Gulf Journal of Mathematics
Vol 20, Issue 2 (2025) 39-56

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i2.3046

A NUMERICAL TECHNIQUE TO SOLVE LINEAR STOCHASTIC VOLTERRA INTEGRAL EQUATION

PRIT PRITAM PAIKARAY[1], SANGHAMITRA BEURIA[2*] and NIGAM CHANDRA PARIDA[3]

Abstract. This article introduces a numerical technique using Walsh function approximation and the corresponding operational matrix of integration to address linear stochastic Volterra integral equation. The main objective of the approach is to transform the integral equation into a system of algebraic equations, which is then solved to obtain an approximate solution to the specified linear stochastic Volterra integral equation. The rate of convergence of the proposed method for stochastic Volterra integral equations with Lipschitz continuous functions is discussed and the corresponding error analysis shows a linear order of convergence. The proposed approach exhibits better precision in addressing linear stochastic Volterra integral equations relative to existing methodologies, as shown by multiple examples with readily available analytical solutions.

Keywords. Brownian motion, Walsh approximation, Itô integral, Stochastic Volterra integral equation, Lipschitz function.
2020 Mathematics Subject Classification. Primary 60H05, 60H35, 65C30. Secondary 44B20.

1. Introduction

Stochastic differential equations (SDE), as utilised in several disciplines, such as the biological sciences, finance mathematics (specifically option pricing), agricultural sciences and physical sciences play a significant role [21, 14, 26, 7]. Stochastic Volterra integral equations (SVIE) are highly significant in these particular areas of study[13, 8]. But the most challenging part is to solve many of such problems similar to those in deterministic differential equations which increases the complexity of the problem.

Hence, the utilisation of numerical approximation approaches becomes significant when tackling these problems. The numerical estimation of the solution to a wide range of SVIEs, including both linear SVIEs, linear stochastic Volterra Fredholm integral equations or nonlinear SVIEs, may be achieved applying several numerical techniques [19, 20, 9, 1, 15]. Orthogonal functions like the Haar wavelet,

Date: Received: May 19, 2025; Accepted: Jun 15, 2025.
* Corresponding author

block pulse function (BPF), Laguerre polynomials, Chebyshev's polynomials, Legendre polynomials and many others have recently been used to obtain an approximate solution of the given integral equation [22, 3, 16, 18, 2, 25, 24, 17, 4, 23, 5].

One such set of orthonormal functions is the Walsh functions, which represent an orthonormal basis exclusively accommodating the values -1 and 1. Due to this reason, numerous mathematicians view the Walsh functions formulated in 1923 [27] as an algorithmically created orthonormal system in the field of digital technology. The computer's ability to precisely calculate the exact value for discrete functions at any given moment provides it with a notable advantage over conventional trigonometric functions. In 1975, to solve the variational problem, Hsiao and Chen used the Walsh function approximation method [3]. In 1979, they utilised an identical concept to solve the integral equation [12].The primary benefit of this technique is its ability to convert the issue into an algebraic system, which is subsequently solved to obtain an approximate solution.

In this work, we utilise the Walsh function [27] to approximate the expression $x(t)$ which represents the solution to the SVIE [16] given by the equation:

$$x(t) = f(t) + \int_0^t k_1(s,t)x(s)ds + \int_0^t k_2(s,t)x(s)dB(s) \tag{1.1}$$

where $x(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s,t \in [0,T)$, represent the stochastic processes based on the same probability space (Ω, F, P) and $x(t)$ is unknown. Here $B(t)$ is a Brownian motion [14, 21] and $\int_0^t k_2(s,t)x(s)dB(s)$ is the Itô integral.

The evaluation in most of the earlier works is predominantly based on the assumption that the derivatives $f'(t)$ and $\frac{\partial^2 k_i}{\partial s \partial t}$ for $i = 1,2$ exist and are bounded whereas in this article, we consider Lipschitz continuous functions $f(t), k_1(s,t)$, and $k_2(s,t)$. This approach allows for a linear rate of convergence and enables the consideration of a more general form of SVIE that needs to be integrated. The last stage involves a numerical comparison between the estimated answer and the actual solution to assess the method's validity. To ensure that the procedure is legitimate, the approximate and exact solutions are numerically compared in the last part. This method can be helpful to obtain an approximation of the solution to SVIEs like Lotka-Volterra integral equation, stiff chemical Langevian equation, Black-Scholes model and many others that can be used in study of various real world problems.

2. Walsh Function and its Properties

Definition 2.1 (Rademacher Function)**.** The i-th Rademacher function $r_i(t)$, $i = 1, 2, \ldots$, for $t \in [0,1)$ is defined by [27]

$$r_i(t) = \begin{cases} 1 & i = 0, \\ sgn(sin(2^i \pi t)) & \text{otherwise} \end{cases}$$

where

$$sgn(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

Definition 2.2 (Walsh Function)**.** The n^{th} Walsh function for $n = 0, 1, 2, \cdots$, denoted by $w_n(t)$, $t \in [0, 1)$ is defined [27] as

$$w_n(t) = \prod_{i=1}^{q} (r_i(t))^{b_i}$$

where $n = b_q 2^{q-1} + b_{q-1} 2^{q-2} + b_{q-2} 2^{q-3} + \ldots + b_1 2^0$ is the binary expression of n. Hence, the binary representation of n has $q = \left[\log_2 n\right] + 1$ digits, where $\left[\,\cdot\,\right]$ is the greatest integer less than or equal to $'\cdot'$.

The first m Walsh functions for $m \in \mathbb{N}$ can be written as an m-vector by

$$W(t) = \begin{bmatrix} w_0(t) & w_1(t) & w_2(t) \ldots w_{m-1}(t) \end{bmatrix}^T .$$

The following properties are satisfied by Walsh functions.

Orthonormality. The set of Walsh functions $w_0(t), w_1(t), w_2(t), \ldots$ is orthonormal. i.e.,

$$\int_0^1 w_i(t) w_j(t) dt = \begin{cases} 1 & \text{i=j}, \\ 0 & \text{otherwise}. \end{cases}$$

Completeness. For every $f \in L^2[0, 1)$

$$\int_0^1 f^2(t) dt = \sum_{i=0}^{\infty} f_i^2 ||w_i(t)||^2$$

where $f_i = \int_0^1 f(t) w_i(t) dt$.

Walsh Function Approximation. Since $\{w_i(t)\}$ is a complete orthonormal set, any real-valued function $f(t) \in L^2[0, 1)$ can be approximated by the function $f_m(t)$ given by,

$$f(t) \approx f_m(t) = \sum_{i=0}^{m-1} c_i w_i(t) \quad \textit{where}, \quad c_i = \int_0^1 f(s) w_i(s) ds. \tag{2.1}$$

is called the Walsh coefficient of $f(t)$. This implies,

$$\begin{aligned} f_m(t) &= \sum_{i=0}^{m-1} \left(\int_0^1 f(s) w_i(s) ds \right) w_i(t) \\ &= \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \int_{jh}^{(j+1)h} f(s) w_i(s) ds \right) w_i(t) \end{aligned}$$

As for $s \in (jh, (j+1)h)$, $w_i(s)$ is constant, so we can replace $w_i(s)$ by $w_i(\eta_j)$. where η_j is any point in $(jh, (j+1)h)$. Hence,

$$\begin{aligned} f_m(t) = \quad & \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} w_i(\eta_j) \int_{jh}^{(j+1)h} f(s)ds \right) w_i(t) \\ = \quad & \left(\sum_{j=0}^{m-1} w_0(\eta_j) \int_{jh}^{(j+1)h} f(s)ds \right) w_0(t) \\ + \quad & \left(\sum_{j=0}^{m-1} w_1(\eta_j) \int_{jh}^{(j+1)h} f(s)ds \right) w_1(t) \\ + \quad & \dots \\ + \quad & \left(\sum_{j=0}^{m-1} w_{m-1}(\eta_j) \int_{jh}^{(j+1)h} f(s)ds \right) w_{m-1}(t) \end{aligned}$$

$$\begin{aligned} f_m(t) \quad = \quad & \begin{bmatrix} \int_0^h f(s)ds & \int_h^{2h} f(s)ds & \dots & \int_{(m-1)h}^1 f(s)ds \end{bmatrix} \begin{bmatrix} w_0(\eta_0) \\ w_0(\eta_1) \\ \vdots \\ w_0(\eta_{m-1}) \end{bmatrix} w_0(t) \\ + \quad & \begin{bmatrix} \int_0^h f(s)ds & \int_h^{2h} f(s)ds & \dots & \int_{(m-1)h}^1 f(s)ds \end{bmatrix} \begin{bmatrix} w_1(\eta_0) \\ w_1(\eta_1) \\ \vdots \\ w_1(\eta_{m-1}) \end{bmatrix} w_1(t) + \dots \\ + \quad & \begin{bmatrix} \int_0^h f(s)ds & \int_h^{2h} f(s)ds & \dots & \int_{(m-1)h}^1 f(s)ds \end{bmatrix} \begin{bmatrix} w_{m-1}(\eta_0) \\ w_{m-1}(\eta_1) \\ \vdots \\ w_{m-1}(\eta_{m-1}) \end{bmatrix} w_{m-1}(t) \end{aligned}$$

which can be written as

$$f_m(t) = F^T T_W^T W(t) \tag{2.2}$$

where,

$$F = \begin{bmatrix} f_0 & f_1 & f_2 \dots f_{m-1} \end{bmatrix}^T, \qquad f_i = \int_{ih}^{(i+1)h} f(s)ds \tag{2.3}$$

and the (i,j)-th entry of the matrix T_W is given by the following formula:

$$T_W(i,j) = w_i(\eta_j), i, j = 0, 1, \dots, m-1. \tag{2.4}$$

It can found from [4] that,

$$T_W T_W^T = mI \,\text{and}\, T_W^T = T_W, \text{ which implies that } T_W^{-1} = \frac{1}{m} T_W \tag{2.5}$$

Hence (2.2) becomes

$$f_m(t) = F^T T_W W(t) \tag{2.6}$$

Similarly, approximation of $k(s,t) \in L^2([0,1)\times[0,1))$ can be calculated as

$$k(s,t) \approx k_m(s,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} w_i(s) w_j(t)$$

where, $c_{ij} = \int_0^1 \int_0^1 k(s,t) w_i(s) w_j(t) dt ds$.
with the matrix form as

$$k(s,t) \approx k_m(s,t) = W^T(s) T_W K T_W W(t) = W^T(t) T_W K^T T_W W(s) \tag{2.7}$$

where $K = [k_{ij}]_{m\times m}, k_{ij} = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} k(s,t) dt ds$.

A relation between Walsh function and BPF is developed in the next section which is used subsequently for converting the SVIE to a system of algebraic equations.

3. Relationship between Block Pulse Functions and Walsh Function

Definition 3.1 (Block Pulse Functions)**.** An m-set of BPF $\phi_i(t), t \in [0,1)$ for $i = 0,1,...,m-1$ for a given positive integer m is defined as

$$\phi_i(t) = \begin{cases} 1 & \text{if } \frac{i}{m} \le t < \frac{(i+1)}{m}, \\ 0 & \text{otherwise} \end{cases} .$$

The function ϕ_i is known as the ith BPF.

The vector form of the first m BPFs denoted by $\Phi(t)$ is given by $\Phi(t) = \begin{bmatrix}\phi_0(t) & \phi_1(t) & \phi_2(t)\ldots\phi_{m-1}(t)\end{bmatrix}^T$, $t \in [0,1)$.

The BPFs are orthogonal,disjoint and complete [11].

The vector form of BPF satisfies [11]

$$\Phi(t)\Phi(t)^T X = \tilde{X}\Phi(t) \text{ and } \Phi^T(t) A \Phi(t) = \hat{A}\Phi(t) \tag{3.1}$$

where $X \in \mathbb{R}^{m\times 1}, \tilde{X}$ is the $m\times m$ diagonal matrix with $\tilde{X}(i,i) = X(i)$ for $i = 1,2,3\cdots m, A \in \mathbb{R}^{m\times m}$ and $\hat{A} = \begin{bmatrix} a_{11} & a_{22} & \ldots & a_{mm}\end{bmatrix}^T$ is the m-vector which contains the diagonal entries of A. The BPF vector $\Phi(t)$, $t \in [0,1)$ can be integrated as

$$\int_0^t \Phi(\tau) d\tau = P\Phi(t), t \in [0,1), \tag{3.2}$$

where

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

is called deterministic operational matrix of integration. Hence, for every function $f(t) \in L^2[0,1)$, the integral of $f(t)$ can be approximated as

$$\int_0^t f(s) ds = F^T P \Phi(t)$$

In a similar way, it can be observed that Itô integral of BPF vector $\Phi(t)$, $t \in [0,1)$ can be carried out as [16]

$$\int_0^t \Phi(\tau)dB(\tau) = P_S\Phi(t), t \in [0,1) \tag{3.3}$$

where

$$P_S = \begin{bmatrix} B(\frac{h}{2}) & B(h) & \dots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & \dots & B(2h) - B(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{bmatrix}$$

is known as the stochastic operational matrix of integration. Therefore, the approximation of Itô integral of every function $f(t) \in L^2[0,1)$ can be estimated as [16] by

$$\int_0^t f(s)dB(s) = F^T P_S \Phi(t).$$

A relationship between the the BPF and the Walsh function is described in the next theorem.

Theorem 3.2. *Let the m-set of BPF vector and Walsh function vectors be $\Phi(t)$ and $W(t)$ respectively. Then the Walsh function $W(t)$ can be approximated by using the BPF vectors $\Phi(t)$ as $W(t) = T_W\Phi(t)$, $m = 2^k$, and $k = 0, 1, \dots,$ where $T_W = \big[c_{ij}\big]_{m\times m}$, $c_{ij} = w_i(\eta_j)$, for some $\eta_j = \big(\frac{j}{m}, \frac{j+1}{m}\big)$ and $i, j = 0, 1, 2, \dots m-1$.*

Proof. Suppose $m = 2^k$ and $w_i(t)$, $i = 0, 1, 2, \dots m-1$ is the i^{th} component of the Walsh function vector. Expansion of $w_i(t)$ into vector form of BPFs with m components yields $w_i(t) = \sum_{j=0}^{m-1} c_{ij}\phi_j(t) = C_i^T\Phi(t)$, $i = 0, 1, 2, \dots m-1$, where the i-th row of T_W is C_i^T and the $(i,j)^{th}$ entry of T_W is c_{ij}.

$$c_{ij} = \frac{1}{h}\int_0^1 w_i(t)\phi_j(t)dt = \frac{1}{h}\int_{jh}^{(j+1)h} w_i(t)dt.$$

Applying Mean value theorem for integration we obtain

$$c_{ij} = \frac{1}{h}\int_{jh}^{(j+1)h} w_i(t)dt = \frac{1}{h}\big((j+1)h - jh\big)w_i(\eta_j) = w_i(\eta_j)$$

where $\eta_j \in \big(\frac{j}{m}, \frac{j+1}{m}\big)$, $m = \frac{1}{h}$.
Since $w_i(t)$ is constant in the interval $\big(\frac{j}{m}, \frac{j+1}{m}\big)$, we choose $c_{ij} = w_i(\frac{2j+1}{2m})$, $i, j = 0, 1, 2, \dots m-1$.

Hence $W(t) = T_W\Phi(t)$. $\square$

Theorem 3.2 implies that $\Phi(t) = \frac{1}{m}T_W W(t)$.
The following theorem is an application of the above condition:

Lemma 3.3 (Walsh function integration)**.** *The integral of the Walsh function $W(t)$ is given by*
$\int_0^t W(s)ds = \wedge W(t)$, *where* $\wedge = \frac{1}{m} T_W P T_W$ *and*

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Proof. The integration of the Walsh function vector $W(t)$ with respect to t is

$$\int_0^t W(s)ds = \int_0^t T_W \Phi(s)ds = T_W \int_0^t \Phi(s)ds$$
$$= T_W P \Phi(t) = \Big(T_W P \frac{1}{m} T_W^T\Big) W(t) = \wedge W(t)$$

where $\wedge = \frac{1}{m}\Big(T_W P T_W^T\Big) = \frac{1}{m}\Big(T_W P T_W\Big)$(by (2.5)) □

Here, $\wedge$ is called as the Walsh operational matrix of integration.

Lemma 3.4 (Walsh function and its Stochastic Integration)**.** *The Itô integral of the Walsh function $W(t)$ is* $\int_0^t W(s)dB(s) = \wedge_S W(t)$, *where* $\wedge_S = \frac{1}{m} T_W P_S T_W$ *and*

$$P_S = \begin{bmatrix} B(\frac{h}{2}) & B(h) & \dots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & \dots & B(2h) - B(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{bmatrix}.$$

Proof. The Itô integral of the Walsh function $W(t)$ is

$$\int_0^t W(s)dB(s) = \int_0^t T_W \Phi(s)dB(s) = T_W \int_0^t \Phi(s)dB(s)$$
$$= T_W P_S \Phi(t) = \Big(T_W P_S \frac{1}{m} T_W^T\Big) W(t) = \wedge_S W(t),$$

where $\wedge_S = \frac{1}{m}\Big(T_W P_S T_W^T\Big) = \frac{1}{m}\Big(T_W P_S T_W\Big)$.(Using (2.5)) □

Here, $\wedge_S$ is called the Walsh operational matrix for Itô integral.

4. Numerical Solution of Linear Stochastic Volterra Integral Equation

We deal with the following linear SVIE

$$x(t) = f(t) + \int_0^t k_1(s,t)x(s)ds + \int_0^t k_2(s,t)x(s)dB(s) \tag{4.1}$$

where $x(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) and $x(t)$ is unknown. Also $B(t)$

is Brownian motion process and $\int_0^t k_2(s,t)x(s)dB(s)$ is the Ito Integral.
Using equation (2.6) and(2.7) in (4.1) we have

$$\begin{aligned} X^T T_W W(t) &= F^T T_W W(t) + \int_0^t W^T(t) T_W K_1^T T_W W(s) W^T(s) T_W X ds \\ &\quad + \int_0^t W^T(t) T_W K_2^T T_W W(s) W^T(s) T_W X dB(s) \\ &= F^T T_W W(t) + W^T(t) T_W K_1^T T_W \int_0^t W(s) W^T(s) T_W X ds \\ &\quad + W^T(t) T_W K_2^T T_W \int_0^t W(s) W^T(s) T_W X dB(s) \end{aligned} \tag{4.2}$$

Now

$$\begin{aligned} &\int_0^t W(s) W^T(s) T_W X ds \\ &= \int_0^t T_W \Phi(s) \Phi^T(s) T_W T_W X ds \quad \text{(By Theorem:3.2)} \\ &= m T_W \int_0^t \Phi(s)\Phi^T(s) X ds \quad \text{(By (2.5))} \\ &= m T_W \tilde{X} \int_0^t \Phi(s) ds \quad \text{(By (3.1))} \\ &= m T_W \tilde{X} P \frac{1}{m} T_W W(t). \quad \text{(By (3.2))} \end{aligned}$$

Hence

$$\int_0^t W(s) W^T(s) T_W X ds = T_W \tilde{X} P T_W W(t) \tag{4.3}$$

Similarly,

$$\int_0^t W(s) W^T(s) T_W X dB(s) = m T_W \tilde{X} P_S \frac{1}{m} T_W W(t) = T_W \tilde{X} P_S T_W W(t) \tag{4.4}$$

Putting (4.3) and (4.4) in (4.2) and using the orthonormality condition, we get

$$\begin{aligned} X^T T_W W(t) &= F^T T_W W(t) + m W^T(t) T_W K_1^T \tilde{X} P T_W W(t) \\ &\quad + m W^T(t) T_W K_2^T \tilde{X} P_S T_W W(t) \\ &= F^T T_W W(t) + W^T(t) T_W H_1 T_W W(t) \\ &\quad + W^T(t) T_W H_2 T_W W(t) \end{aligned}$$

where $H_1 = m K_1^T \tilde{X} P$, $H_2 = m K_2^T \tilde{X} P_S$.
which implies that

$$X^T T_W W(t) = F^T T_W W(t) + m \hat{H_1}^T T_W W(t) + m \hat{H_2}^T T_W W(t)$$

Here,

$$\left(X^T - F^T - m\hat{H_1}^T - m\hat{H_2}^T \right) T_W W(t) = 0 \tag{4.5}$$

Hence, a non trivial solution of SVIE (4.1) can be computed by solving

$$\left(X - F - m\hat{H}_1 - m\hat{H}_2\right) = [0]_{m\times 1} \tag{4.6}$$

5. Error Analysis

This section emphasises examining a disparity between the exact solution and the approximate solution of the SVIE. Before start of the analysis, we define $\|X\|_2 = E(|X|^2)^{\frac{1}{2}}$, where E is the expectation of the random variable $|X|^2$.

Theorem 5.1. *Suppose $f \in L^2[0,1)$ is a Lipschitz function with Lipschitz constant C. Then $\|e_m(t)\|_2 = O(h)$, where $e_m(t) = |f(t) - \sum_{i=0}^{m-1} c_i w_i(t)|$ and $c_i = \int_0^1 f(s)w_i(s)ds$.*

Proof. Let $f_m(t) = \sum_{i=0}^{m-1} c_i w_i(t)$ where $c_i = \int_0^1 f(s)w_i(s)ds$ be the approximation of the Lipschitz function $f(t)$.
Now,

$$e_m(t) = |f(t) - f_m(t)| \leq \omega(\frac{1}{2^k}, f) \leq Ch.$$

Here $\omega(\frac{1}{2^k}, f)$ is called the modulus of continuity of the function f [10].
Therefore,

$$\begin{aligned} \|e_m(t)\|_2 &= \left(\int_0^1 |f(t) - f_m(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq Ch\int_0^1 dt = O(h). \end{aligned}$$

□

Theorem 5.2. *Suppose $k \in L^2\big([0,1)\times[0,1)\big)$ is a Lipschitz function with Lipschitz constant L. If $k_m(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} w_i(x) w_j(y)$, $c_{ij} = \int_0^1\int_0^1 k(s,t)w_i(s)w_j(t)dtds$, then $\|e_m(x,y)\|_2 = O(h)$, where $|e_m(x,y)| = |k(x,y) - k_m(x,y)|$.*

Proof. It is clear from [10] that

$$\begin{aligned} k_m(x,y) &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\left(\int_0^1\int_0^1 k(s,t)w_i(s)w_j(t)dtds\right)w_i(x)w_j(y) \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(\int_0^1\int_0^1 k(s,t)w_i(s)w_i(x)w_j(t)w_j(y)dtds) \\ &= \int_0^1\int_0^1 k(s,t)D_m(t\oplus y)D_m(s\oplus x)dtds \end{aligned}$$

where $D_m(t) = \sum_{i=0}^{m-1} w_i(t)$ is called the Dirichlet kernel of order m [10]. Then by the definition of Dirichlet kernel and translation invariance of the integral, we have

$$k_m(x,y) = 2^k.2^k\int_{\Delta_i^{(k)}}\int_{\Delta_j^{(k)}} k(s,t)dtds$$

where $\Delta_i^{(k)} = \left[\frac{i}{2^k}, \frac{(i+1)}{2^k}\right)$. Hence,

$$|k_m(X) - k(X)| \leq 2^{2k} \int_{\Delta_i^{(k)}} \int_{\Delta_j^{(k)}} |k(T) - k(X)| dT$$

where $T = (s,t)$ and $X = (x,y)$. Also observe that if k is uniformly Lipschitz with Lipschitz constant L, then

$$|k_m(X) - k(X)| \leq 2^{2k} \int_{\Delta_i^{(k)}} \int_{\Delta_j^{(k)}} L|T - X| dT$$

Therefore,

$$\|k_m(X) - k(X)\|_2 \leq \sqrt{2} L h = O(h).$$

$\square$

Theorem 5.3. *Suppose $x_m(t)$ is the approximate solution of the linear SVIE (1.1). If*

a) $k_1(s,t), \quad k_2(s,t) \in L^2\big([0,1) \times [0,1)\big)$ *and $f \in L^2[0,1)$ are Lipschitz functions with Lipschitz constants L_1, L_2 and C respectively,*

b) $|x(t)| \leq \sigma$, $|k_1(s,t)| \leq \rho_1$ *and* $|k_2(s,t)| \leq \rho_2$

then

$$\|x(t) - x_m(t)\|_2^2 = O(h^2)$$

Proof. Let (1.1) be the given SVIE and $x_m(t)$ be the approximation to the solution using the Walsh function. Then

$$\begin{aligned} x(t) - x_m(t) &= f(t) - f_m(t) \\ &+ \int_0^t \big(k_1(s,t)x(s) - k_{1m}(s,t)x_m(s)\big) ds \\ &+ \int_0^t \big(k_2(s,t)x(s) - k_{2m}(s,t)x_m(s)\big) dB(s) \end{aligned}$$

implies

$$\begin{aligned} |x(t) - x_m(t)| &\leq |f(t) - f_m(t)| \\ &+ \left| \int_0^t \big(k_1(s,t)x(s) - k_{1m}(s,t)x_m(s)\big) ds \right| \\ &+ \left| \int_0^t \big(k_2(s,t)x(s) - k_{2m}(s,t)x_m(s)\big) dB(s) \right|. \end{aligned}$$

We know that $(a+b+c)^2 \leq 5a^2 + 5b^2 + 5c^2$.
Therefore,

$$\begin{aligned} |x(t) - x_m(t)|^2 &\leq 5|f(t) - f_m(t)|^2 \\ &+ 5\left| \int_0^t \big(k_1(s,t)x(s) - k_{1m}(s,t)x_m(s)\big) ds \right|^2 \\ &+ 5\left| \int_0^t \big(k_2(s,t)x(s) - k_{2m}(s,t)x_m(s)\big) dB(s) \right|^2. \end{aligned}$$

This implies

$$E\big(|x(t)-x_m(t)|^2\big) \;\leq\; 5E\bigg(|f(t)-f_m(t)|^2\bigg)+5I_1+5I_2.$$

where

$$I_1 = E\bigg(\bigg|\int_0^t \big(k_1(s,t)x(s)-k_{1m}(s,t)x_m(s)\big)ds\bigg|^2\bigg),$$

and

$$I_2 = E\bigg(\bigg|\int_0^t \big(k_2(s,t)x(s)-k_{2m}(s,t)x_m(s)\big)dB(s)\bigg|^2\bigg) \tag{5.1}$$

Now for $i=1,2$, we have

$$\begin{aligned}|k_i(s,t)x(s)-k_{im}(s,t)x_m(s)| \leq \quad & |k_i(s,t)||x(s)-x_m(s)| \\ + \quad & |k_i(s,t)-k_{im}(s,t)||x(s)| \\ + \quad & |k_i(s,t)-k_{im}(s,t)||x(s)-x_m(s)|\end{aligned}$$

For $i=1,2$, let $|k_i(s,t)|\leq\rho_i$, $|x(s)|\leq\sigma$ and using Theorem 5.2, we get

$$|k_i(s,t)x(s)-k_{im}(s,t)x_m(s)| \leq \sqrt{2}L_ih\sigma+(\rho_i+\sqrt{2}L_ih)|x(s)-x_m(s)| \tag{5.2}$$

which give

$$\begin{aligned}I_1 \;&\leq\; E\bigg(\bigg(\int_0^t\Big|k_1(s,t)x(s)-k_{1m}(s,t)x_m(s)\Big|ds\bigg)^2\bigg)\\ &\leq\; E\bigg(\bigg(\int_0^t\big(\sqrt{2}L_ih\sigma+(\rho_i+\sqrt{2}L_ih)|x(s)-x_m(s)|\big)ds\bigg)^2\bigg)\end{aligned} \tag{5.3}$$

By Cauchy- Schwarz inequality, for $t>0$ and $f\in L^2[0,1)$

$$\bigg|\int_0^t f(s)ds\bigg|^2 \leq t\int_0^t |f|^2ds. \tag{5.4}$$

Considering the fact that $t\in[0,1)$ and $\int_0^t |f|^2ds\geq 0$, the above equation implies that

$$\bigg|\int_0^t f(s)ds\bigg|^2 \leq \int_0^t |f|^2ds. \tag{5.5}$$

This implies (5.3) becomes

$$I_1 \;\leq\; E\bigg(2\int_0^t\Big((\sqrt{2}L_1h\sigma)^2+(\rho_1+\sqrt{2}L_1h)^2|x(s)-x_m(s)|^2\Big)ds\bigg)$$

Therefore, using the condition that $(a+b)^2\leq 2a^2+2b^2$, the above equation can be rewritten as

$$I_1 \;\leq\; 2(\sqrt{2}L_1h\sigma)^2+2(\rho_1+\sqrt{2}L_1h)^2E\bigg(\int_0^t |x(s)-x_m(s)|^2ds\bigg) \tag{5.6}$$

Now, using the Itô isometry, (5.1) reduced to

$$\begin{aligned} I_2 = \quad & E\bigg(\int_0^t \bigg| k_2(s,t)x(s) - k_{2m}(s,t)x_m(s)\bigg|^2 ds\bigg) \\ \leq \quad & 2E\bigg(\int_0^t \big((\sqrt{2}L_2h\sigma)^2 + (\rho_2 + \sqrt{2}L_2h)^2|x(s) - x_m(s)|^2\big)ds\bigg) \text{ (Using (5.2))} \end{aligned}$$

Hence,

$$I_2 \leq \quad 2(\sqrt{2}L_2h\sigma)^2 + 2(\rho_2 + \sqrt{2}L_2h)^2 E\bigg(\int_0^t |x(s) - x_m(s)|^2 ds\bigg) \tag{5.7}$$

Using Theorem 5.1, equation (5.6) and (5.7) in (5.1), we get

$$\begin{aligned} E\big(|x(t) - x_m(t)|^2\big) \ \leq \ & 5C^2h^2 \\ + \ & 5\bigg(2(\sqrt{2}L_1h\sigma)^2 + 2(\rho_1 + \sqrt{2}L_1h)^2 E\bigg(\int_0^t |x(s) - x_m(s)|^2 ds\bigg)\bigg) \\ + \ & 5\bigg(2(\sqrt{2}L_2h\sigma)^2 + 2(\rho_2 + \sqrt{2}L_2h)^2 E\bigg(\int_0^t |x(s) - x_m(s)|^2 ds\bigg)\bigg) \end{aligned}$$

$$E\big(|x(t) - x_m(t)|^2\big) \ \leq \ R_1 + R_2 \int_0^t E\big(|x(s) - x_m(s)|^2\big) ds \tag{5.8}$$

where

$$R_1 = 5\bigg(C^2h^2 + 2(\sqrt{2}L_1h\sigma)^2 + 2(\sqrt{2}L_2h\sigma)^2\bigg)$$

and

$$R_2 = 5\bigg(2(\rho_1 + \sqrt{2}L_1h)^2 + 2(\rho_2 + \sqrt{2}L_2h)^2\bigg)$$

By using Gronwall's inequality, we have

$$E\big(|x(t) - x_m(t)|^2\big) \ \leq \ R_1 \exp\bigg(\int_0^t R_2 ds\bigg). \tag{5.9}$$

which implies that,

$$\|x(t) - x_m(t)\|_2^2 = E\big(|x(t) - x_m(t)|^2\big) \leq R_1 e^{R_2} = O(h^2) \tag{5.10}$$

□

6. Numerical Examples

In this section we implement the suggested approach to address various SVIEs. The first two examples illustrate the convergence of the procedure by comparing approximate and analytical solution. Due to the impracticality of finding an analytical solution for every SVIE, the last example demonstrates approximate solutions for $m = 32, 64$, and 128 to illustrate convergence.The calculations are performed using Matlab 2013(a).

Here E represent the error, defined as the maximum absolute difference between the Walsh coefficients(X_i) of the exact solution ($x(t)$) and the Walsh

TABLE 1. Mean error $\bar{x}_E$, standard deviation of error s_E, and confidence interval for mean error in Example 6.1 with $m = 8$

n	$\bar{x}_E$	s_E	95% confidence interval for mean error.	
			Lower	Upper
30	0.00543042339	0.00472214581	0.00374062521	0.00712022157
50	0.00626993437	0.00442989552	0.00504202998	0.00749783876
75	0.00705047567	0.00481071208	0.00596170903	0.00813924231
100	0.00640558992	0.00481776079	0.00546130880	0.00734987103
125	0.00689936851	0.00500025280	0.00602278555	0.00777595148
150	0.00686900260	0.00580316061	0.00594030348	0.00779770171
200	0.00682115439	0.00600805207	0.00598848085	0.00765382792

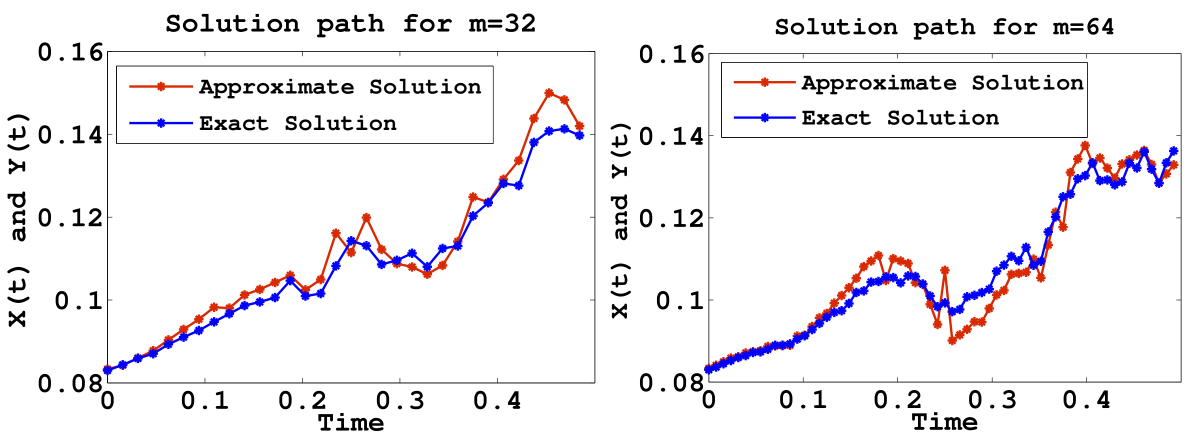

FIGURE 1. Example 6.1's exact solution$(x(t))$ and approximate solution$(y(t))$ for $m = 32$ and $m = 64$.

coefficients(Y_i) approximation solution $(y(t))$ as mentioned in (2.1) that is obtained by solving (4.6), denoted as $\|E\|_\infty = \max_{1\le i\le m} |X_i - Y_i|$, where $x(t) = \sum_{i=0}^{m-1} X_i w_i(t)$ In the following examples, n represents the number of iterations, $\bar{x}_E$ denotes the mean of error E, and s_E represents the standard deviation of error E.

Example 6.1. [18, 16] Consider the linear SVIE

$$x(t) = \frac{1}{12} + \int_0^t \cos(s)x(s)ds + \int_0^t \sin(s)x(s)dB(s), s, t \in [0, 0.5)$$

with the exact solution $x(t) = \frac{1}{12} e^{\frac{-t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s)}$, for $0 \le t < 0.5$.

We can see from Figure 1 that as the value of m increases from 32 to 64, the approximate solution and the exact solution comes close to each other justifying the feasibility and effectiveness of the method. Also figure 2 gives the error trend of the method for $m = 32$ with 30 and 100 iterations. Tables 1 and

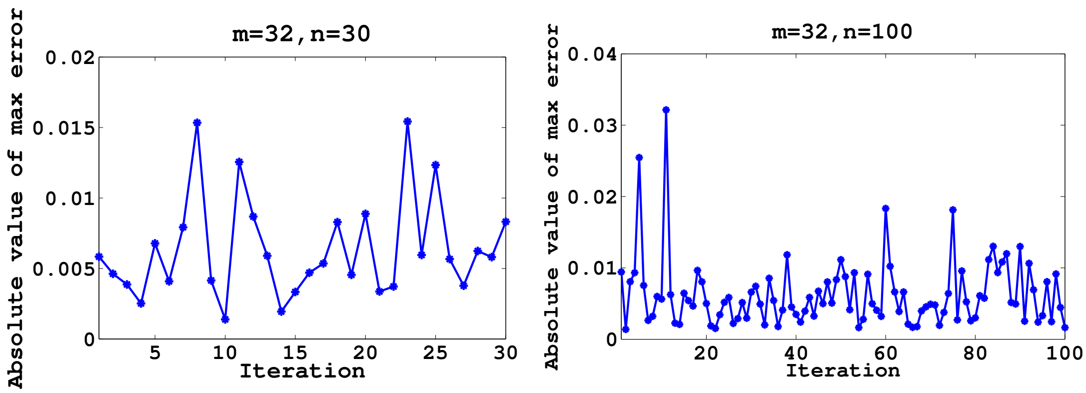

FIGURE 2. Error trend of Example 6.1 for $m = 32$, $n = 30$, and $n = 100$.

TABLE 2. Mean error $\bar{x}_E$, standard deviation of error s_E, and confidence interval for mean error in Example 6.1 with $m = 32$

n	$\bar{x}_E$	s_E	95% confidence interval for mean error.	
			Lower	Upper
30	0.00637765274	0.00360745366	0.00508674202	0.00766856345
50	0.00720684095	0.00586365605	0.00558151841	0.00883216349
75	0.00649984610	0.00488908128	0.00539334285	0.00760634936
100	0.00625583011	0.00474702145	0.00532541390	0.00718624631
125	0.00675880050	0.00523369353	0.00584129357	0.00767630743
150	0.00650117417	0.00478655986	0.00573516505	0.00726718328
200	0.00627666571	0.00451326428	0.00565115920	0.00690217223

2 gives a numerical date of the mean error, standard deviation of error and the confidence interval of mean error for $m = 8$ and $m = 32$ respectively with different itterations.

Example 6.2. [5] Consider the linear stochastic Volterra integral equation

$$x(t) = \frac{1}{3} + \int_0^t \ln(s+1)x(s)ds + \int_0^t \sqrt{\ln(s+1)}x(s)dB(s), s, t \in [0, 0.5)$$

with the exact solution $x(t) = \frac{1}{3}e^{\frac{-t}{2}+\frac{1}{2}\ln(t+1)t+\frac{1}{2}\ln(t+1)+\int_0^t \sqrt{\ln(s+1)}dB(s)}$, for $0 \leq t < 0.5$.

Figure 3 illustrates that when the value of m increases from 32 to 64, the approximate solution converges with the exact solution, hence validating the method's feasibility and efficacy. Figure 4 illustrates the error trend of the approach for $m = 32$ with 30 and 100 iterations. Tables 3 and 4 present numerical data on the mean error, standard deviation of error, and the confidence interval of mean error for $m = 8$ and $m = 32$, respectively, over different iterations.

TABLE 3. Mean error, standard deviation of error, and interval of confidence for mean error in Example 6.2 with m=8

n	$\bar{x}_E$	s_E	95% interval of confidence for error mean.	
			Lower	Upper
30	0.02438967318	0.01686869621	0.01835328777	0.03042605860
50	0.02133558510	0.01927290494	0.01599340841	0.02667776180
75	0.02481916653	0.02431454695	0.01931626652	0.03032206654
100	0.03117403473	0.02410897354	0.02644867591	0.03589939354
125	0.02798218759	0.02469707194	0.02365259998	0.03231177519
150	0.02820020531	0.02390153628	0.02437516286	0.03202524775
200	0.02393113488	0.02147434937	0.02095494190	0.02690732786

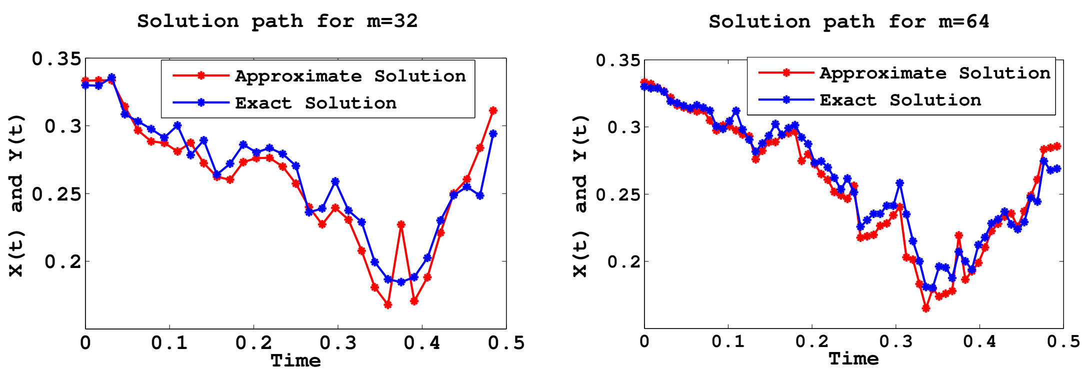

FIGURE 3. Example 6.2's solution($x(t)$) and approximate solution($y(t)$) exact for m=32 and m=64

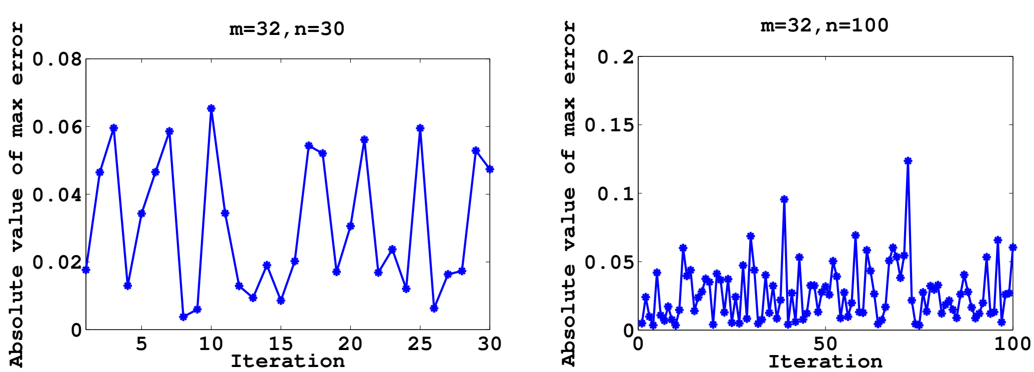

FIGURE 4. Example 6.2's error trend for m=32,n=30, and n=100

TABLE 4. Mean error, standard deviation of error, and interval of confidence for mean error in Example 6.2 with m=32

n	$\bar{x}_E$	s_E	95% interval of confidence for error mean.	
			Lower	Upper
30	0.03060433176	0.02005909861	0.02342627551	0.03778238801
50	0.02486336513	0.02210900512	0.01873506157	0.03099166869
75	0.02616748611	0.01892441632	0.02188448750	0.03045048472
100	0.02703621216	0.02121478100	0.02287811508	0.03119430924
125	0.02533150170	0.02023967788	0.02178332972	0.02887967367
150	0.03001731579	0.02289160166	0.02635389655	0.03368073503
200	0.02681006075	0.02044064753	0.02397713153	0.02964298996

7. Conclusion

Numerical methods are necessary to tackle the challenges associated with determining the precise solution for a large number of SVIEs. In the past, various numerical methods have been devised to estimate the solution of Stochastic Volterra Integral Equations (SVIEs). This research article presents a numerical technique for estimating solutions to the linear SVIEs. Additionally, it includes numerical evaluations for certain SVIEs. An error analysis of the methodology has been performed to verify its reliability. The Walsh function approximation is believed to be more precise than existing techniques for solving linear SVIEs, as evidenced by the numerical analysis conducted in several preexisting examples. The scope of this concept can be broadened to encompass nonlinear SVIEs and SVIEs with singular kernels, which have the potential to address a wide range of physical challenges.

Acknowledgement. The authors would like to thank the reviewers for their helpful suggestions and comments, which improved the quality of this paper.

References

[1] Abdelkawy, M. (2023). Shifted Legendre spectral collocation technique for solving stochastic Volterra–Fredholm integral equations. International Journal of Nonlinear Sciences and Numerical Simulation, 24(1), 123-136. https://doi.org/10.1515/ijnsns-2020-0144

[2] Ahmadinia, M., Afshariarjmand, H., & Salehi, M. (2023). Numerical solution of Itô–Volterra integral equations by the QR factorization method. Journal of Applied Mathematics and Computing, 69(4), 3171-3188. https://doi.org/10.1007/s12190-023-01873-9

[3] Chen, C. F., & Hsiao, C. H. (1975). A Walsh series direct method for solving variational problems. Journal of the Franklin Institute, 300(4), 265-280. https://doi.org/10.1016/0016-0032(75)90199-4

[4] Cheng, C. F., Tsay, Y. T., & Wu, T. T. (1977). Walsh operational matrices for fractional calculus and their application to distributed systems. Journal of the Franklin Institute, 303(3), 267-284. https://doi.org/10.1016/0016-0032(77)90029-1

[5] Cheraghi, T. A. A., Khodabin, M., & Ezzati, R. (2019). Numerical solution of linear stochastic Volterra integral equations via new basis functions. Filomat, 33(18), 5959-5966. https://doi.org/10.2298/FIL1918959C

[6] Etheridge, A. (2002). A course in financial calculus. Cambridge University Press.

[7] Evans, L. C. (2012). An introduction to stochastic differential equations (Vol. 82). American Mathematical Soc.

[8] Fabiano, N., & Radenovic, S. (2021). Geometric Brownian motion and a new approach to the spread of Covid-19 in Italy. Gulf Journal of Mathematics, 10(2), 25-30. https://doi.org/10.56947/gjom.v10i2.516

[9] Feng, Q., & Zhang, J. (2023). Cubature Method for Stochastic Volterra Integral Equations. SIAM Journal on Financial Mathematics, 14(4), 959-1003. https://doi.org/10.1137/22M146889X

[10] Golubov, B., Efimov, A., & Skvortsov, V. (2012). Walsh series and transforms: theory and applications (Vol. 64). Springer Science & Business Media.

[11] Hatamzadeh-Varmazyar, S., Masouri, Z., & Babolian, E. (2016). Numerical method for solving arbitrary linear differential equations using a set of orthogonal basis functions and operational matrix. Applied Mathematical Modelling, 40(1), 233-253. https://doi.org/10.1016/j.apm.2015.04.048

[12] Hsiao, C. H., & Chen, C. F. (1979). Solving integral equations via Walsh functions. Computers & Electrical Engineering, 6(4), 279-292. https://doi.org/10.1016/0045-7906(79)90034-X

[13] Izaddine, H. G., Deme, S., & Dabye, A. S. (2025). Analysis of fractional Black-Scholes, Ornstein-Uhlenbeck, and Langevin models: a minimum distance estimation approach. Gulf Journal of Mathematics, 19(1), 217-250. https://doi.org/10.56947/gjom.v19i1.2545

[14] Kloeden, P. E., Platen, E., Kloeden, P. E., & Platen, E. (1992). Stochastic differential equations (pp. 103-160). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-662-12616-5

[15] Maleknejad, K., Basirat, B., & Hashemizadeh, E. (2011). Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations. Computers & Mathematics with applications, 61(9), 2821-2828. https://doi.org/10.1016/j.camwa.2011.03.055

[16] Maleknejad, K., Khodabin, M., & Rostami, M. (2012). Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions. Mathematical and computer Modelling, 55(3-4), 791-800. https://doi.org/10.1016/j.mcm.2011.08.053

[17] Maleknejad, K., Sohrabi, S., & Rostami, Y. (2007). Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials. Applied Mathematics and Computation, 188(1), 123-128. https://doi.org/10.1016/j.amc.2006.09.099

[18] Mohammadi, F. (2016). Numerical solution of stochastic Ito-Volterra integral equations using Haar wavelets. Numerical mathematics: theory, methods and applications, 9(3), 416-431. https://doi.org/10.4208/nmtma.2016.m1425

[19] Momenzade, N., Vahidi, A. R., & Babolian, E. (2023). A numerical method for solving stochastic volterra-fredholm integral equation. Iranian Journal of Mathematical Sciences and Informatics, 18(1), 145-164. http://dx.doi.org/10.52547/ijmsi.18.1.145

[20] Oduselu–Hassan, E. O., Njoseh, I. N., & Tsetimi, J. (2023) . A Numerical Approximation of the Stochastic Ito-Volterra Integral Equation. Asian Research Journal of Mathematics, 19 (11). pp. 61-68. https://doi.org/10.9734/ARJOM/2023/V19I11753

[21] Øksendal, B., & Øksendal, B. (2003). Stochastic differential equations (pp. 65-84). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-14394-6

[22] Paley, R. E. A. C. (1932). A remarkable series of orthogonal functions. Proc. London Math. Soc, 34(2), 241-279.

[23] Rao, G. P. (Ed.). (1983). Piecewise constant orthogonal functions and their application to systems and control. Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/BFb0041228

[24] Saha Ray, S., & Singh, S. (2020). New stochastic operational matrix method for solving stochastic Itô–Volterra integral equations characterized by fractional Brownian motion. Stochastic Analysis and Applications, 39(2), 224-234. https://doi.org/10.1080/07362994.2020.1794892

[25] Singh, S., & Saha Ray, S. (2019). Stochastic operational matrix of Chebyshev wavelets for solving multi-dimensional stochastic Itô–Volterra integral equations. International Journal of Wavelets, Multiresolution and Information Processing, 17(03), 1950007. https://doi.org/10.1142/S0219691319500073

[26] Tudor, C., & Tudor, M. (1995). Approximation schemes for Itô-Volterra stochastic equations. Boletín de la Sociedad Matemática Mexicana: Tercera Serie, 1(1), 73-85.

[27] Walsh, J. L. (1923). A closed set of normal orthogonal functions. American Journal of Mathematics, 45(1), 5-24. https://doi.org/10.2307/2387224

[1] Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, India

Email address: paikaraypritpritam@gmail.com

[2] Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, India

Email address: sbeuria@ouat.ac.in

[3] Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, India

Email address: ncparida@ouat.ac.in