>0 \text{ satisfy the inequality (3.3)}$$

Lemma 3.3. *The operator $\mathfrak{T}_2$ is continuous on $\mathcal{B}_\varrho$ and satisfies the following inequality*

$$\|\mathfrak{T}_2\sigma\| \leq |\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)$$

Proof. Let $(\sigma_n)_n$ be convergence sequence to σ as $n\longrightarrow\infty$ in $\mathcal{B}_\varrho$.

$$\begin{aligned}|\mathfrak{T}_2\sigma_n - \mathfrak{T}_2\sigma| \leq{}& |\lambda(\sigma_n)-\lambda(\sigma)| + \frac{1-\nu}{B(\nu)}|\chi(\xi,\sigma_n(\xi)) - \chi(\xi,\sigma(\xi))| \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^\xi(\xi-z)^{\nu-1}|\chi(z,\sigma_n(z)) - \chi(z,\sigma(z))|dz\end{aligned}$$

taking the continuity of the functions χ and λ into consideration, we get

$$\lim_{n\longrightarrow+\infty}\chi(z,\sigma_n(z)) = \chi(z,\sigma(z)) \text{ and } \lim_{n\longrightarrow+\infty}\lambda(\sigma_n) = \lambda(\sigma)$$

On the other hand, the hypothesis $\mathcal{H}_3$) has been employed

$$(\xi-z)^{\nu-1}|\chi(z,\sigma_n(z)) - \chi(z,\sigma(z))| \leq 2\|\varpi\|\Omega(\varrho)(\xi-z)^{\nu-1}$$

We notice that since the function $z\mapsto 2\|\varpi\|\Omega(\varrho)(\xi-z)^{\nu-1}$ is Lebesgue integrable over $[a,\xi]$. The dominated convergence theorem ensures that

$$\int_a^\xi(\xi-z)^{\nu-1}\left(\chi(z,\sigma_n(z)) - \chi(z,\sigma(z))\right)dz \xrightarrow{n\to\infty} 0$$

It follows that $\|\mathfrak{T}_2\sigma_n - \mathfrak{T}_2\sigma\|\longrightarrow 0$ as $n\longrightarrow\infty$, which implies the continuity of the operator $\mathfrak{T}_2$.

Next, we show that $\mathfrak{T}_2$ is bounded. For each $\sigma\in\mathcal{B}_\varrho$, we have

$$\begin{aligned}|\mathfrak{T}_2\sigma(\xi)| &\leq |\sigma_0| + |\lambda(\sigma)| + \frac{1-\nu}{B(\nu)}|\chi(\xi,\sigma(\xi))| + \frac{\nu}{B(\nu)\Gamma(\nu)}\left|\int_a^\xi(\xi-z)^{\nu-1}\chi(z,\sigma(z))dz\right| \\ &\leq |\sigma_0| + \lambda^* + \left(\frac{1-\nu}{B(\nu)} + \frac{(b-a)^\nu}{B(\nu)\Gamma(\nu)}\right)\|\varpi\|\Omega(\varrho) \\ &\leq |\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\end{aligned}$$

□

Theorem 3.4. *Assume that the hypotheses $\mathcal{H}_1) - \mathcal{H}_4)$ hold, and moreover, that*

$$\Lambda_\varphi + \Lambda_\phi \left(|\sigma_0| + \lambda^* + G^* \|\varpi\| \Omega(\varrho)\right) < 1 \tag{3.7}$$

Then the nonlocal fractional problem (1.1) has at least one solution $\sigma \in \mathcal{C}$. Moreover, the set of solutions of (1.1) is bounded in $\mathcal{C}$.

Proof. Consider the set $\mathcal{B}_\varrho$ and the operators $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ defined previously Clearly, $\mathcal{B}_\varrho$ is a bounded, closed and convex subset of $\mathcal{C}$.

By using the hypothesis $\mathcal{H}_1)$, it is easy to show that $\mathfrak{T}_1$ and $\mathfrak{T}_3$ are Lipschitzian with Lipschitz constants Λ_ϕ and Λ_φ, respectively. In order to show the compactness of $\mathfrak{T}_2$ we need to show that $\mathfrak{T}_2 \mathcal{B}_\varrho$ is relatively compact in $\mathcal{C}$ and we use the Arzelà-Ascoli Theorem [13]. For this purpose we prove that $\mathfrak{T}_2$ is equicontinuous, for $a \leq \xi_1 \leq \xi_2 \leq b$, we have

$$\begin{aligned}
|\mathfrak{T}_2\sigma(\xi_1) - \mathfrak{T}_2\sigma(\xi_2)| \leq & \frac{1-\nu}{B(\nu)} |\chi(\xi_1, \sigma(\xi_1)) - \chi(\xi_2, \sigma(\xi_2))| \\
& + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^{\xi_1} \left| (\xi_1 - z)^{\nu-1} - (\xi_2 - z)^{\nu-1} \right| |\chi(z, \sigma(z))| dz \\
& + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_{\xi_1}^{\xi_2} (\xi_2 - z)^{\nu-1} |\chi(z, \sigma(z))| dz \\
\leq & \frac{1-\nu}{B(\nu)} |\chi(\xi_1, \sigma(\xi_1)) - \chi(\xi_2, \sigma(\xi_2))| \\
& + \frac{\|\varpi\| \Omega(\varrho)}{B(\nu)\Gamma(\nu)} \left(|(\xi_2 - a)^\nu - (\xi_1 - a)^\nu| + 2 |(\xi_2 - \xi_1)^\nu| \right)
\end{aligned}$$

From the continuity of χ, we get

$$\lim_{\xi_1 \to \xi_2} |\mathfrak{T}_2\sigma(\xi_2) - \mathfrak{T}_2\sigma(\xi_1)| = 0$$

So, $\mathfrak{T}_2$ is equicontinuous on $\mathcal{B}_\varrho$. Moreover, by using Lemma 3.3 $\mathfrak{T}_2$ is continuous and bounded. Thus, the Arzelà-Ascoli theorem yields that the mapping $\mathfrak{T}_2$ is compact on $\mathcal{B}_\varrho$.

We must now demonstrate the last condition of Theorem 2.5. Let $\sigma \in \mathcal{C}$ and $\mu \in \mathcal{B}_\varrho$, such that

$$\sigma = \mathfrak{T}_1(\sigma)\mathfrak{T}_2(\mu) + \mathfrak{T}_3(\sigma).$$

We aim to show that $\sigma \in \mathcal{B}_\varrho$, we have

$$\begin{aligned}
|\sigma(\xi)| = & \Bigg| \varphi(\xi, \sigma(\xi)) + \phi(\xi, \sigma(\xi)) \left(\sigma_0 + \lambda(\sigma) + \frac{1-\nu}{B(\nu)} \chi(\xi, \mu(\xi)) \right. \\
& \left. + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^\xi (\xi - z)^{\nu-1} \chi(z, \mu(z)) dz \right) \Bigg| \\
& \leq |\varphi(\xi, \sigma(\xi))| + |\phi(\xi, \sigma(\xi))| \left(|\sigma_0| + |\lambda(\sigma)| + \frac{1-\nu}{B(\nu)} |\chi(\xi, \mu(\xi))| \right. \\
& \left. + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^\xi (\xi - z)^{\nu-1} |\chi(z, \mu(z))| dz \right) \\
& \leq \Lambda_\varphi |\sigma(\xi)| + |\varphi(\xi, 0)| + (\Lambda_\phi |\sigma(\xi)| + |\phi(\xi, 0)|) \left(|\sigma_0| + \lambda^* + \left(\frac{1-\nu}{B(\nu)} + \frac{(b-a)^\nu}{B(\nu)\Gamma(\nu)} \right) \|\varpi\| \Omega(\varrho) \right) \\
& \leq \Lambda_\varphi |\sigma(\xi)| + \varphi^* + (\Lambda_\phi |\sigma(\xi)| + \phi^*) \left(|\sigma_0| + \lambda^* + G^* \|\varpi\| \Omega(\varrho) \right)
\end{aligned}$$

Taking supremum over ξ

$$\|\sigma\| \leq \Lambda_{\varphi}\|\sigma\| + \varphi^* + (\Lambda_{\phi}\|\sigma\| + \phi^*)\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)$$

Thus

$$\|\sigma\| \leq \frac{\varphi^* + \phi^*\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)}{1 - \Lambda_{\varphi} - \Lambda_{\phi}\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)}$$

Using hypothesis $\mathcal{H}_4$), we have $\sigma \in \mathcal{B}_{\varrho}$.

Lastly, we show that $\Lambda_{\phi}M + \Lambda_{\varphi} < 1$. Let $\sigma \in \mathcal{B}_{\varrho}$, then we have

$$\begin{aligned}|\mathfrak{T}_2\sigma(\xi)| &\leq |\sigma_0| + |\lambda(\sigma)| + \frac{1-\nu}{B(\nu)}\|\varpi\|\Omega(\varrho) + \frac{1}{B(\nu)\Gamma(\nu)}(b-a)^{\nu}\|\varpi\|\Omega(\varrho)\\ &\leq |\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\end{aligned}$$

Thus $M = \sup_{\sigma\in\mathcal{B}_{\varrho}}\sup_{a\leq\xi\leq b}|B\sigma(\xi)| \leq |\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)$. Then, by the inequality (3.7), the argument is complete. Hence, as a consequence of the Theorem 2.5, the operator equation $\sigma = \mathfrak{T}_1\sigma\mathfrak{T}_2\sigma + \mathfrak{T}_3\sigma$ has a solution in $\mathcal{B}_{\varrho}$. As a result, the problem (1.1) has a solution on $\mathcal{T}$. □

4. An illustrative example

In this section, we provide an example to illustrate our main result. Consider the following Nonlinear fractional problem

$$\begin{cases} {}^{ABC}D_0^{\nu}\left(\frac{\sigma(\xi) - \frac{\sigma(\xi)}{(t+7)(|\sigma(\xi)|+1)} - \cos(\sqrt{\xi}-17)+2}{1+\exp(-\xi-5)|\sigma(\xi)|}\right) = \sigma^2(\xi)\frac{\ln(3+\xi)}{6+25\xi}, \quad \xi\in[0,4]\\ \left.\left(\frac{\sigma(\xi) - \frac{\sigma}{(t+7)(|\sigma(\xi)|+1)} - \cos(\sqrt{\xi}-17)+2}{1+\exp(-\xi-5)|\sigma(\xi)|}\right)\right|_{\xi=0} = \pi + \frac{\sigma(0)}{|\sigma(0)|+1}\end{cases} \tag{4.1}$$

In this example we set $a = 0$, $b = 4$, $\nu = \frac{1}{4}$, $\sigma_0 = \pi$, $\lambda(\sigma) = \frac{\sigma(0)}{|\sigma(0)|+1}$, $B(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)}$, $\chi(\xi,\sigma(\xi)) = \sigma^2(\xi)\frac{\ln(3+\xi)}{6+25\xi}$, $\varphi(\xi,\sigma(\xi)) = \frac{\sigma(\xi)}{(\xi+7)(|\sigma(\xi)|+1)} - \cos(\sqrt{\xi}-17)+2$, $\phi(\xi,\sigma(\xi)) = 1+\exp(-\xi-5)|\sigma(\xi)|$

We can show that

$$\begin{aligned}|\left(\phi(\xi,\sigma_1(\xi)) - \phi(\xi,\sigma_2(\xi)\right)|| &\leq \exp(-\xi-5)||\sigma_1(\xi)| - |\sigma_2(\xi)||\\ &\leq \exp(-5)\,|\sigma_1(\xi) - \sigma_2(\xi)|\end{aligned}$$

and

$$\begin{aligned}|\varphi(\xi,\sigma_1(\xi)) - \varphi(\xi,\sigma_2(\xi))|\,| &\leq \left|\frac{\sigma_1(\xi)}{(\xi+7)(|\sigma_1(\xi)|+1)} - \frac{\sigma_2(\xi)}{(\xi+7)(|\sigma_2(\xi)|+1)}\right|\\ &\leq \frac{1}{7}\,|\sigma_1(\xi) - \sigma_2(\xi)|\,.\end{aligned}$$

It follows that $\Lambda_{\varphi} = \frac{1}{7}$, $\Lambda_{\phi} = \exp(-5)$, $\varphi* = \sup_{\xi\in[0,4]}|\varphi(\xi,0)| = 3$, $\phi* = \sup_{\xi\in[0,4]}|\phi(\xi,0)| = e^{-5}$, hence

$$|\chi(\xi,\sigma(\xi))| = \left|\sigma^2(\xi)\frac{\ln(3+\xi)}{6+25\xi}\right| \leq \|\varpi\|\Omega(\|\sigma\|)$$

where $\varpi(\xi) = \frac{\ln(3+\xi)}{6+25\xi}$, $\Omega(\sigma) = \sigma^2$, it is easy to verify that $\|\varpi\| = 0.073$.

Taking $\varrho = 10$, we obtain

$$\Lambda_{\varphi} + \Lambda_{\phi}\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right) = 0.2396 < 1$$

and

$$\Phi(\varrho) = \frac{\varphi^* + \phi^*\left(\sigma_0 + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)}{1 - \Lambda_{\varphi} - \Lambda_{\phi}\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)} = 8.668 \leq 10,$$

according to Theorem 3.4, the problem (4.1) has at least one solution $\vartheta \in \mathcal{C}$ such that $\|\vartheta\| \leq 10$. Moreover, the graphical representation in Figure 1 shows that $\Lambda_\varphi + \Lambda_\phi\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)$ is less than 1 and the Figure 2 shows that $\Phi(\varrho) \leq \varrho$ at various values of ν when $\varrho \in [0, 20]$.

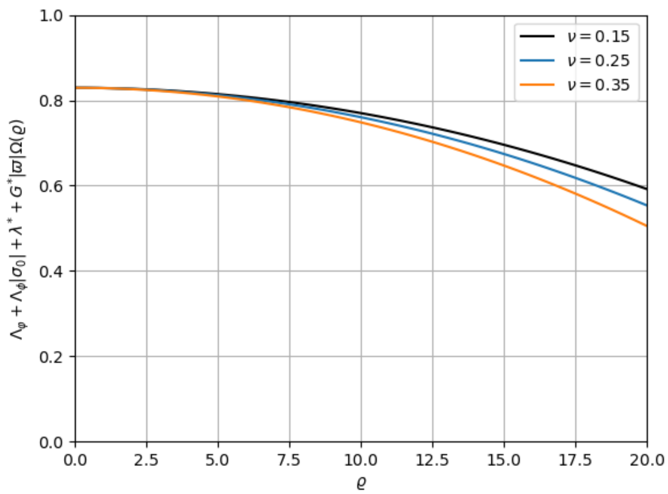

FIGURE 1. Graphical representation of $\Lambda_\varphi + \Lambda_\phi\left(|\sigma_0| + \lambda^* + G^*\|\varpi\|\Omega(\varrho)\right)$, for various values of ν and $\varrho \in [0, 20]$

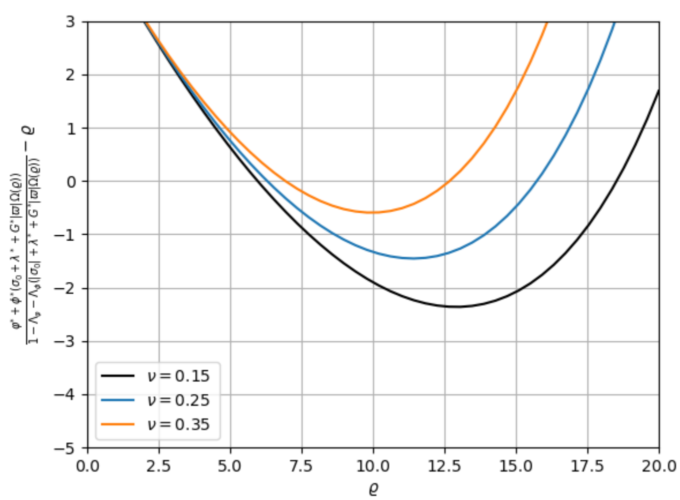

FIGURE 2. Graphical representation of $\Phi(\varrho) - \varrho$, when $\varrho \in [0, 20]$

To solve Problems (4.1), we employ the method introduced by Mekkaoui and Atangana [18], which is based on a two-step Lagrange polynomial interpolation. The approximate solution for various values of ν is presented graphically in Figure 3.

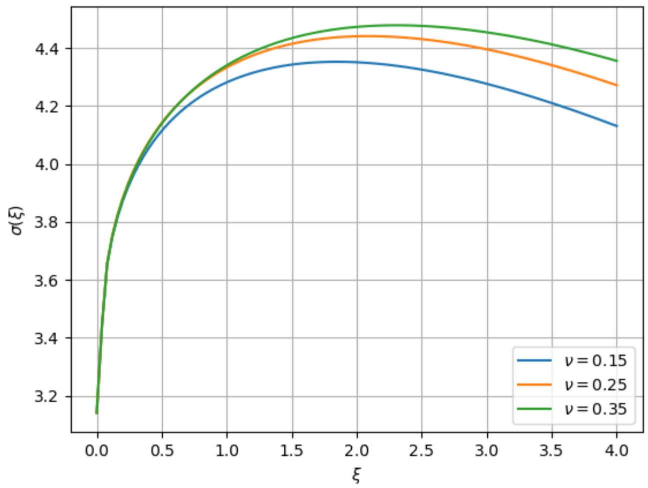

Figure 3. Graphs of the approximate solutions for various values of ν

5. Conclusion

In this study, we have established the existence of solutions for a class of fractional hybrid integro-differential equations with nonlocal boundary conditions. By employing a generalized form of Dhage's hybrid fixed point theorem tailored to the sum of three fractional operators, we have provided a robust theoretical framework for addressing such problems. The theoretical findings were further supported by a concrete example, for which numerical simulations and graphical representations were conducted using Python. These results not only demonstrate the applicability of our approach but also highlight its potential for solving complex problems in fractional calculus involving hybrid and nonlocal structures. Future work may focus on extending this framework to broader classes of fractional systems and exploring stability and uniqueness properties.

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